19th Brazilian Congress of Thermal Sciences and Engineering
November 06th-10th, 2022, Bento Gonçalves - RS - Brazil

ENC-2022-0211
THERMAL CONTACT CONDUCTANCE ESTIMATION USING THE RECIPROCITY FUNCTIONAL AND AN ORTHOGONAL DECOMPOSITION FOR THE MEASUREMENTS

Matheus A. M. Campos
Marcelo J. Colaço
Gabriel L. Verissimo
Department of Mechanical Engineering, POLI/COPPE, Federal University of Rio de Janeiro, Cidade Universitária, Cx. Postal 68503, Rio de Janeiro, 21941-972, Brazil
matheusamc@poli.urj.br, mc@pobox.com and gabrielverissimo@mecanica.coppe.ufrj.br

Abstract. *Heat transfer between two solids in contact is a problem of great interest, appearing in many practical situations in diverse fields such as electronics, nuclear, aerospace, and biomedical engineering. In actual applications, the contact between the bodies is not perfect, leading to a discontinuity in temperature and affecting the heat flux in the interface between them. An approach to this problem is to consider that there is a thermal contact conductance in the interface of the bodies and defining it as the ratio between heat flux and the temperature jump in this region. The thermal contact conductance estimate turns out to be both an important and challenging task, and inverse problem techniques seem to be very useful to solve it. A previous methodology has been developed applying the Reciprocity Gap Functional which involves the solution of a set of auxiliary problems, and use their results to perform an integral to find the thermal contact conductance. There are many options to solve the auxiliary problems, both numerical and analytical. The Classical Integral Transform Method is a suitable technique, and it is capable to generate an analytical solution to the estimation problem. Combining the Reciprocity Gap Functional and the Classical Integral Transform Method leads to a non-intrusive and non-iterative analytical method that can estimate the thermal contact conductance using some temperature measurements taken on the exterior surface of the bodies at a low computational cost. Further improvements can be made in the methodology leading to simpler solutions in the estimation and making it easier to analyze solutions convergence conditions. To achieve it, we propose in this work to decompose the temperature measurements in a proper orthogonal base, which may simplify even more the previously presented technique.*

Keywords: *Inverse problems, Reciprocity functional and Contact resistance*

1. INTRODUCTION

In many engineering applications, there is the need to put two different solid material bodies in contact. When these bodies are brought into contact, heat is conducted from one to another, but since the contact between these bodies is not actually perfect, there is a discontinuity in the temperature field and the heat flux is affected in the interface between them. This kind of problem appears in many applications of great interest concerning situations in diverse fields such as electronics (Xuan, Cui and Zang, 2014), nuclear (Milosevic, Raynaud and Maglic, 2002), aerospace (Gilmore, 2002), and biomedical engineering (McWaid and Marschall, 1992). To model the heat transfer in this situation turns out to be both an important and challenging task. An approach to this problem is to consider that there is a thermal contact conductance in the interface of the bodies and defining it as the ratio between heat flux and the temperature jump in this region (Ozisik, 1993). When using this model, the problem turns to the estimation of the thermal contact conductance in the contact interface. Inverse problem techniques seem to be very useful to solve it, and a previous methodology has been developed applying the Reciprocity Gap Functional (Colaço and Alves, 2012; Colaço and Alves, 2013; Abreu et al., 2014; Colaço, Alves and Orlande, 2014; Colaço and Alves, 2015; Abreu et al., 2015; Padilha and Colaço, 2015; Padilha et al., 2016; Abreu et al., 2016).

The Reciprocity Gap Functional (RF) concept was developed by Andrieux and Ben (1993), starting from the Maxwell-Betti reciprocity theorem. This concept is centered in the idea that a field in equilibrium inside a material body will give difference responses when disturbed according to the existence or not of discontinuities inside the body. An adaptation in this concept was made by Colaço and Alves (2012), resulting in a non-intrusive and non-iterative method that can estimate the thermal contact conductance using some temperature measurements taken on the exterior surface of a body. This method involves two main parts: (i) the solution of a set of auxiliary problems, and (ii) the use of their results to perform an integral to find the thermal contact conductance. There are many options to solve the auxiliary problems, both numerical and analytical. Previous works (Padilha and Colaço, 2015; Padilha et al., 2016) showed that the Classical Integral Transform Method (CITT) is a suitable technique, and it is capable to generate an analytical solution to the estimation problem. Combining the RF and the CITT leads to an analytical solution to the inverse problem. This new methodology was first proposed to regular geometries, but it was later generalized to irregular geometries (Freitas, 2019). This combined technique provided good results at a low computational cost.

Although the combining of these methods results in an analytical expression for the estimation of the thermal contact conductance, there remains an integral in these expressions, which must be solved numerically. These integrals involve the temperature measurements and the eigenfunctions that arise from the CITT implementation. If these temperature measurements would be properly approximated by functions in the same function space as the CITT solution, further simplifications could be made. Our aim in this work is to follow this idea and to find a properly estimation to the temperature measurements in order to obtain a simpler solution for the estimation problem and provide a full analytical expression.

2. PHYSICAL PROBLEM

In this work, the same problem initially proposed by Colaço and Alves (2012), and for which an analytical solution for the thermal contact conductance estimation was achieved by Padilha et al. (2016), is considered. It consists of a two-dimensional steady-state heat transfer problem in two isotropic rectangular bodies in contact, as shown in Fig. 1. These bodies will be referred respectively as Ω_1 and Ω_2, and the region between them, where there is a thermal contact conductance, h_c, to be estimated, will be referred to as Γ . We will assume that the materials have constant thermal conductivities, k_1 and k_2, and that their lateral boundary surfaces Γ_1 and Γ_2 are thermally insulated, while a known heat flux, q, is applied to the top surface, Γ_0, of material Ω_1 and the temperature, T, at the bottom surface, Γ_{00}, of the second material is fixed.

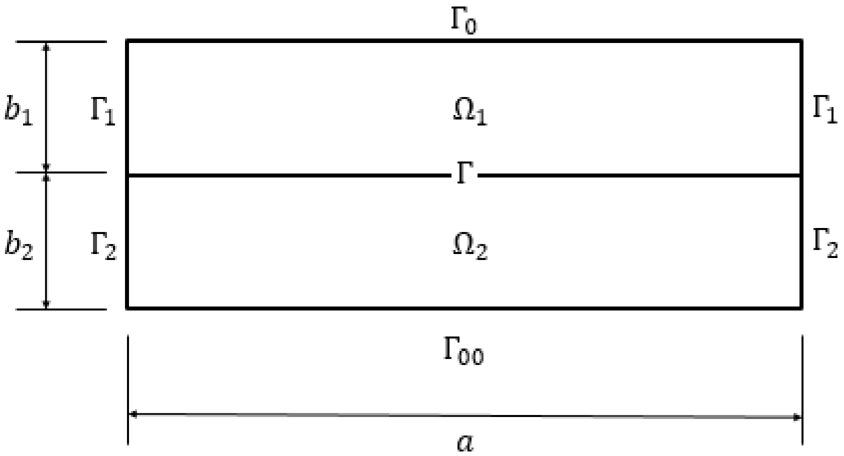

Figure 1. Studied body scheme.

This problem will lead to the following set of equations, from Eq. (1) to Eq. (8).

$$\nabla^2 T_1 = 0 \qquad \text{in } \Omega_1 \tag{1}$$

$$-k_1 \frac{\partial T_1}{\partial y} = 0 \qquad \text{at } \Gamma_0 \tag{2}$$

$$\frac{\partial T_1}{\partial x} = 0 \qquad \text{at } \Gamma_1 \tag{3}$$

$$k_1 \frac{\partial T_1}{\partial y} = h_c (T_1 - T_2) \qquad \text{at } \Gamma \tag{4}$$

$$\nabla^2 T_2 = 0 \qquad \text{in } \Omega_2 \tag{5}$$

$$T_2 = T_{ref} \quad \text{at } \Gamma_{00} \tag{6}$$

$$\frac{\partial T_2}{\partial x} = 0 \quad \text{at } \Gamma_2 \tag{7}$$

$$-k_1 \frac{\partial T_1}{\partial y} = k_2 \frac{\partial T_2}{\partial y} \quad \text{at } \Gamma \tag{8}$$

2.1 Physical problem simulation

To solve the inverse problem, it is needed to generate simulated temperature measurements at the surface of the body. The direct problem presented above is solved through the finite difference method and its results is used as simulated temperature measurements. Measurement noises will also be simulated by adding a Gaussian noise to the solution of the direct problem as presented in Eq. (9).

$$Y = T(\Gamma_0) + \epsilon\sigma \tag{9}$$

The measurements are considered have a standard deviation, σ, and present a random behavior that can be modeled by a Gaussian distribution with zero mean and unitary standard deviation.

2.2 Test-cases

Our analysis will be carried out considering three test-cases. All cases have the same dimensions but consider different materials for the first and second bodies. The dimensions and materials combinations for each test-case are summarized in Tab. 1 and Tab. 2, respectively. The thermal conductivities are presented in Tab. 3.

Table 1. Test-cases dimensions.

Dimensions	Value (m)
a	0.04
b_1	0.01
b_2	0.01

Table 2. Test-cases materials.

Test-case	Ω_1	Ω_2
TC1	AISI 1050 Steel	AISI 1050 Steel
TC2	AISI 1050 Steel	Inconel
TC3	Inconel	AISI 1050 Steel

Table 3. Thermal conductivities.

Material	Thermal conductivity (W/m ºC)
AISI 1050 Steel	54
Inconel	14

3. INVERSE PROBLEM

To solve the inverse problem concerning the estimate of the thermal contact conductance, h_c , through the RF, first two auxiliary problems must be defined: the first one to determine the temperature jump, $T_1 - T_2$, across the contact surface Γ ; and the second one to determine the heat flux, $-k_1\frac{\partial T_1}{\partial y}$, at the same surface. The auxiliary problems are formulated for the same domains as the direct problem, Ω_1 and Ω_2, but considering auxiliary harmonic functions F_1, F_2 and G_1, as follows.

3.1 First auxiliary problem: $T_1 - T_2$ at Γ

To obtain the temperature jump $T_1 - T_2$ at the surface Γ, the following problem for some auxiliary functions $F_1 \in C^2(\Omega_1)$ and $F_2 \in C^2(\Omega_2)$ where the orthonormal basis system $\psi_j \in L^2(\Gamma_0)$ was initially proposed by (Colaço and Alves, 2013) and later modified by (Abreu et al., 2014):

$$\nabla^2 F_{1,j} = 0 \quad \text{in } \Omega_1 \tag{10}$$

$$-k_1 \frac{\partial F_{1,j}}{\partial y} = \psi_j \quad \text{at } \Gamma_0 \tag{11}$$

$$\frac{\partial F_{1,j}}{\partial x} = 0 \quad \text{at } \Gamma_1 \tag{12}$$

$$F_{1,j} = F_{2,j} \quad \text{at } \Gamma \tag{13}$$

$$\nabla^2 F_{2,j} = 0 \quad \text{in } \Omega_2 \tag{14}$$

$$F_{2,j} = 0 \quad \text{at } \Gamma_{00} \tag{15}$$

$$\frac{\partial F_{2,j}}{\partial x} = 0 \quad \text{at } \Gamma_2 \tag{16}$$

$$-k_1 \frac{\partial F_{1,j}}{\partial y} = k_2 \frac{\partial F_{2,j}}{\partial y} \quad \text{at } \Gamma \tag{17}$$

3.2 Second auxiliary problem: $-k_1\, \partial T_1/\partial y$ at Γ

To obtain the heat flux $-k_1 \frac{\partial T_1}{\partial y}$ at the surface Γ, the following problem for some auxiliary function $G_1 \in C^2(\Omega_1)$ where the same orthonormal basis system $\psi_j \in L^2(\Gamma_0)$ initially proposed by (Colaço and Alves, 2013) and later modified by (Abreu et al., 2014) was used:

$$\nabla^2 G_{1,j} = 0 \quad \text{in } \Omega_1 \tag{18}$$

$$G_{1,j} = \psi_j \quad \text{at } \Gamma_0 \tag{19}$$

$$\frac{\partial G_{1,j}}{\partial x} = 0 \quad \text{at } \Gamma_1 \tag{20}$$

$$\frac{\partial G_{1,j}}{\partial y} = 0 \quad \text{at } \Gamma \tag{21}$$

3.3 Inverse problem solution

3.3.1 Temperature jump $T_1 - T_2$ estimation

The solution or the inverse problem start from the identities presented in Eq. (22) and Eq. (23). After applying Green's second identity to them, we obtain Eq. (24) and Eq. (25).

$$\int_{\Omega_1} \left[F_{1,j} \nabla^2 T_1 - T_1 \nabla^2 F_{1,j} \right] d\Omega_1 = 0 \tag{22}$$

$$\int_{\Omega_2}\left[F_{2,j}\nabla^2 T_2 - T_2\nabla^2 F_{2,j}\right]d\Omega_2 = 0 \tag{23}$$

$$\int_{\partial\Omega_1}\left[F_{1,j}\frac{\partial T_1}{\partial \boldsymbol{n}_1} - T_1\frac{\partial F_{1,j}}{\partial \boldsymbol{n}_1}\right]d\partial\Omega_1 = 0 \tag{24}$$

$$\int_{\partial\Omega_2}\left[F_{2,j}\frac{\partial T_2}{\partial \boldsymbol{n}_2} - T_2\frac{\partial F_{2,j}}{\partial \boldsymbol{n}_2}\right]d\partial\Omega_2 = 0 \tag{25}$$

Doing the same manipulations as presented by Colaço and Alves (2012) and considering both the physical problem and the first auxiliary problem, Eq. (26) is found, where Y is the temperature measurement on the surface Γ_0.

$$k_1\int_{\Gamma_0}\left[F_{1,j}\left(\frac{-q}{k_1}\right) - Y\frac{\partial F_{1,j}}{\partial \boldsymbol{n}_1}\right]d\Gamma_0 = \int_{\Gamma} k_1\frac{\partial F_{1,j}}{\partial \boldsymbol{n}_1}(T_1 - T_2)\,d\Gamma \tag{26}$$

The integral appearing on the left side of Eq. (26) will be called the reciprocity functional $\Re(F_{1,j})$, as proposed by Colaço and Alves (2012), and so Eq. (27) can be rewritten. It can also be seen that the right side of Eq. (26) may be rewritten as the inner product of the temperature jump, $T_1 - T_2$, and the flux of function $F_{1,j}$ on the surface Γ, resulting in Eq. (28).

$$k_1\Re(F_{1,j}) = \int_{\Gamma} k_1\frac{\partial F_{1,j}}{\partial \boldsymbol{n}_1}(T_1 - T_2)\,d\Gamma \tag{27}$$

$$k_1\Re(F_{1,j}) = \langle T_1 - T_2, k_1\frac{\partial F_{1,j}}{\partial \boldsymbol{n}_1}\rangle_{L^2(\Gamma)} \tag{28}$$

From this result it is interesting to observe that $\Re(F_{1,j})$ depends only on the heat flux, q, the temperature measurements, Y, and the auxiliary function $F_{1,j}$. The first two quantities are known from the physical problem, while the last one can be found solving the first auxiliary problem. On the right side of Eq. (27), the flux $\frac{\partial F_{1,j}}{\partial \boldsymbol{n}_1}$ can also be found from the auxiliary problem solution. Hence, it is possible to estimate the temperature jump, $T_1 - T_2$, solving Eq. (27) for different functions from the orthonormal basis ψ_j.

3.3.2 Heat flux $-k_1\,\partial T_1/\partial y$ estimation

In a similar way, the same procedure used in Section 3.3.1 can be used to estimate the heat flux $-k_1\frac{\partial T_1}{\partial y}$ on the surface Γ. This time, we start from the identity presented in Eq. (29) that after Green's second identity application turns into Eq. (30).

$$\int_{\Omega_1}\left[G_{1,j}\nabla^2 T_1 - T_1\nabla^2 G_{1,j}\right]d\Omega_1 = 0 \tag{29}$$

$$\int_{\partial\Omega_1}\left[G_{1,j}\frac{\partial T_1}{\partial \boldsymbol{n}_1} - T_1\frac{\partial G_{1,j}}{\partial \boldsymbol{n}_1}\right]d\partial\Omega_1 = 0 \tag{30}$$

Manipulating Eq. (30) as proposed by Colaço and Alves (2012) and considering the physical problem and the second auxiliary problem as proposed on this work, we find that:

$$k_1\int_{\Gamma_0}\left[G_{1,j}\left(\frac{-q}{k_1}\right) - Y\frac{\partial G_{1,j}}{\partial \boldsymbol{n}_1}\right]d\Gamma_0 = \int_{\Gamma} -k_1 G_{1,j}\frac{\partial T_1}{\partial \boldsymbol{n}_1}\,d\Gamma \tag{31}$$

The integral on the left side of Eq. (31) can be defined as the reciprocity functional $\Re(G_{1,j})$, and we can rewrite the right side as an inner product, resulting into Eq. (32). Solving Eq. (32), it is possible estimate the heat flux, $-k_1\frac{\partial T_1}{\partial y}$, on the surface Γ, once the auxiliary for $G_{1,j}$ function is solved.

$$k_1\Re(G_{1,j}) = \langle G_{1,j}, -k_1\frac{\partial T_1}{\partial \mathbf{n}_1}\rangle_{L^2(\Gamma)} \tag{32}$$

3.3.3 Thermal contact conductance h_c estimation

Once both temperature jump, $T_1 - T_2$, and the heat flux, $-k_1\frac{\partial T_1}{\partial y}$, on the surface Γ, are estimated, it is possible to estimate the thermal contact conductance, h_c, on the same surface as:

$$h_c = \frac{-k_1\frac{\partial T_1}{\partial y}}{T_1 - T_2} \tag{33}$$

From Eqs. (28) and (32), it is possible to rewrite the temperature jump and the heat flux using new generic basis functions, β_j and γ_j, where:

$$\beta_j = k_1\frac{\partial F_{1,j}}{\partial y}\bigg|_\Gamma \tag{34}$$

$$\gamma_j = G_{1,j}\big|_\Gamma \tag{35}$$

The temperature jump and heat flux can therefore be written as Eq. (36) and Eq. (37). Using these results, Eq. (28) and Eq. (32) may be rewritten as Eq. (38) and Eq. (39), respectively.

$$[(T_1 - T_2)]_\Gamma = \sum_{i=1}^{N} \alpha_i \beta_i \tag{36}$$

$$\left[-k_1\frac{\partial T_1}{\partial y}\right]_\Gamma = \sum_{i=1}^{N} \delta_i \gamma_i \tag{37}$$

$$k_1\Re(F_{1,j}) = \sum_{i=1}^{N} \alpha_i \langle \beta_i, \beta_j\rangle_{L^2(\Gamma)} \tag{38}$$

$$k_1\Re(G_{1,j}) = \sum_{i=1}^{N} \delta_i \langle \gamma_i, \gamma_j\rangle_{L^2(\Gamma)} \tag{39}$$

Solution of Eqs. (38) and (39) will determine constants α_i and δ_i, and, once they are found, the thermal contact conductance will be given as:

$$h_c = \frac{\sum_{i=1}^{N}\delta_i \gamma_i}{\sum_{i=1}^{N}\alpha_i \beta_i} \tag{40}$$

3.3.4 Solution of the auxiliary problems using CITT

As proposed by Padilha et al. (2016), it is possible to find the functions F_1, F_2 and G_1 by solving the auxiliary problems using CITT. The use of this technique will both improve the accuracy of the method to solve the inverse problem and its performance, since an analytical solution will be found.

The use of this technique on these auxiliary problems can be found in (Padilha et al., 2016), and results to:

$$F_{1j}(x,y) = \frac{1}{a}\frac{(k_1-k_2)b_2+k_2y}{k_1b_2+k_2b_1}\int_{x=0}^{a}\psi_j dx + \sum_{m=1}^{\infty}\left[\frac{2}{a}cos(\lambda_m x)\frac{\left[k_2-k_1 tanh^2(\lambda_m b_2)\right]sinh(\lambda_m y)+(k_1-k_2)sinh(\lambda_m b_2)cosh(\lambda_m y)}{k_2 sinh(\lambda_m b_1)+k_1 cosh(\lambda_m b_1)tanh(\lambda_m b_2)}\int_{x=0}^{a}cos(\lambda_m x)\psi_j dx\right] \tag{41}$$

$$F_{2j}(x,y) = \frac{1}{a}\frac{k_2y}{k_1b_2+k_2b_1}\int_{x=0}^{a}\psi_j dx + \sum_{m=1}^{\infty}\left[\frac{2}{a}cos(\lambda_m x)\frac{k_1 sech(\lambda_m b_2)sinh(\lambda_m y)}{k_2 sinh(\lambda_m b_1)+k_1 cosh(\lambda_m b_1)tanh(\lambda_m b_2)}\int_{x=0}^{a}cos(\lambda_m x)\psi_j dx\right] \tag{42}$$

$$G_{1j}(x,y) = \frac{1}{a}\int_{x=0}^{a}\psi_j dx + \sum_{m=1}^{\infty}\left[\frac{2}{a}cos(\lambda_m x)\frac{cosh(\lambda_m[b_2-y])}{cosh(\lambda_m b_1)}\int_{x=0}^{a}cos(\lambda_m x)\psi_j dx\right] \tag{43}$$

$$\lambda_m = \frac{m\pi}{a} \tag{44}$$

3.3.5 Inverse problems solution combining RF and CITT

From the auxiliary problems solutions presented in Eqs. (41)-(44), β_i and γ_i may be found as:

$$\beta_i = -\frac{1}{a}\frac{k_1k_2}{k_1b_2+k_2b_1}\int_{x=0}^{a}\psi_j dx + \sum_{m=1}^{\infty}\left[-2\lambda_m cos(\lambda_m x)\frac{k_1k_2}{k_2 sinh(\lambda_m b_1)+k_1 cosh(\lambda_m b_1)tanh(\lambda_m b_2)}\int_{x=0}^{a}cos(\lambda_m x)\psi_j dx\right] \tag{45}$$

$$\gamma_i = \frac{1}{a}\int_{x=0}^{a}\psi_j dx + \sum_{m=1}^{\infty}\left[\frac{2}{a}cos(\lambda_m x)\frac{1}{cosh(\lambda_m b_1)}\int_{x=0}^{a}cos(\lambda_m x)\psi_j dx\right] \tag{46}$$

Different choices can be made for the functions that form the basis ψ_i. Based on the work of (Padilha et al., 2016), these functions were chosen to:

$$\psi_j = \begin{cases} \sqrt{\frac{1}{a}} & j=0 \\ \sqrt{\frac{1}{a}}cos(\lambda_j x) & j=1,2,3,\ldots \end{cases} \tag{47}$$

$$\lambda_j = \frac{j\pi}{a} \tag{48}$$

Rearranging Eqs. (38) and (39), applying the definitions of the reciprocity functional presented in Eqs. (26) and (31), and using the results shown in Eqs. (41)-(47), it is found that:

$$\alpha_j = \begin{cases} \dfrac{k_1}{\frac{k_1k_2}{k_1b_2+k_2b_1}}\sqrt{\frac{1}{a}}\left[a\left(\frac{-q}{k_1}\right)-\frac{k_1k_2}{k_1b_2+k_2b_1}\int_{x=0}^{a}Y dx\right] & j=0 \\ -\dfrac{1}{k_2\lambda_j A_{1,2}}\sqrt{\frac{2}{a}}\left[k_2 cosh(\lambda_j b_1)+k_1 sinh(\lambda_j b_1)tanh(\lambda_j b_2)\right]\left(\int_{x=0}^{a}Y cos(\lambda_j x)dx\right) & j=1,2,3,\ldots \end{cases} \tag{49}$$

$$\delta_j = \begin{cases} -q\sqrt{a} & j=0 \\ -\sqrt{\frac{2}{a}} k_1 \lambda_j \sinh(\lambda_j b_1)\cosh(\lambda_j b_1)\left(\int_{x=0}^{a} Y\cos(\lambda_j x)dx\right) & j=1,2,3,\dots \end{cases} \tag{50}$$

$$A_{1,2} = \frac{k_1 k_2}{k_2 \sinh(\lambda_m b_1) + k_1 \cosh(\lambda_m b_1)\tanh(\lambda_m b_2)} \tag{51}$$

Using these results, the temperature jump and the heat flux will be given by:

$$[(T_1 - T_2)]_\Gamma = \frac{q}{A_{1,2}} + \frac{1}{a}\int_{x=0}^{a} Y dx + \frac{2}{ak_2}\sum_{j=1}^{\infty} C_j \cos(\lambda_j x)[k_2\cosh(\lambda_j b_1) + k_1 \sinh(\lambda_j b_1)\tanh(\lambda_j b_2)]\left(\int_{x=0}^{a} Y\cos(\lambda_j x)dx\right) \tag{52}$$

$$\left[-k_1\frac{\partial T_1}{\partial y}\right]_\Gamma = -\left[q + \frac{2k_1}{a}\sum_{j=1}^{\infty}\lambda_j \cos(\lambda_j x)\sinh(\lambda_j b_1)\left(\int_{x=0}^{a} Y\cos(\lambda_j x)dx\right)\right] \tag{53}$$

From the results found in Eqs. (52) and (53), it is possible to estimate the thermal contact conductance according to Eq. (33) once the integrals appearing there are solved.

3.3.6 Orthonormal measurement decompostion

As proposed by Padilha et al. (2016), there is an analytical solution to the estimation of the temperature jump and the heat flux and, therefore, the thermal contact conductance at surface Γ. However, there still appear two integrals to be calculated using the temperature measurement, Y, on the top surface Γ_0. If these measurements are approximated by a sum of functions, as, for example, a Fourier Series, an analytical expression to the integral can be found, and then it is possible to find a full analytical solution to the estimation of the thermal contact conductance.

A proper function series selection would make it simpler to solve the integral if the function basis is orthonormal to functions appearing in the integral. Thus, the temperature measurements are approximated by an even Fourier Series, where λ_j is the same eigenvalues found in the earlier expressions, as follows:

$$Y(x) = C_0 + \sum_{j=1}^{\infty} C_j \cos(\lambda_j x) \tag{54}$$

Replacing this expression in Eqs. (52) and (53), full analytical expressions are obtained for the estimations. These new expressions eliminate the need to numerically calculate the integrals that appears in the solutions found by Padilha et al. (2016), but introduce a new cost finding the constants C.

$$[(T_1 - T_2)]_\Gamma = \frac{q}{A_{1,2}} + C_0 + \frac{1}{k_2}\sum_{j=1}^{\infty} C_j \cos(\lambda_j x)[k_2\cosh(\lambda_j b_1) + k_1 \sinh(\lambda_j b_1)\tanh(\lambda_j b_2)] \tag{55}$$

$$\left[-k_1\frac{\partial T_1}{\partial y}\right]_\Gamma = -\left[q + k_1\sum_{j=1}^{\infty}\lambda_j C_j \cos(\lambda_j x)\sinh(\lambda_j b_1)\right] \tag{56}$$

Replacing Eqs. (55) and (56) into Eq. (33) the following analytical estimation of the thermal contact conductance, h_c, can be found

$$[h_c]_\Gamma = \frac{-[q + k_1\sum_{j=1}^{\infty}\lambda_j C_j \cos(\lambda_j x)\sinh(\lambda_j b_1)]}{\frac{q}{A_{1,2}} + C_0 + \frac{1}{k_2}\sum_{j=1}^{\infty} C_j \cos(\lambda_j x)[k_2\cosh(\lambda_j b_1) + k_1\sinh(\lambda_j b_1)\tanh(\lambda_j b_2)]} \tag{57}$$

4. RESULTS

As presented in Section 2.1, simulated measurements are used to verify the proposed solution to the inverse problem. For the error modelling, a noiseless scenario where the measured data match exactly the simulated temperature (σ = 0 ºC) and two noisy scenarios with σ = 0.5 ºC and σ = 2 ºC will be considered. These three cases will be put under the same conditions in which the boundary conditions are summarized in Tab. 4 and six different profiles for the thermal contact conductance are evaluated, as presented in Tab. 5. These cases and profiles are the same as presented by (Padilha, 2016).

Table 4. Boundary conditions.

Quantity	Value
Heat flux q at top surface (W/m^2)	-100000
Reference temperature T_{ref} at bottom surface (ºC)	0
Maximum thermal contact conductance $h_{c,max}$ (W/m^2 ºC)	1000

Table 5. Thermal contact conductance profiles.

Case	Profile expression
1	$h_{c,max}$ for $(x < a/4)$ and $(x > 3a/4)$ 0 for $(a/4 \leq x \leq 3a/4)$
2	$h_{c,max}$ for $(x < a/4)$ and $(a/2 < x < 3a/4)$ 0 for $(a/4 \leq x \leq a/2)$ and $(x \geq 3a/4)$
3	$h_{c,max}\, sin\left(\frac{2\pi}{a}\right)$
4	$h_{c,max}\, sin\left(\frac{2\pi}{a}\right)$
5	$h_{c,max}$
6	$h_{c,max}$ for $(x < a/4)$ and $(a/2 < x < 3a/4)$ $h_{c,max}/2$ for $(a/4 \leq x \leq a/2)$ 0 for $(x \geq 3a/4)$

After the temperature measurements were simulated and noises were added to them, the *symfit* library available in *Python* was used to fit the measurements to the proposed function, Eq. (54). Using these fit measurements, thermal contact conductance is estimated as presented in Figs. 2-4. During simulations, it is found that for each case there is an optimum number of terms in the series that make the result converge. Until now, no adequate method was found to justify the existence of this optimum number of terms or a reason for that to occur, and further investigations are occurring. The number of terms used in each case is summarized in Tabs. 6 – 8.

Table 6. Number of terms used in the series to find convergence with test-case 1.

Case	$\sigma = 0$ ºC	$\sigma = 0.5$ ºC	$\sigma = 2$ ºC
1	9	5	4
2	6	5	4
3	6	4	3
4	6	5	5
5	25(1)	1	1
6	6	5	4

(1) no limit was found for this case, so we chose to keep it as 25.

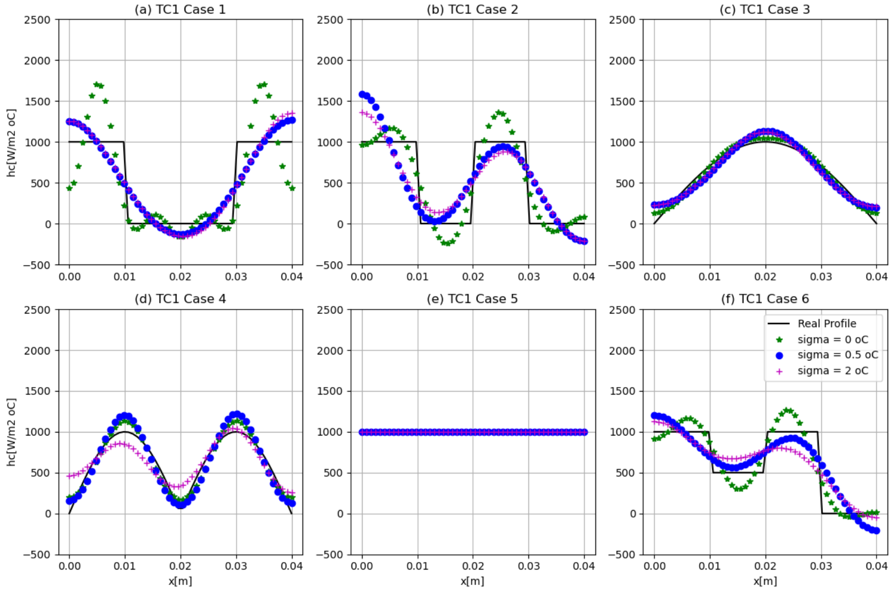

Figure 2. Thermal contact conductance estimation for test-case 1.

Table 7. Number of terms used in the series to find convergence with test-case 2.

Case	$\sigma = 0$ °C	$\sigma = 0.5$ °C	$\sigma = 2$ °C
1	5	5	3
2	6	5	4
3	5	3	3
4	6	5	5
5	25(1)	1	1
6	6	5	4

(1) no limit was found for this case, so we chose to keep it as 25.

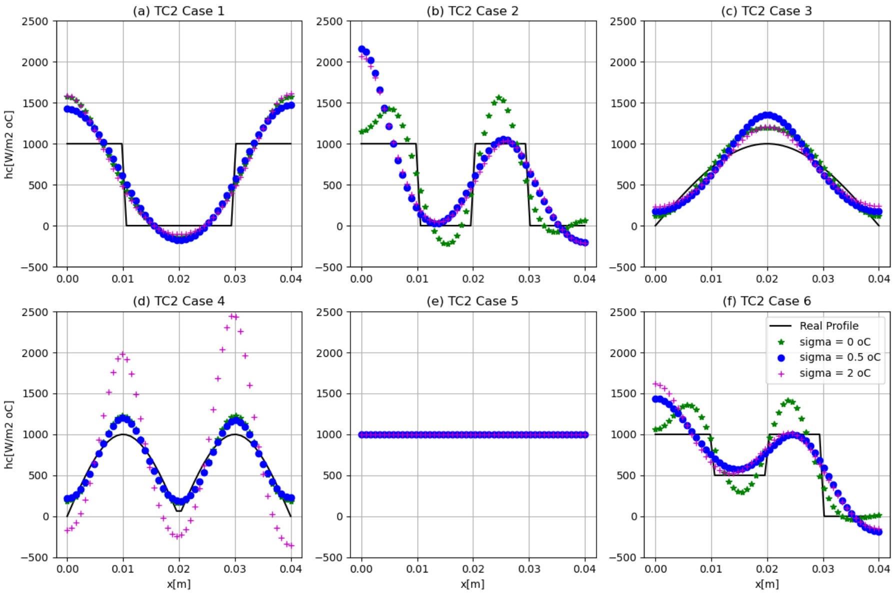

Figure 3. Thermal contact conductance estimation for test-case 2.

Table 8. Number of terms used in the series to find convergence with test-case 3.

Case	$\sigma = 0$ °C	$\sigma = 0.5$ °C	$\sigma = 2$ °C
1	7	5	4
2	7	5	6
3	6	5	4
4	7	5	5
5	25(1)	1	1
6	7	5	4

(1) no limit was found for this case, so we chose to keep it as 25.

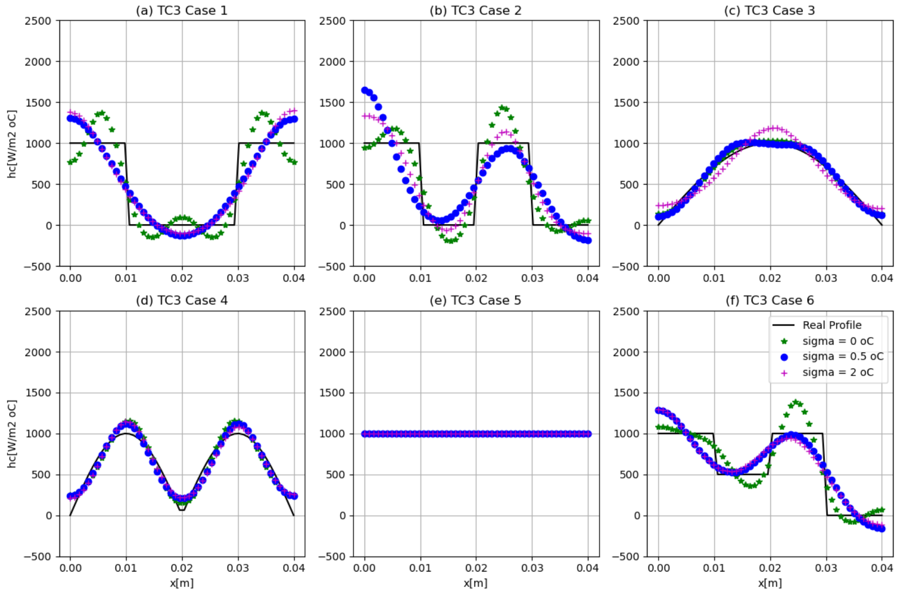

Figure 4. Thermal contact conductance estimation for test-case 3.

For every test-case, we can see that the estimated thermal contact conductance, h_c, better matches the real profile when they are continuous, as in Cases 3 and 5, especially for noiseless measurements. For the constant Case 5 with noiseless measurements, Figs. (1e, 2e and 3e), the thermal contact conductance is perfectly estimated, but as we add noise to the measurements, we could only find the profile when using only the first term ($j = 0$) in the summations. For Case 3 with noiseless measurements, Figs. (1c, 2c and 3c), the real profiles are reasonable followed by the estimation, although they do not match the limits where $h_c = 0$. Even when we add noise to the measurements, the estimation does reasonably match the real profile. For the Cases 1, Figs. (1a, 2a and 3a), 2, Figs. (1b, 2b and 3b), and 6, Figs. (1f, 2f and 3f), where functions have a step like profile, we can see that the discontinuities have a significant effect, and the estimation could not follow the real function profile. However, the estimation is capable to reasonably find the steps levels and their centers, especially when using noiseless measurements. In Case 4, Figs. (1d, 2d and 3d), although the real profile shows a discontinuity, the function is smoother than in Cases 1, 2 and 6. In this case, we see that there is a reasonable agreement between the real profile and the estimated one when there is no noise, or its standard deviation is held as 0.5 °C. When applying the higher standard deviation, we see that in this fourth case for the first test-case, Fig. 2, the thermal contact conductance estimated level is close, but there is a deformation in the profile. With the second test-case, both estimated profile and thermal contact conductance values are significantly far from the real ones. As in Case 4, for Case 3, Figs. (1c, 2c and 3c), the estimation cannot match the profile in it ends, regardless the added noise.

When comparing results, we see that for cases where less terms in the series were admitted on this work than by Padilha (2016) to achieve stable solutions, poorer results were found. It happens more evidentially in noiseless cases 1, 2, 3 and 5, for which Padilha (2016) could use significantly more terms. For the same cases but with added noise, the difference in the number of terms used is not significant, then there is a better match in results. Nevertheless, results found on this work still provide good estimation to the real profiles.

On this work, the estimation was made with a *Python* code on an Intel Core i7-3635QM CPU at 2.40GHz with 4 Cores and 8 Logical Processors. The fastest estimation took 0.001 s and occurred for the noisy cases 5 for the test-cases

1, 2 and 3. The slowest estimation was 0.038 s and occurred for the noiseless case 6 of test-case 3. (Padilha et al., 2016) could estimate a function like the one used in case 1 in 0.2 s using an Intel Atom™ CPU N450 1.66 GHz. Although, the solution proposed on this work took significantly lower time to estimate the thermal contact conductance, no trustworthy computational cost comparison between the methods can be made.

5. CONCLUSION

The results presented in this work have shown that the Reciprocity Gap Functional with the CITT is a very good inverse problem method, capable of estimating thermal parameters using only a couple of measurements on a chosen surface. The proposed modifications in this paper lead to further simplifications, when decomposing the measurements in a Fourier Series, reducing the integrals to be solved to a set of constants that could be find applying simple curve fit techniques. This simplification in the method is capable to provide great estimation for the thermal contact conductance and could keep the low computational cost. Although not trustworthy comparison could be made, results suggest that the proposed modification accelerates the estimation process.

When comparing results to the ones found by Padilha (2016), there is a loss in accuracy for some cases, although both solutions could provide good estimation to the real profiles. The lost of accuracy is related to the number of terms in the series appearing in the solutions, which are lesser when using the measurement decomposition. There is still a lack of knowledge when dealing with the optimum number of terms to be considered during estimations, and why after a certain number of terms the results start to diverge.

Future research should extend this work by examining the reason why the measurements decomposition reduced the number of terms admitted in the series when comparing to the expressions proposed by Padilha et al. (2016) that use a numerical solution to the integrals. The new analytical solution proposed on this work should also make it easier to analyze conditions that will lead to the series convergence and most accurate solutions.

6. ACKNOWLEDGEMENTS

This paper was partially funded by the following Brazilian agencies: CNPq (Conselho Nacional para o Desenvolvimento Científico e Tecnológico), FAPERJ (Rio de Janeiro State Agency for Research), CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) and ANP-PRH8 (Brazilian National Agency for Oil, Gas and BioFuels – Human Research Program number 8).

7. REFERENCES

Abreu, L. A. S., Alves, C. J. S., Colaço, M. J., and Orlande, H. R. B., 2014, "A non-intrusive inverse problem technique for the identification of contact failures in double-layered composites". In *Proceedings of the Fifteenth International Heat Transfer Conference – IHTC-15*. Kyoto, Japan.

Abreu, L. A. S., Colaço, M. J., Orlande, H. R. B., and Alves, C. J. S., 2015, "Detection of contact failures in composites using infrared thermography and the reciprocity functional approach". In *Proceedings of the Sixteenth International Conference on Mechanics and Materials in Design*. Ponta Delgada, Azores.

Abreu, L. A. S., Colaço, M. J., Orlande, H. R. B., and Alves, C. J. S., 2016, "Thermography detection of contact failures in double layered materials using the reciprocity functional approach". *Applied Themal Engineering*, Vol. 100, pp. 1173-1178.

Colaço, M., J., and Alves, J. S., 2012, "Estimation of unknown contact resistance by means of reciprocity function approach". In *J. Eberhardsteiner et al. (Eds.), Proceedings of the European Congress on Computational Methods in Applied Sciences and Engineering – ECCOMAS*. Vienna, Austria, pp. 10-14.

Colaço, M., J., and Alves, J. S., 2013, "A fast non-intrusive method for estimating spatial thermal conductivity by means of the reciprocity functional approach and the method of fundamental solutions". *International ournal of Heat and Mass Transfer*, Vol. 60, pp. 653-663

Colaço, M., J., and Alves, J. S., 2015, "A backward reciprocity function approach to the estimation of spatial and transient thermal contact conductance in doublelayered materials using non-intrusive measurements". *Numerical Heat Transfer Part A: Applications*, Vol. 68, pp. 117-132.

Colaço, M. J., J. S., and Orlande, H. R. B., 2014, "Transient non-intrusive method for estimating thermal contact conductance by means of the reciprocity functional approach". *Inverse Problems in Science and Engineering*, Vol. 23, pp. 688-717.

Freitas, G. C., 2019, "Estimativa de Condutâncias Térmicas de Contato em interfaces irregulares usando a Técnica da Transformada Integral Clássica e o Método dos Funcionais de Reciprocidade", M.Sc. thesis, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brasil.

Gilmore, D. G., 2002, *Spacecraft Thermal Control Handbook*, The Aerospace Press, El Segundo, CA, 2nd edition.

McWaid, T. H., and Marschall, E., 1992, "Application of the modified greenwood willianson contact modell for the prediction of thermal contact resistance". *Wear*, Vol. 152, pp. 263-277

Milosevic, N. D., Raynaud, M., and Maglic, .D., 2002. "Estimation of thermal contact resistance between the materials of double-layer sample using the laser flash method". *Inverse Problems in Engineering*, Vol. 10, No. 1, pp. 85–103.

Ozisik, M., N., 1993, *Heat Conduction*, John Wiley & Sons, New York.

Padilha, R. S., 2016, "Uma Formulação Analítica para a Estimativa Não Intrusiva de Condutâncias Térmicas de Contato Usando o Conceito do Funcional de Reciprocidade", D.Sc. thesis, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brasil.

Padilha, R. S., and Colaço, M. J., 2015, "Analytical non-intrusive technique to estimate thermal contact conductances in double layered materials by combining the reciprocity functional and the integral transform approaches". In *Proceedings of the 23rd International Congress of Mechanical Engineering – COBEM*. Rio de Janeiro, Brazil.

Padilha, R. S., Colaço, M. J., Abreu, L. A. S., and Orlande, H. R. B., 2016, "Experimental validation of a fully analytical method for determining thermal contact conductances in double-layered materials using the reciprocity functional and the integral transform approach". In *Proceedings of the First Pacific Rim Thermal Engineering Conference – PRTEC*. Hawaii's Big Island, USA.

Xuan, Y., Cui, T., and Zhang, P., 2014. "Multiscale simulation of thermal contact resitance in electronic packing". *International Journal of Thermal Sciences*, Vol. 83, pp. 16–24.

8. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper