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THERMAL CONTACT CONDUCTANCE ESTIMATION USING A TIME-DEPENDENT METHOD OF FUNDAMENTAL SOLUTIONS

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Abstract. *Heat conduction in composite solids is strongly dependent on the thermal contact conductance (TCC) between their different layers. This property is very important to obtain the heat transfer behavior in many engineering applications such as electronic packaging, nuclear reactors, aerospace, and biomedicine, among others. The numerical solution of the transient heat transfer model can be performed by mesh-based methods e.g., the Finite Difference Method, which might require a great effort for very accurate space discretization. On the other hand, meshless methods are gaining attention by their simple implementation and good accuracy. One of these methodologies is the Method of Fundamental Solutions (MFS), which combines the knowledge about the boundary and initial conditions, named collocation points, with points that may be placed outside the studied domain, called source points. This method represents the solution (temperature in this case) as a linear combination of fundamental solutions, whose unknown coefficients must be obtained through the solution of a linear system. In this work, a time-dependent MFS is applied to the solution of a transient heat conduction problem, considering different TCC configurations. Results shown good accuracy, although the resulting matrix is highly ill-conditioned, and a Tikhonov regularization scheme had to be used to solve it.*

Keywords: *Method of Fundamental Solution, Inverse Heat Conduction Problem, Thermal Contact Conductance, Tikhonov Regularization, Singular Value Decomposition*

1. INTRODUCTION

Failures in composite materials are important in several applications such as nuclear-reactor cooling, aerodynamic heating of supersonic aircraft and missiles, packaging of electronics, and others. The use of these materials is strongly dependent on the contact between their different layers. One parameter commonly used to evaluate a contact interface is the thermal contact conductance (TCC), which is defined as the ratio between the heat flux and the temperature jump at such interface.

The Method of Fundamental Solutions (MFS) is a meshless method initially proposed by Kupradze and Aleksidze (1964) for obtaining the solution of elliptic PDE. Its numerical implementation can be found in several texts, as for instance in the paper of Mathon and Johnston (1997). The MFS has recently gained popularity to solve heat transfer problems due to its simplicity and accuracy.

The MFS defines the solution of a partial differential equation (PDE) as a linear combination of some basis functions (the fundamental solutions) that automatically satisfy the governing equation, and considers a “pseudo-boundary” outside the study domain, where the so-called source points are placed. The coefficients of such expansion are obtained by requiring that the solution is also satisfied on the boundaries and/or for the initial time. The source points placement significantly influences the solution of the MFS, as described by Chen et al. (2016). The resulting system is naturally ill-conditioned, and it is common to use some regularization to solve it.

The MFS can be applied to both direct and inverse problems due to its formulation. For instance, Mera (2005) proposed the solution of a backward heat conduction problem (BHCP) combined with a Tikhonov regularization, and Johansson et al. (2011) proved its denseness results for the solution of the BHCP using a time-dependent fundamental solution for some given source points placement.

Other approaches use the MFS to solve Inverse Heat Conduction Problems (IHCP) combined with other methods, as in Colaço and Alves (2013), where the MFS was used combined with the reciprocity functional approach to estimate a spatial thermal contact conductance between two solids, and in Silva et al. (2021a) and Silva et al. (2021b), where the MFS was combined with the Particle Filter to estimate a time-dependent heat transfer coefficient.

This paper extends the MFS solution for layered materials, presented in Johansson and Lesnic (2009), to determining the TCC between two solids using a single set of source points with non-intrusive measurements neither iterative procedures.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

In this paper, two contacting blocks of the same material and dimension are considered. The thermal properties are assumed constant, and the thermal contact conductance between the two contact surfaces (on $x = L/2$) is known as h. The governing equations of the direct problem are given by Eqs. (1) - (7) .

$$\frac{\partial^2 T_1(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_1}{\partial t}, \qquad 0 \le x \le \frac{L}{2}, \qquad 0 < t \le t_f \tag{1}$$

$$\frac{\partial^2 T_2(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_2}{\partial t}, \qquad \frac{L}{2} \le x \le L, \qquad 0 < t \le t_f \tag{2}$$

$$-k\frac{\partial T_1}{\partial \boldsymbol{n}} = q, \qquad x = 0, \qquad 0 < t \le t_f \tag{3}$$

$$k\frac{\partial T_1}{\partial \boldsymbol{n}} = -k\frac{\partial T_2}{\partial \boldsymbol{n}}, \qquad x = \frac{L}{2}, \qquad 0 < t \le t_f \tag{4}$$

$$-k\frac{\partial T_1}{\partial \boldsymbol{n}} = h(T_1 - T_2), \qquad x = \frac{L}{2}, \qquad 0 < t \le t_f \tag{5}$$

$$\frac{\partial T_2}{\partial \boldsymbol{n}} = 0, \qquad x = L, \qquad 0 < t \le t_f \tag{6}$$

$$T_1(x,t) = T_2(x,t) = T_0, \qquad 0 \le x \le L, \qquad t = 0 \tag{7}$$

For the inverse formulation, the TCC h from Eq. (5) is unknown and needs to be determined. Therefore, additional data must be supplied to fully determine it, which is assumed to be known by some measurements at $x = L$ in the inverse procedure.

3. METHOD OF FUNDAMENTAL SOLUTIONS

The MFS approximates the solution of the governing differential equation by a linear combination of its fundamental solutions, as described by Eq. (8).

$$\hat{T}(x,t) = \sum_{s=1}^{S} \phi_s G(x,t;y_s,\tau_s) \tag{8}$$

where S represents the number of source points, (x,t) are respectively the spatial and temporal coordinates of a point from the studied domain, and (y,τ) are respectively the spatial and temporal coordinates of each source point located on a fictitious boundary outside the solution domain. The time-dependent basis function (a proposed fundamental solution) $G(x,t;y_s,\tau_s)$ for one-dimensional heat conduction problems is given by Eq. (9) as in Johansson and Lesnic (2008).

$$G(x,t;y_s,\tau_s)=\frac{\exp\left[-\dfrac{|x-y_s|^2}{4\alpha(t-\tau_s)}\right]}{4\pi(t-\tau_s)}H(t-\tau_s) \tag{9}$$

where H is the Heaviside step function, and α is the thermal diffusivity.

The approximate solution $\hat{T}(x,t)$ automatically satisfies the one-dimensional parabolic partial differential equation, but not necessarily the associated initial (IC) and boundary conditions (BC). This can be achieved by enforcing Eq. (15) by means of collocation. For instance, temperatures T_1 and T_2 on some boundary conditions 1 and 2 can be represented by the system of algebraic equations as in Eq. (10) and Eq. (11) , respectively.

$$T_1(x)=\sum_{s=1}^{S}\boldsymbol{\phi}_{s,1}\mathbf{G}(x,t;y_s,\tau_s) \tag{10}$$

$$T_2(x)=\sum_{s=1}^{S}\boldsymbol{\phi}_{s,2}\mathbf{G}(x,t;y_s,\tau_s) \tag{11}$$

Similarly, expressions for the boundary conditions of the second kind can be reached by taking the normal derivative of the expressions from Eq. (10) and Eq. (11) and enforcing them on the prescribed boundary conditions.

Then, the unknown coefficients $\boldsymbol{\phi}$ can be determined by the collocation points, defined by n initial conditions and $2m$ boundary conditions for each domain, where $n=2m-2$ as in Johansson and Lesnic (2008). To deal with two domains in contact, this paper extends the MFS formulation to layered materials proposed by Johansson and Lesnic (2009) to deal with the TCC. The resultant system given by the collocation points is shown in Eq. (12) .

$$\begin{bmatrix} \mathbf{G}(IC_1)_{n\times S} & [0] \\ \mathbf{G}(BC_1)_{m+1\times S} & [0] \\ \mathbf{G}(BC_{if})_{2(m+1)\times S} & \mathbf{G}(BC_{if})_{2(m+1)\times S} \\ [0] & \mathbf{G}(BC_2)_{m+1\times S} \\ [0] & \mathbf{G}(IC_2)_{n\times S} \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}_{1,1} \\ \vdots \\ \boldsymbol{\phi}_{S,1} \\ \boldsymbol{\phi}_{1,2} \\ \vdots \\ \boldsymbol{\phi}_{S,2} \end{bmatrix} = \begin{bmatrix} \mathbf{b}[IC_1]_{n\times 1} \\ \mathbf{b}[BC_1]_{m+1\times 1} \\ \mathbf{b}[BC_{if}]_{2(m+1)\times 1} \\ \mathbf{b}[BC_2]_{m+1\times 1} \\ \mathbf{b}[IC_2]_{n\times 1} \end{bmatrix} \tag{12}$$

where the index 1 and 2 denotes the correspondent domain and "if" is the interface. The system from Eq. (12) requires that $S\le 2\big(n+2(m+1)\big)$ to have a unique solution. Finally, it is possible to solve the system $\mathbf{G}\boldsymbol{\phi}=\mathbf{b}$. Notice that in this approach the MFS can be applied to both domains using the same source points, considering that each domain has its own coefficients.

Once the coefficients are determined, it is possible to estimate the TCC using the information of the heat flux and the temperatures in each material in contact, as in Eq. (13)

$$h=\frac{q}{\Delta \mathrm{T}}=\frac{-\sum_{s=1}^{S}\dfrac{\partial\boldsymbol{\phi}_{s,1}}{\partial\boldsymbol{n}}\phi_{s,1}\mathbf{G}\big(x_{if},t,y_s,\tau_s\big)}{\sum_{s=1}^{S}\boldsymbol{\phi}_{s,2}\mathbf{G}\big(x_{if},t,y_s,\tau_s\big)-\sum_{s=1}^{S}\boldsymbol{\phi}_{s,1}\mathbf{G}\big(x_{if},t,y_s,\tau_s\big)} \tag{13}$$

3.1 Singular Value Decomposition with Tikhonov regularization

The resulting system from Eq. (12) is ill-conditioned, and therefore some regularization procedure must be is applied. In this paper we used the Tikhonov method, where the regularized solution is defined to be the solution of the least squares problem described by Eq. (14) .

$$\min_{\phi}\left\{\left|\left|\mathbf{G}\boldsymbol{\phi}-\mathbf{b}\right|\right|_2+\lambda\left|\left|\boldsymbol{\phi}\right|\right|_2\right\} \tag{14}$$

where $||\,.\,||$ denotes the usual Euclidean norm and λ is called the regularization parameter. In terms of Singular Value Decomposition (SVD) it is possible to represent the regularized solution by Eq. (15) as

$$\boldsymbol{\phi}_s=\sum_{s=1}^{S}g_s\frac{w_s^T b}{\sigma_s}v_s \tag{15}$$

where σ_s are the singular values, w_s is the left-side of $\mathbf{G}$, v_s is the right-side of $\mathbf{G}$, and g_s are the so-called Weiner Weights given by Eq. (16)

$$g_s = \frac{\sigma^2}{\sigma^2 + \lambda^2} \tag{16}$$

In this work the regularization parameter λ is chosen using the L-Curve criterion.

3.2 Source points placement

In this paper two different source points distribution strategies were used and will be referred to as Case 1 and Case 2.

The Case 1 configuration is based on a common strategy for time-dependent one-dimensional problems, as can be seen in the works of Johansson and Lesnic (2009) and Grabski et al. (2016), where the source points y are placed at a given distance δd from the study domain Ω. To avoid singularities the source points were placed below the x-axis, as can be noticed in Figure 1. Note that "SP' indicates the source points and Ω_1 and Ω_2 indicates each material in contact. The expression for the source points of Case 1 is described in Eq. (17) .

$$(\mathrm{y_s}, \mathrm{\tau_s}) = \begin{cases} \left(\mathrm{y}_s = -\delta d, \quad \tau_j = -(2s-1)\frac{t_f}{S}\right), & s = 1,2, \dots, \frac{S}{2} \\ \left(\mathrm{y}_s = 1 + \delta d, \quad \tau_j = -\left[2\left(s - \frac{S}{2}\right) - 1\right]\frac{t_f}{S}\right), & s = \frac{S}{2} + 1, \frac{S}{2} + 2, \dots, S \end{cases} \tag{17}$$

Case 2 consists of fixing a negative time coordinate of the source points and varying the spatial components, resulting in a domain parallel to the x-axis, as suggested by Young et al. (2004). This strategy has shown good results for one-dimensional parabolic problems in recent studies by Grabski (2019) and Kopperschmidt et al. (2020) due to its robustness and the fact that it depends on two parameters: δd and δt. A representation of Case 2 is shown in Figure 2. The expression for the source points is described in Eq. (18) .

$$(\mathrm{y_s}, \mathrm{\tau_s}) = \left\{\mathrm{y}_s = -\delta d + \frac{(2s-1)(1+2\delta d)}{2S}, \quad \tau_s = -\delta t\right\}, \quad s = 1,2, \dots, S \tag{18}$$

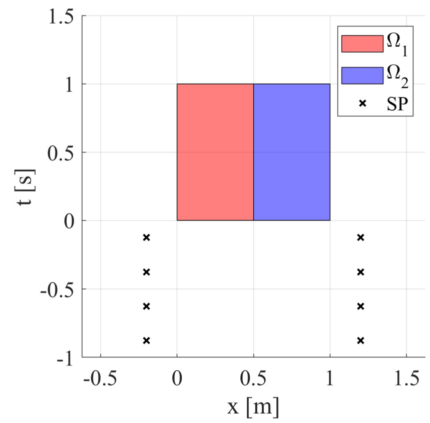

Figure 1. MFS collocation and source points distribution in Case 1 using $m = 4$, and $\delta d = 0.2$.

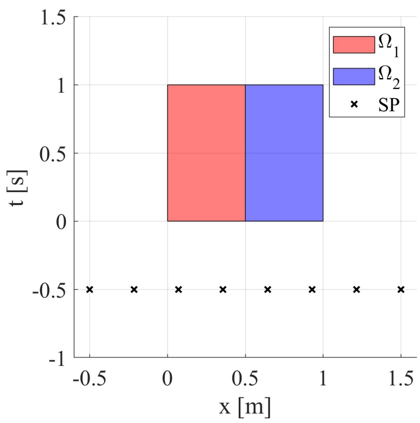

Figure 2. MFS collocation and source points distribution in Case 2: $m = 4$, $\delta d = 0.5$, and $\delta t = -0.5$.

4. NUMERICAL EXPERIMENTS

Following the works of Johansson and Lesnic (2008), we assume unitary thermal diffusivity, $\alpha = 1$, domain length $L = 1$, boundary heat flux $q = 1$, and final time $t_f = 1$. Three different values of thermal contact conductances were considered, $h = 0.02$, $h = 0.2$, and $h = 2$.

Synthetic measurements on $x = L$ were considered, as given by Eq. (19) .

$$T_{meas} = T^*(1 + \varepsilon\zeta) \tag{19}$$

where T^* is the analytical response obtained by the solution of the direct problem given by Eq. (1) - (7) described by Tsai and Crane (1992), ε dictates the level of noise, and ζ is a normally distributed random variable with zero mean and unit standard deviation, obtained by the Matlab® function randn. The noise levels applied to this work were $\varepsilon = [0.1\%, 1\%, 2.5\%]$. To enable the reproducibility of the results, a seed function was applied to the measurements.

To test the accuracy of the approximate solution, we used the root mean squared error ($RMSE$) defined in Eq. (20) , and the relative root mean square error (RES), defined in Eq. (21) , where N_t is the total number of test points, and h and h^* are, respectively, the approximate and the exact values of the TCC at these points. As in Yan et al. (2008), we used $N_t = 21^2$ points considered in the time domain $[t_i, t_f]$ to analyze the TCC estimation.

$$RMSE(h) = \sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}(h_i - h_i^*)^2} \tag{20}$$

$$RES(h) = \sqrt{\frac{\sum_{i=1}^{N_t}(h_i - h_i^*)^2}{\sum_{i=1}^{N_t} h_i^{*2}}} \tag{21}$$

To analyze the influence of the number of measurements, we used $m = [12, 26, 50]$, where m represents the number of collocation points placed on each considered boundary condition.

The parameter δd of Case 1 was fixed on 0.5, because generally when δd tends to 0 or is much higher than 1 the MFS produces instabilities due to the fundamental solution as in Johansson and Lesnic (2008). The parameters δd and δt of Case 2 were, respectively, 2 and -0.1, as suggested by Kopperschmidt et al. (2020).

Each regularization parameter λ was determined by the L-Curve analysis. An example of L-Curve is shown in Fig. 3 for $h = 2$ (Case 1), using $m = 50$ and $\varepsilon = 2.5\%$. The results were sensible to the regularization due to its ill-conditioning.

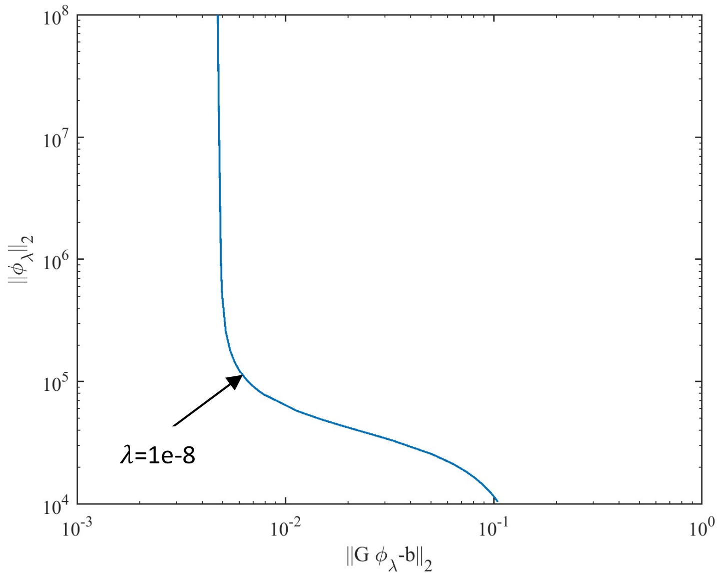

Figure 3. L-Curve for $h = 2$, Case 1, $m = 50$, and $\varepsilon = 2.5\%$.

In general, the methodology presented unstable results at the beginning of the estimation of every studied case, resulting in high values of RMSE and RES which decreased as an initial time far from zero is taken, as shown in Fig. 3 and Fig. 4, respectively. The results of this paper disregard the TCC estimated before 0.2 seconds, or 20% of the final time due to these instabilities.

The RMSE and RES for each different case is shown in Tab. 1 for $h = 0.02\ [kW/m^2K]$, in Tab. 2 for $h = 0.2\ [kW/m^2K]$, and in Tab. 3 for $h = 2\ [kW/m^2K]$.

It can be noticed that the RMSE and RES for Case 1 and Case 2 were close for each studied case. Related to the number of measurements, which is equivalent to m on this approach, it can be verified that, in general, all cases were able to estimate the TCC, even for a few measurements, and the errors decreased as a higher number of measurements were used.

The results for $h = 0.02$ on Tab. 1 showed that Case 1 presented higher errors when using few measurements, while presented better results do $m = 26$ and $m = 50$. Similar conclusions can be drawn from the results of Tab. 2 for $h = 0.2$, while for $h = 2$, Case 1 presented worse results than every case, as can be seen in Tab. 3.

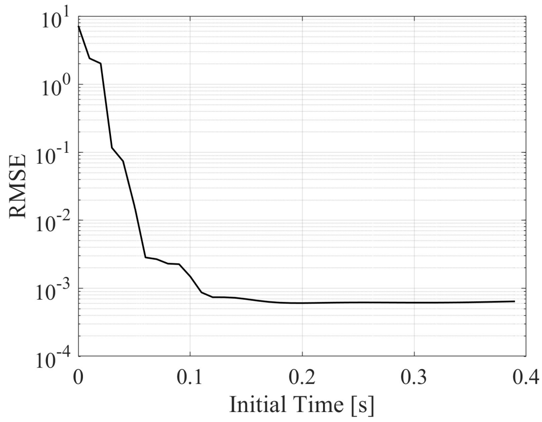

Figure 4. RMSE for $h = 0.02$, Case 1, $m = 50$, $\lambda = 5e-9$, and $\varepsilon = 0.1\%$.

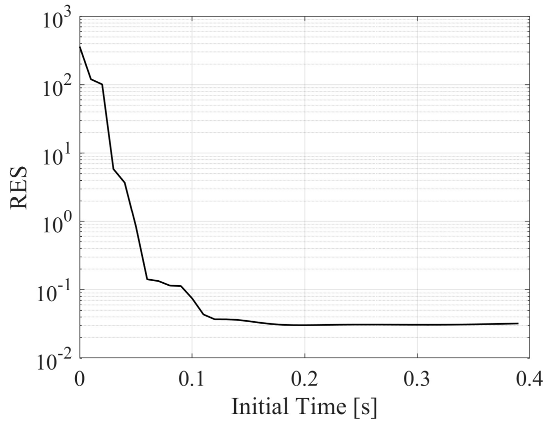

Figure 5. RES for $h = 0.02$, Case 1, $m = 50$, $\lambda = 5e-9$, and $\varepsilon = 0.1\%$.

Table 1. Results for $h = 0.02$ $[kW/m^2K]$

$N_{meas} = m$	ε [%]	Case 1			Case 2		
		RMSE	RES	λ	RMSE	RES	λ
12	0.1	0.0087	0.4356	1e-9	0.0067	0.3363	1e-11
	1	0.0093	0.4650	1e-9	0.0069	0.3430	1e-11
	2.5	0.0104	0.5194	1e-9	0.0112	0.5619	1e-9
26	0.1	0.0012	0.0576	5e-9	0.0024	0.1194	1e-12
	1	0.0011	0.0574	5e-9	0.0025	0.1238	1e-12
	2.5	0.0012	0.0591	5e-9	0.0026	0.1316	1e-12
50	0.1	6.0674e-04	0.0303	5e-9	0.0016	0.0776	1e-13
	1	7.0136e-04	0.0351	5e-9	0.0016	0.0819	1e-13
	2.5	8.9157e-04	0.0446	5e-9	0.0018	0.0898	1e-13

Table 2. Results for $h = 0.2$ $[kW/m^2K]$

$N_{meas} = m$	ε [%]	Case 1			Case 2		
		RMSE	RES	λ	RMSE	RES	λ
12	0.1	0.0551	0.2757	1e-8	0.0202	0.1009	1e-9
	1	0.0614	0.3073	1e-8	0.0213	0.1064	1e-9
	2.5	0.0728	0.3638	1e-8	0.0232	0.1158	1e-9
26	0.1	0.0112	0.0561	5e-9	0.0158	0.0788	1e-11
	1	0.0113	0.0564	1e-8	0.0177	0.0884	1e-11
	2.5	0.0119	0.0593	1e-8	0.0209	0.1045	1e-11
50	0.1	0.0066	0.0328	1e-8	0.0068	0.0341	1e-9
	1	0.0082	0.0412	1e-8	0.0076	0.0381	1e-9
	2.5	0.0112	0.0561	1e-8	0.0096	0.0482	1e-9

Table 3. Results for $h = 2\ [kW/m^2K]$

$N_{meas} = m$	ε [%]	Case 1			Case 2		
		RMSE	RES	λ	RMSE	RES	λ
12	0.1	0.7470	0.3735	1e-3	0.4486	0.2243	1e-8
	1	0.8561	0.4280	1e-3	0.4716	0.2358	1e-8
	2.5	1.0710	0.5355	1e-3	0.5527	0.2764	1e-8
26	0.1	0.2495	0.1247	5e-9	0.2253	0.1126	1e-8
	1	0.3267	0.1633	1e-7	0.2971	0.1485	1e-8
	2.5	0.4839	0.2419	1e-7	0.4454	0.2227	1e-8
50	0.1	0.1838	0.0919	1e-8	0.1141	0.0571	1e-9
	1	0.2667	0.1334	1e-8	0.1717	0.0858	1e-9
	2.5	0.4265	0.2133	1e-8	0.2957	0.1479	1e-9

The level of noise ε caused an increase in the RMSEs for all cases, as expected. The influence of increasing ε can be seen in Fig. 6, Fig. 7, and Fig. 8 for $h = 0.02, 0.2,$ and $2\ [kW/m^2K]$, respectively, for Case 1 using $m = 50$, where is possible to notice that the estimation deviates from the expected the greater the noise levels.

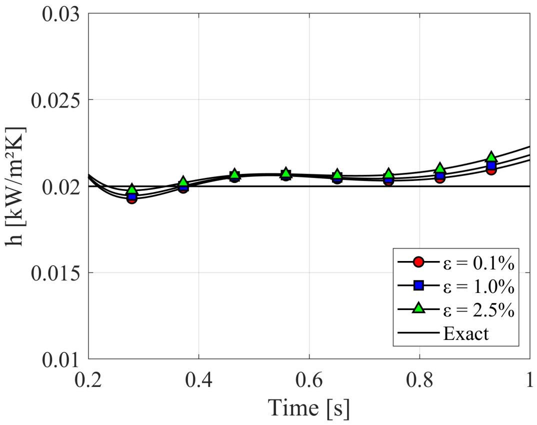

Figure 6. Estimated TCC for Case 1 using $m = 50$, when $h = 0.02\ [kW/m^2K]$.

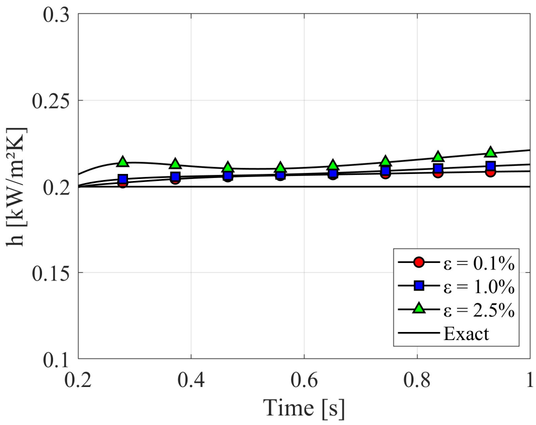

Figure 7. Estimated TCC for Case 1 using $m = 50$, when $h = 0.2\ [kW/m^2K]$.

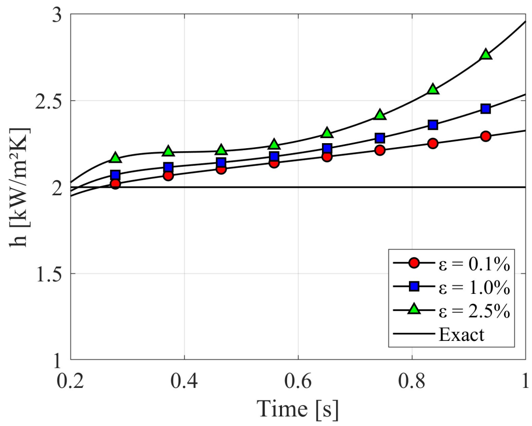

Figure 8. Estimated TCC for Case 1 using $m = 50$, when $h = 2\ [kW/m^2K]$.

5. CONCLUSIONS

This paper extended the MFS using time-dependent fundamental solutions with Tikhonov regularization to solve an inverse heat conduction problem of determining the thermal contact conductance between two materials in a one-dimensional mathematical model. Two different source points placement strategies were used to compare the solutions, having only negative time components, avoiding problems with singularities, and being possible to estimate the TCC at any time on the studied time domain.

The MFS resulted in accurate results, even for relatively high noise levels being applied to the boundary input data, and for few measurements, except near the initial time, which presented unstable results. The computational cost for the examples is of the order of seconds, even using the higher number of measurements.

There are several potential extensions of the method, as being adapted to two and three-dimensions, then consider a spatial and a time-dependent thermal contact conductance.

The estimation results highly depend on the regularization parameter λ, being necessary to choose it carefully. Other approaches to choosing the regularization parameter as the cross-validation (GCV) criterion can be used to make the method more automated.

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