19th Brazilian Congress of Thermal Sciences and Engineering
November 06th-10th, 2022, Bento Gonçalves - RS - Brazil

ENC-2022-0178
ANALYTICAL TRANSIENT HEAT CONDUCTION IN 2-D POLAR COORDINATES IN A COMPOSITE CYLINDER DOMAIN APPLIED TO PLUG AND ABANDONMENT OF OIL WELLS

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Abstract. *Plug and Abandonment (P&A) procedure is a mandatory regulatory requirement when the economical capacity of the reservoir is depleted and need to be decommissioned, being a non-profitable and expensive activity. New technologies are being studied in order to shrink costs by doing P&A focusing on the minimization of maneuvers and rig operations to accomplish this operation. The thermite reaction approach is one of these emerging studies, consisting on an exothermic reaction capable of surpass the melting point of the oil well structure and, after the cooldown and solidification of the melted product, a seal is formed on the well cross-section accomplishing P&A requirements. An analytical mathematical framework based on the Separation Of Variables (SOV) is presented in this work in order to solve the transient heat conduction governing equation in 2-D polar coordinates in a solid composite, which eigenvalue problem in radial direction is implicitly dependent on the polar transverse eigenvalues where only real eigenvalues are computed, differently when considering the analysis on 2-D axisymmetric domain. Also, non-homogeneous boundary conditions on the radial direction of the first, second and third kind can be prescribed on the composite cylinder domain, whereas on the angular direction only homogeneous boundary condition of first and second kind should be considered. On the transient governing equation, the heat source term is spatial-dependent on the radial and angular directions and is time-independent, which can be arbitrated in any layer that constitutes the composite cylinder domain. The solution is computed taking into account thermophysical and geometrical specifications applied to real case oil wells and the heat source is determined by the stoichiometric mixture of the thermite reaction. Validation with numerical solution is presented in order to verify the mathematical framework correctness. Lastly, SOV approach can be applied to solid or hollow cylinders composite media by performing minor modifications in the mathematical modeling*

Keywords: *Oil & Gas, Analytical Non-steady Heat Conduction, Generalized Eigenvalues and Eigenvectors Problems, 2-D Composite Hollow Cylinder Medium.*

1. INTRODUCTION

Many modern engineering applications include cylindrical geometries as a mean of exchanging heat efficiently. Moreover, the association of multiple cylindrical layers adds the advantage of combining thermal and mechanical properties. Important examples of appliances are related to tunnel biotechnology [1], tunnel heating systems in cold regions [2], improvement of materials [3], ground heat exchangers [4], automotive, space, chemical, civil, nuclear and Oil & Gas industries. Aiming thermal phenomena for these applications, the heat transfer analysis throughout the multi-layered domain of both non-steady and steady-state approaches are fundamentals concepts to achieve expected project demands.

Although numerical approaches are commonly employed to investigate most engineering phenomena due to its accuracy and consolidated basis, analytical frameworks provide thorough understanding about the occurring phenomena [5] and assists numerical modeling for the verification between both schemes.

Several solution methods as Laplace transform method [6-7], Green's function approach [8], generalized orthogonal and quasi-orthogonal expansion techniques [9-11] finite integral transform technique [12], Distributed Transfer Function Method (DTFM) [13-20] and Separation of Variables Method (SVM) [21-22] achieves reliable transient heat conduction results for composite components. The aforementioned methods are capable of providing precise results for conduction heat transfer. However, the techniques require decoupling procedures when nonhomogeneous boundary and source terms are included in the governing equation. Finally, very few cases of engineering applications with proper numerical verification are currently found in open literature, with even fewer illustrative examples making use of engineering design thermophysical and geometry properties.

In this work, an investigation of a transient heat conduction developed in 2-D (r,θ) polar coordinates is presented. The multi-layer geometry that represents the physical domain was conceived as a pie slice $(\phi < 2\pi)$. The mathematical frameworks, namely SVM and eigenfunction expansion method [23], were applied to solve the decoupled transient homogeneous and steady state inhomogeneous problems, respectively. Here, the final results yielded a closed form analytical double-series solution. The presented methodology enables path to the application of nonhomogeneous boundary conditions of the first, second or third kinds on the radial direction. Additionally, for the azimuthal direction, the method is applicable when prescribing a homogenous boundary condition of the first or second kind.

The process of sealing an oil well is a non-profitable and expensive mandatory activity for the oil industry with supervision performed by environmental regulatory agencies [24]. The traditional tamponing technique is called Plug and Abandonment (P&A) and consists in establishing several barriers along the wellbore in desirables depths [25] where a specific rock formation is found, the cap-rock. The final plug must attend the requirements for an eternal perspective sealing and fully occupy the section of the well [24], [26-32].

However, this method has significant disadvantages since it not rarely results in expending large amounts of time and money [33]. Well abandonment operation can amount to about two thirds of all decommissioning expenditures, which prompts oil industries to look for new technologies capable of shrinking the costs of P&A [34].

Alternatively, an emergent technology under development named Thermal Plug and Abandonment (TP&A) consists in the use of a heat emitter device. The equipment is descended inside the production tube by a “through tubing” operation and is able to melt the production tubing, the first layer of the well structure. This technology allows the possibility of leaving the production tube inside the well, reducing the number of required maneuvers to accomplish P&A, saving time and hence diminishing costs [35].

On the development of TP&A concepts, the proposed final milestone of this study predicts that a plug is formed by the solidification of thermite and oil well structures previously melted. However, due to concerns about leaving the production tube in the borehole, as mentioned in Vrålstad et al (2019) [36], an intermediate milestone was established based on the combination of TP&A and the usual cement procedures. Hence, gaps on the production tube are formed due to the melted sections, enabling path for the insertion of cement by the conventional method, fully occupying the whole cross section of the borehole without removing the production pipe.

In previous work, another TP&A analysis was performed, where a new hybrid analytical/numerical approach using DTFM method was implemented for transient heat conduction in 1-D (radial) composite hollow cylinders. Back then, a method capable of solving non-differentiable boundary conditions profiles was introduced by De Andrade et al [37]. The heat flux profile applied in the referenced work was estimated from Abdelal et al. (2015) [38] data, who developed an experimental work based on the thermite reaction. The physical apparatus was capable of measuring the transient temperature response of the exothermic thermite reaction.

Although transient heat conduction is a well stablished subjected, a literature survey demonstrated that applications of transient heat conduction in 2-D (r,θ) polar coordinates applied to composite solid cylinders and the use of these mathematical frameworks devoted to engineering applications are scarce.

2. MATHEMATICAL MODEL

Consider a multi-layer hollow cylinder geometry represented by 2-D (r,θ) polar coordinates with n number of layers, perfect thermal contact between interfacial boundaries and isotropic properties. The transient heat conduction governing equation for a space dependent and time-independent heat source is cast as

$$\frac{1}{\alpha_i}\frac{\partial T_i}{\partial t}(r,\theta,t) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_i}{\partial r}(r,\theta,t)\right) + \frac{1}{r^2}\frac{\partial^2 T_i}{\partial \theta^2}(r,\theta,t) + \frac{g_i(r,\theta)}{k_i},\ r_{i-1} \le r \le r_i,\ 1 \le i \le n \tag{1}$$

where T_i, α_i, k_i and g_i are the temperature distribution, thermal diffusivity, thermal conductivity and heat source, respectively, for the *ith*-layer. Additionally, r is the radius, θs is the azimuthal coordinate and t is the time. Inner and outer boundary conditions and initial condition are given, respectively, by

$$A_{in}\frac{\partial T_1}{\partial r}(r_0,\theta,t) + B_{in}T_1(r_0,\theta,t) = C_{in} \tag{2}$$

$$A_{out}\frac{\partial T_n}{\partial r}(r_n,\theta,t) + B_{out}T_n(r_n,\theta,t) = C_{out} \tag{3}$$

$$T_i(r,\theta,t=0) = f_i(r,\theta) \tag{4}$$

being A_{in}, B_{in}, C_{in} the prescribed inner boundary conditions, A_{out}, B_{out}, C_{out} the prescribed outer boundary conditions and $f_i(r,\theta)$ the initial temperature distribution in the *i*th layer at $t=0$.

Dirichlet, Neumann and Robin inhomogeneous boundary conditions can be applied in Eqs. (2) and (3). The assumption of perfect contact between layers and any interfacial inner $(i \neq 1)$ and outer $(i \neq n)$ conditions guarantee the heat flux and temperature continuity, respectively. These conditions may be mathematically described as

$$T_i\left(r_{i-1},\theta,t\right)=T_{i-1}(r_{i-1},\theta,t) \quad (5)$$

$$k_i\frac{\partial T_i}{\partial r}\left(r_{i-1},\theta,t\right)=k_{i-1}\frac{\partial T_{i-1}}{\partial r}(r_{i-1},\theta,t) \quad (6)$$

$$T_i\left(r_i,\theta,t\right)=T_{i+1}(r_i,\theta,t) \quad (7)$$

$$k_i\frac{\partial T_i}{\partial r}\left(r_i,\theta,t\right)=k_{i+1}\frac{\partial T_{i+1}}{\partial r}(r_i,\theta,t) \quad (8)$$

In the angular surfaces, namely $\theta=0$ and $\theta=\phi$, on the other hand, only homogeneous boundary conditions of the first and second kind may be prescribed, which are written as

$$T_i\left(r,\theta=0,t\right)=0 \text{ and } \frac{\partial T_i}{\partial \theta}\left(r,\theta=0,t\right)=0 \quad (9a)$$

$$T_i\left(r,\theta=\varphi,t\right)=0 \text{ and } \frac{\partial T_i}{\partial \theta}\left(r,\theta=0,t\right)=0 \quad (9b)$$

The analytical solution SVM is a decoupling method and is only applicable to homogeneous problems. Hence, the non-homogeneous problem has to be split into a homogeneous transient problem and a non-homogeneous steady-state problem as follows

$$T_i\left(r,\theta,t\right)=\overline{T}_i\left(r,\theta,t\right)+T_{ss,i}\left(r,\theta\right) \quad (10)$$

2.1. Homogeneous Transient Problem

The analogous homogeneous transient problem $\overline{T}_i\left(r,\theta,t\right)$ of Eqs. (1)-(10) is developed as

$$\frac{1}{\alpha_i}\frac{\partial \overline{T_i}}{\partial t}\left(r,\theta,t\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \overline{T_i}}{\partial r}\left(r,\theta,t\right)\right)+\frac{1}{r^2}\frac{\partial^2 \overline{T_i}}{\partial \theta^2}\left(r,\theta,t\right),\ r_{i-1}\le r\le r_i,\ 1\le i\le n \quad (11)$$

where the source term $g_i\left(r,\theta\right)$ was vanished. The equivalence of the inner and outer boundary and initial conditions are, respectively

$$A_{in}\frac{\partial \overline{T_1}}{\partial r}\left(r_0,\theta,t\right)+B_{in}\overline{T_1}\left(r_0,\theta,t\right)=0 \quad (12)$$

$$A_{out}\frac{\partial \overline{T_n}}{\partial r}\left(r_n,\theta,t\right)+B_{out}\overline{T_n}\left(r_n,\theta,t\right)=0 \quad (13)$$

$$\overline{T}_i\left(r,\theta,t=0\right)=f_i(r,\theta)-T_{ss,i}\left(r,\theta\right) \quad (14)$$

Interfacial conditions at the inner interface of the *i*th layer and outer interface of the *i*th layer are respectively given as,

$$\overline{T_i}\left(r_{i-1},\theta,t\right)=\overline{T}_{i-1}(r_{i-1},\theta,t)s \quad (15)$$

$$k_i\frac{\partial \overline{T}_i}{\partial r}\left(r_{i-1},\theta,t\right)=k_{i-1}\frac{\partial \overline{T}_{i-1}}{\partial r}(r_{i-1},\theta,t) \quad (16)$$

$$\overline{T}_i\left(r_i,\theta,t\right)=\overline{T}_{i+1}(r_i,\theta,t) \quad (17)$$

$$k_i\frac{\partial \overline{T}_i}{\partial r}\left(r_i,\theta,t\right)=k_{i+1}\frac{\partial \overline{T}_{i+1}}{\partial r}(r_i,\theta,t) \quad (18)$$

Lastly, the homogenization of the complementary transient corresponding to the boundary conditions for the azimuthal surfaces $\theta=0$ and $\theta=\phi$ is cast, respectively, as

$$\overline{T_i}\left(r,\theta=0,t\right)=0 \text{ and } \frac{\partial\overline{T_i}}{\partial\theta}\left(r,\theta=0,t\right)=0 \tag{19}$$

$$\overline{T_i}\left(r,\theta=\varphi,t\right)=0 \text{ and } \frac{\partial\overline{T_i}}{\partial\theta}\left(r,\theta=\varphi,t\right)=0 \tag{20}$$

The transient temperature distribution in the product form is given by Eq. (21).

$$\overline{T_i}\left(r,\theta,t\right)=R_i\left(r\right)\Theta_i\left(\theta\right)\Gamma_i\left(t\right) \tag{21}$$

Substituting in Eq. (11) and applying the SVM yields the Ordinary Differential Equations (ODEs) represented as below

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial R_i}{\partial r}+\left(-\frac{\beta_i^2}{r^2}+\lambda_i^2\right)R_i=0 \tag{22}$$

$$\frac{\partial^2\Theta_i}{\partial\theta^2}+\beta_i^2\Theta_i=0 \tag{23}$$

$$\frac{1}{\alpha_i}\frac{\partial\Gamma_i}{\partial t}+\lambda_i^2\Gamma_i=0 \tag{24}$$

where R_i, Θ_i and Γ_i are the separated variables accounting for the transverse (radial) eigenfunction, azimuthal eigenfunctions and temporal dependence of the temperature distribution, respectively. λ_i^2 and β_i^2 are constants of separation.

The following general solution for each separated variable was determined considering the prescribed assumption for the heat flux and temperature continuity throughout the domain, given by Eqs. (16) and (18). For all values of t, the constants of separation λ_i and β_i are the radial and azimuthal eigenvalues, respectively, being cast as [10] [23] [39] [40]

$$\lambda_{imp}=\lambda_{imp}\sqrt{\frac{\alpha_1}{\alpha_i}},i=1,2,\ldots,n \tag{25}$$

$$\Theta_i=\Theta\Rightarrow\beta_i=\beta,i=1,2,\ldots,n \tag{26}$$

Firstly, the solution of the ODEs given in Eq. (22) is

$$R_{imp}\left(\lambda_{imp}r\right)=a_{imp}J_{\beta_m}\left(\lambda_{imp}r\right)+b_{imp}Y_{\beta_m}\left(\lambda_{imp}r\right) \tag{27}$$

where functions J_{β_m} and Y_{β_m} in Eq. (18) are the β_m-order Bessel functions of the first and second kind, respectively. Bessel functions occurrence is due to the variation of cross section of an extended surface [41]. Because a hollow cylinder is considered herein, the Bessel function Y_{β_m} is nonsingular for any $r\geq r_0>0$ [20].

The key concept for the solution consists in determining the azimuthal and radial problems. Application of boundary conditions given in Eqs. (12) and (13) along with interfacial conditions given by Eqs. (15) - (18) to the eigencondition in Eq. (27), for $m=0,\ 1,\ 2,\ldots$, results in the following matrix representation

$$\begin{bmatrix} C_{1in} & C_{2in} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ x_{11} & x_{12} & x_{13} & x_{14} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ y_{11} & y_{12} & y_{13} & y_{14} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & x_{i1} & x_{i2} & x_{i3} & x_{i4} & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & y_{i1} & y_{i2} & y_{i3} & y_{i4} & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & x_{n-1,1} & x_{n-1,2} & x_{n-1,3} & x_{n-1,4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & y_{n-1,1} & y_{n-1,2} & y_{n-1,3} & y_{n-1,4} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & C_{1out} & C_{2out} \end{bmatrix} \begin{bmatrix} a_{1mp} \\ b_{1mp} \\ \ldots \\ \ldots \\ b_{1mp} \\ b_{imp} \\ \ldots \\ \ldots \\ a_{nmp} \\ b_{nmp} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \\ 0 \\ \ldots \\ \ldots \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

where C_{1in}, C_{2in}, C_{1out} and C_{2out} are the inner and outer boundary conditions coefficients for the radial eigencondition, respectively, and x_{ij} and y_{ij} are the interfacial boundary conditions coefficients for the radial eigencondition with $i=1,\ 2,\ldots,\ n$ and $j=1,\ 2,\ 3,\ 4$, which quantities are determined by means of the boundary conditions given in Eqs. (15)-(21). Setting the determinant of the coefficient matrix in Eq. (28) equal to zero, one can verify the existence of an infinite number of roots for the eigenvalues in the form $\lambda_{1mp} < \lambda_{2mp} < \lambda_{3mp} < ... < \lambda_{nmp} < ...$ [42].

For $i = n$, $m = m_{\max}$ and $p = p_{\max}$ the eigenvalue assumes its maximum value. In this work, the coefficient matrix involves only Bessel functions whose behavior is oscillatory. The root finder toll used was the Rolle's theorem. It is expected that, when dealing with continuous oscillatory functions containing zeros it will exist several points where the function assumes zero values. Evaluating this function by choosing two points sufficiently close to each other, namely $f(a)$ and $f(b)$, with $a < b$, (a,b) and considering a control parameter N of the minimum distance between two zeros, there is an increment of $f(a) - f(b)/N$ which assures the existence of zeros along (a,b).

The coefficients a_{imp} and b_{imp} in Eq. (27) are obtained using the ith interface condition given by Eqs. (15) and (16). The following relation is valid for $i \in [1, n-1]$

$$\begin{pmatrix} a_{i+1,mp} \\ b_{i+1,mp} \end{pmatrix} = \begin{pmatrix} J_{\beta_m}\left(\lambda_{i+1,mp} r_i\right) & Y_{\beta_m}\left(\lambda_{i+1,mp} r_i\right) \\ k_{i+1} J'_{\beta_m}\left(\lambda_{i+1,mp} r_i\right) & k_{i+1} Y'_{\beta_m}\left(\lambda_{i+1,mp} r_i\right) \end{pmatrix}^{-1} \times \begin{pmatrix} J_{\beta_m}\left(\lambda_{imp} r_i\right) & Y_{\beta_m}\left(\lambda_{imp} r_i\right) \\ k_i J'_{\beta_m}\left(\lambda_{imp} r_i\right) & k_i Y'_{\beta_m}\left(\lambda_{imp} r_i\right) \end{pmatrix} \begin{pmatrix} a_{imp} \\ b_{imp} \end{pmatrix} \quad (29)$$

where a_{1mp} is arbitrary and

$$b_{1mp} = -\frac{C_{1in}}{C_{2in}} a_{1mp} \quad (30)$$

secondly, the solution of the ODEs given in Eq. (23) is

$$\Theta_m(\beta_m \theta) = \omega_1 \sin(\beta_m \theta) + \omega_2 \cos(\beta_m \theta) \quad (31)$$

where constants ω_1, ω_2 and β_m are listed in Table 1 and are dependent on the boundary conditions at $\theta = 0$ and $\theta = \phi$.

Table 1 - ω_1, ω_2 and β_m for different combinations of BC's at $\theta = 0$ and $\theta = \phi$ surfaces.

BC at $\theta = 0$	BC at $\theta = \phi$	ω_1	ω_2	β_m
$\overline{T_i}(r, \theta = 0, t) = 0$	$\overline{T_i}(r, \theta = \phi, t) = 0$	1	0	$\frac{m\pi}{\phi}$
$\frac{\partial \overline{T_i}}{\partial \theta}(r, \theta = 0, t) = 0$	$\frac{\partial \overline{T_i}}{\partial \theta}(r, \theta = \phi, t) = 0$	0	1	$\frac{m\pi}{\phi}$
$\overline{T_i}(r, \theta = 0, t) = 0$	$\frac{\partial \overline{T_i}}{\partial \theta}(r, \theta = \phi, t) = 0$	1	0	$\frac{2m-1}{2}\frac{\pi}{\phi}$
$\frac{\partial \overline{T_i}}{\partial \theta}(r, \theta = 0, t) = 0$	$\overline{T_i}(r, \theta = \phi, t) = 0$	0	1	$\frac{2m-1}{2}\frac{\pi}{\phi}$

Finally, the solution of the ODEs given in Eq. (24) is

$$\Gamma_i\left(\lambda_{imp} t\right) = D_{mp} e^{-\alpha_i \lambda_{imp}^2 t} \quad (32)$$

where D_{mp} is a coefficient obtained by the use of both orthogonality conditions in the azimuthal and radial directions [43] being cast, respectively, as

$$\int_0^{\phi} \Theta_m(\beta_m \theta) \Theta_l(\beta_l \theta) d\theta = \begin{cases} 0 & \text{if } m \neq l \\ N_{\theta m} & \text{if } m = l \end{cases} \quad (33)$$

$$\sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_{r_{i-1}}^{r_i} r R_{imp}\left(\lambda_{imp} r\right) R_{imq}\left(\lambda_{imq} r\right) dr = \begin{cases} 0 & \text{if } p \neq q \\ N_{rmp} & \text{if } p = q \end{cases} \quad (34)$$

and accounting the initial condition given in Eq. (14), D_{mp} yields

$$D_{mp} = \frac{1}{N_{\theta m} N_{rmp}} \sum_{i=1}^{n} \frac{k_i}{\alpha_i} \int_0^{\phi} \int_{r_{i-1}}^{r_i} r R_{imp}\left(\lambda_{imp} r\right) \times \Theta_m\left(\beta_m \theta\right) \overline{T}_i\left(r,\theta,t=0\right) dr d\theta \qquad (35)$$

The proof of the orthogonality conditions given by Eqs. (33) and (34) are available in Singh et al. (2008) [43].

Substituting Eqs. (27), (31) and (32) in Eq. (21), a general solution of Eq. (11), yields

$$\overline{T_i}\left(r,\theta,t\right) = \sum_{m=1}^{\infty} \sum_{p=1}^{\infty} D_{mp} e^{-\alpha_i \lambda_{imp}^2 t} R_{imp}\left(\lambda_{imp} r\right) \Theta_m\left(\beta_m \theta\right) \qquad (36)$$

2.2. Inhomogeneous Steady State Problem

The equivalent inhomogeneous steady state problem $T_{ss}\left(r,\theta\right)$ of Eqs. (1)-(3) and Eqs. (5)-(12), accounting for the heat source term $g_i\left(r,\theta\right)$ is cast as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T_{ss,i}}{\partial r}\left(r,\theta,t\right)\right) + \frac{1}{r^2}\frac{\partial^2 T_{ss,i}}{\partial \theta^2}\left(r,\theta,t\right) + \frac{g_i\left(r,\theta\right)}{k_i} = 0, \; r_{i-1} \le r \le r_i, \; 1 \le i \le n \qquad (37)$$

with inner and outer boundary conditions being, respectively

$$A_{in} \frac{\partial T_{ss,i}}{\partial r}\left(r_0,\theta\right) + B_{in} T_{ss,i}\left(r_0,\theta\right) = C_{in} \qquad (38)$$

$$A_{out} \frac{\partial T_{ss,i}}{\partial r}\left(r_n,\theta\right) + B_{out} T_{ss,i}\left(r_n,\theta\right) = C_{out} \qquad (39)$$

The interfacial conditions at the inner interface of the *i*th layer and outer interface of the *i*th layer are, respectively

$$T_{ss,i}\left(r_{i-1},\theta\right) = T_{ss,i-1}(r_{i-1},\theta) \qquad (40)$$

$$k_i \frac{\partial T_{ss,i}}{\partial r}\left(r_{i-1},\theta\right) = k_{i-1} \frac{\partial T_{ss,i-1}}{\partial r}(r_{i-1},\theta) \qquad (41)$$

$$T_{ss,i}\left(r_i,\theta\right) = T_{ss,i+1}(r_i,\theta,t) \qquad (42)$$

$$k_i \frac{\partial T_{ss,i}}{\partial r}\left(r_i,\theta\right) = k_{i+1} \frac{\partial T_{ss,i+1}}{\partial r}(r_i,\theta) \qquad (43)$$

with homogeneous boundary conditions at the azimuthal surfaces $\theta = 0$ and $\theta = \phi$ cast, respectively, as

$$T_{ss,i}\left(r_n,\theta=0\right) = 0 \text{ and } \frac{\partial T_{ss,i}}{\partial \theta}\left(r_n,\theta=0\right) = 0 \qquad (44)$$

$$T_{ss,i}\left(r,\theta=\varphi\right) = 0 \text{ and } \frac{\partial T_{ss,i}}{\partial \theta}\left(r_n,\theta=\varphi\right) = 0 \qquad (45)$$

Solution to the inhomogeneous steady state problem is obtained using the eigenfunction expansion method, written in generalized Fourier series in terms of azimuthal eigenfunction, resulting

$$T_{ss,i}\left(r,\theta\right) = \sum_{m=1}^{\infty} T_{im}\left(r\right) \Theta_m\left(\beta_m \theta\right), \; r_{i-1} \le r \le r_i, \; 1 \le i \le n \qquad (46)$$

Substituting Eq. (46) in Eq. (37) yields

$$\frac{1}{r}\frac{d}{dr}\left(r \frac{dT_{im}\left(r\right)}{dr}\right) - \frac{\beta_m^2}{r^2} T_{im}\left(r\right) + \frac{g_{im}\left(r\right)}{k_i} = 0, r_{i-1} \le r \le r_i, 1 \le i \le n \qquad (47)$$

where

$$g_{im}\left(r\right) = \frac{1}{N_{\theta m}} \int_0^{\phi} g_i\left(r,\theta\right) \Theta_m\left(\beta_m \theta\right) d\theta \qquad (48)$$

Boundary conditions of first, second or third kind at the inner or outer surfaces, namely C_{in} and C_{out}, are also expanded in generalized Fourier series. Solution for a composite hollow cylinder domain is given by

$$T_{im}(r) = A_{ss,i}\, r^{m} + B_{ss,i}\, r^{-m} + f_p(r) \tag{49}$$

where constants $A_{ss,i}$ and $B_{ss,i}$ may be evaluated using boundary and interfacial conditions resulting in a matrixial product evolving a $(2n\times 2n)$ coefficient matrix with a $(2n\times 1)$ boundary conditions dependent vector. Last term on the Right-Hand Side (RHS) of Eq. (49), $f_p(r)$, is determined by a particular integral obtained applying the method of undetermined coefficients.

3. RESULTS AND DISCUSSION

The analysis developed in the next section was based on an oil well configuration depicted in Fig.1, defined from $\theta = 0$ to $\theta = \pi$.

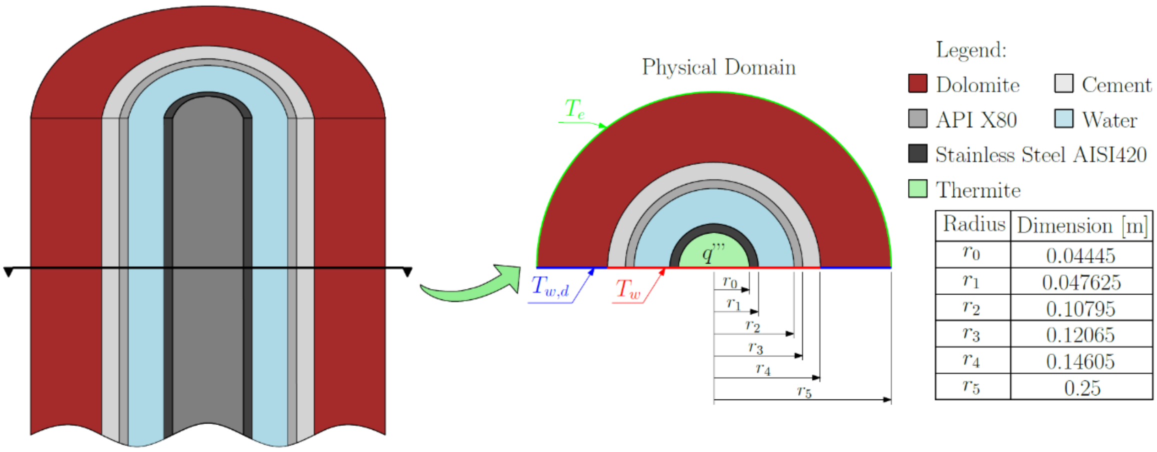

Figure 1. Oil well structural scheme, geometry specifications and materials.

where the thermophysical and geometrical parameters of each layer are cast in Table 2 below

Table 2 - Geometrical, thermophysical, and physical properties of an average offshore oil well structure.

Material	Thermite	SS AISI 420	Water	API X80	Cement	Dolomite
Radius (r) [m]	0.04445	0.047625	0.10795	0.12065	0.14605	0.25
Density [kg/m³]	4142 [46]	7671 [47]	998	7800 [48]	2010 [49]	953 [50]
Thermal Conductivity [W/m.K]	47.88 [46]	26.08 [47]	0.65	54.21 [51]	0.53 [49]	3.34 [50]
Specific Heat [J/kg.K]	748.1 [46]	475.61 [47]	4185	448 [51]	736 [49]	2630 [50]
Melting Point [ºC]	-	1450 [47]	0	1440 [52]	1157 [53]	2570 -4660 [50]

and Boundary conditions prescribed for the case shown in Fig.1 are set in Table 2 below

Table 2 - Investigated cases boundary, interfacial and initial conditions.

Boundary / Interface	Type	Thermal Specification	Initial Temperature
Thermite/SS AISI 420	Heat Source	2-D Volumetric Heat Generation $g_i(r,\theta)$	60. 0 ºC
SS AISI 420/Annular Region	Wall	Heat Conducting	60. 0 ºC
Annular Region /API X80	Wall	Heat Conducting	60. 0 ºC
API X80/Cement	Wall	Heat Conducting	60. 0 ºC
Cement/Dolomite	Wall	Heat Conducting	60. 0 ºC
External Boundary from $\theta=0$ to $\theta=\pi$	Wall	Ambient Heat Dissipation	60. 0 ºC
Flat Boundaries (at $\theta=0$ to $\theta=\pi$)	Wall	Constant Temperature	0. 0 ºC

where $g_i(r,\theta)s$ is the volumetric heat source arising from the thermite reaction with magnitude equals to $S = 10E+08\ W/m^3$. Such value was determined considering the minimum magnitude capable of achieve the SS AISI 420 melting point and is a reachable value during thermite reaction with specifics stoichiometry quantities.

By means of Table 2, end surfaces from $\theta = 0$ to $\theta = \pi$ equals to 0°C. Also, one could determine by Table 3 that $\omega_1 = 1$, $\omega_2 = 0$ and $\beta_m = m$, since $\bar{T}_i(r,\theta=0,t)=0$ and $\bar{T}_i(r,\theta=\pi,t)=0$, and substitutions must be performed in Eq. (31), resulting

$$\Theta_m(\beta_m\theta) = \omega_1 \sin(\beta_m\theta) \tag{50}$$

The initial condition is assumed to be $f_i(r,\theta) = 60°C$, resulting in Eq. (14)

$$\bar{T}_i(r,\theta,t=0) = 60 - T_{ss,i}(r,\theta) \tag{51}$$

Being the physical domain a multi-layer composite solid cylinder, values of coefficients given in Eqs. (2) and (3) are $A_{in} = -k_1$, $B_{in} = 0$, $C_{in} = 0$, $A_{out} = k_6$, $B_{out} = h_6$ and $C_{out} = T_{out} = 60°C$.

Firstly, considering the materials properties available in Table 2 and applying the boundary conditions discussed in Table 3, the eigenvalue problem for the six-layered composite solid cylinder is solved by setting the determinant of the coefficient matrix in Eq. (28) equals to zero, which considers only geometrical, thermophysical and inner and outer boundary prescribed for the physical domain. The infinite series given in Eq. (36) summed over $m \to \infty$ and $p \to \infty$ are really performed over $m \to M$ and $p \to P$, where M and P are the number of transverse eigenvalues used to achieve solution. Maximizing the number of transverse eigenvalues minimizes the truncation error over the achieved results. For this analysis, $M = 20$ and $P = 38$ transverse eigenvalues were used, ensuring convergence along with numerical results.

For the inhomogeneous steady-state problem, results of the ODE given in Eq. (47) are cast as

$$T_{1m}(r) = A_{ss,1}r^m + B_{ss,1}r^{-m} - \frac{2}{\pi}\left(\frac{1-\cos(m\pi)}{m(4-m^2)}\right)\frac{Sr^2}{k_1}, i = 1 \tag{52}$$

$$T_{im}(r) = A_{ss,i}r^m + B_{ss,i}r^{-m}, i \neq 1 \tag{53}$$

where

$$c_{s=}\frac{2}{\pi}\left(\frac{1-\cos(m\pi)}{m(4-m^2)}\right)\frac{S}{k_1} \tag{54}$$

applying the interfacial boundary conditions for Eqs. (52) and (53) leads to a matrixial system where coefficients $A_{ss,i}$ and $B_{ss,i}$ are determined and the steady-state problem given in Eq. (46) is determined.

Substitution of the steady-state solution in Eq. (35) determines the coefficient D_{mp}, which in turn closes the solution of the transient problem as given in Eq. (36). By the solution superposition of both steady-state and transient solution in Eq. (10) determines the final temperature field $T_i(r,\theta,t)$, which are cast in Figures 2.

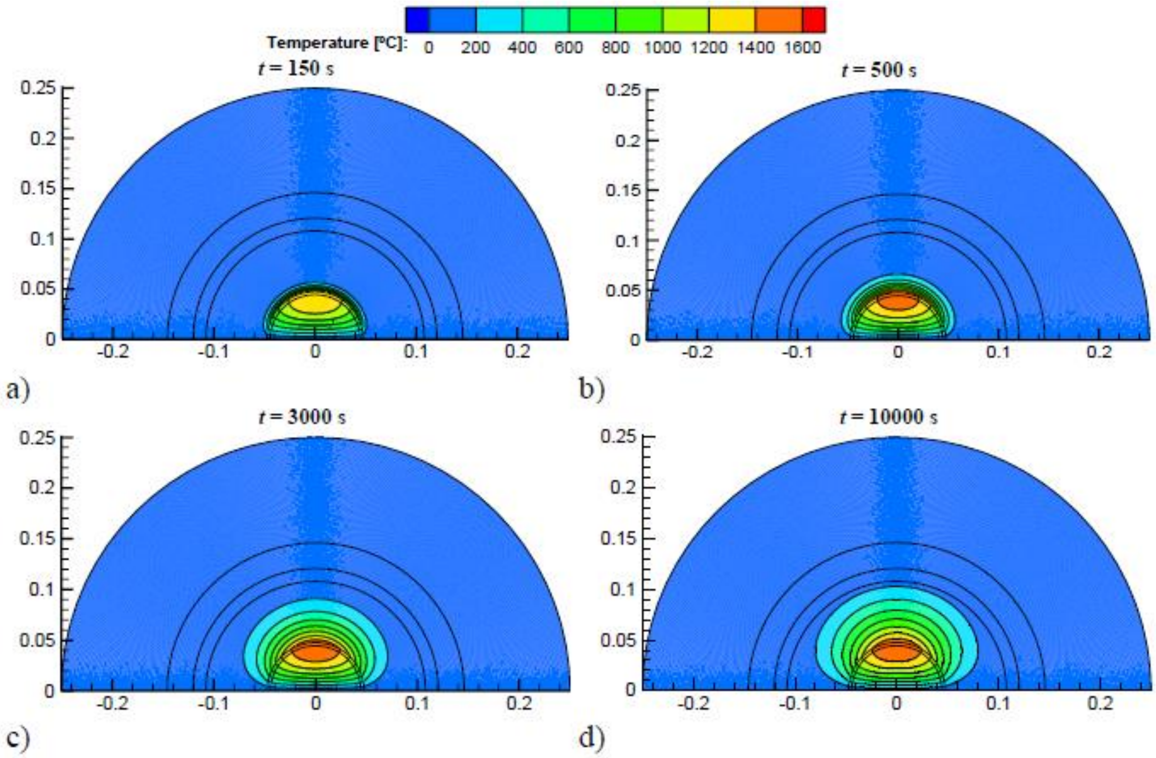

Figure 2 – Analytical temperature contours [ºC] as a function of domain displacement [m] and time (*t*) for: a) $t = 150\ s$, b) $t = 500\ s$, c) $t = 3000\ s$, d) $t = 10000\ s$.

4. ACKNOWLEDGEMENTS

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The authors are thankful to CNPq and CAPES, Brazil, for their financial support during the course of this research.

5. REFERENCES

[1] G. Thews, "Über die mathematische Behandlung physiologischer Diffusionsprozesse in zylinderförmigen Objekten," *Acta Biotheor*, vol. 10, no. 3-4, pp. 105-138, 1953.

[2] G. Zhang, C. Xia, M. Sun, Y. Zou and S. Xiao, "A new model and analytical solution for the heat conduction of tunnel lining ground heat exchangers," *Cold Regions Science and Technology,* vol. 88, pp. 55-66, 2013.

[3] L. A. Dobrzański, M. Drak and B. Ziębowicz, "New possibilities of composite materials application - Materials of specific magnetic properties," *Journal of Materials Processing Technology,* vol. 191, no. 1, pp. 352 - 355, 2007.

[4] Y. Gu and D. L. O'Neal, "An Analytical Solution to Transient Heat Conduction in a Composite Region With a Cylindrical Heat Source," *Journal of Solar Energy Engineering,* vol. 117, no. 3, pp. 242 - 248, 1995.

[5] M. Torabi and K. Zhang, "Analytical solution for transient temperature and thermal stresses within convective multilayer disks with time-dependent internal heat generation, Part I: Methodology," *Journal of Thermal Stresses,* vol. 39, no. 4, pp. 398 - 413, 2016.

[6] H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids: 2nd Edition, London: Oxford University Press, 1959.

[7] X. Lu, P. Tervola and M. Viljanen, "An efficient analytical solution to transient heat conduction in a one-dimensional hollow composite cylinder," *Journal of Physics A: Mathematical and General,* vol. 38, no. 47, pp. 10145 - 10155, 2005.

[8] R. Siegel, "Transient Thermal Analysis of Parallel Translucent Layers by Using Green's Functions," *Journal of Thermophysics and Heat Transfer,* vol. 13, no. 1, pp. 10 - 17, 1999.

[9] M. K. Özisik, Heat Conduction: 2nd Edition, New York: John Wiley & Sons, 1993.

[10] C. W. Tittle, "Boundary value problems in composite media: Quasi-orthogonal functions," *Journal of Applied Physics,* vol. 36, no. 4, pp. 1486 - 1488, 1965.

[11] M. D. Mikhailov, M. N. Özisik and N. L. Vulchanov, "Diffusion in composite layers with automatic solution of the eigenvale problem," *International Journal of Heat and Mass Transfer,* vol. 26, no. 8, pp. 1131-1141, 1983.

[12] S. Singh, K. J. Prashant and Rizwan-Uddin, "Finite integral transform method to solve asymmetric heat conduction

in a multilayer annulus with time-dependent boundary conditions," *Nuclear Engineering and Design,* vol. 241, no. 1, pp. 144 - 154, 2011.

[13] A. G. Butkovskiy, "Some New Results in Distributed Parameter System Control - A Review," *International Federation of Automatic Control, 3rd Symposium,* pp. 29-45, 1982.

[14] A. G. Butkovskiy and L. M. Pustylnikov, Characteristic of Distributed-Parameter Systems, Moscow: Springer, 1993.

[15] B. Yang, "Transfer functions of constrained/combined one-dimensional continuous dynamic systems," *Journal of Sound and Vibration,* vol. 156, no. 3, pp. 425 - 443, 1992.

[16] B. Yang and C. A. Tan, "Transfer Functions of One-Dimensional Distributed Parameter Systems," *Journal of Applied Mechanics,* vol. 59, no. 4, pp. 1009 - 1014, 1992.

[17] B. Yang, "Distributed Transfer Function Analysis of Complex Distributed Parameter Systems," *Journal of Applied Mechanics,* vol. 61, no. 1, pp. 84 - 92, 1994.

[18] B. Yang and H. Fang, "A Transfer-Function Formulation for Nonuniformly Distributed Parameter Systems," *Journal of Vibration and Acoustics,* vol. 116, no. 4, pp. 426 - 432, 1994.

[19] B. Yang, "A distributed transfer function method for heat conduction problems in multilayer composites," *Numerical Heat Transfer, Part B: Fundamentals,* vol. 54, no. 4, pp. 314 - 337, 2008.

[20] B. Yang and S. Liu, "Closed-form analytical solutions of transient heat conduction in hollow composite cylinders with any number of layers," *International Journal of Heat and Mass Transfer,* vol. 108, no. Part A, pp. 907 - 917, 2017.

[21] M. Torabi and K. Zhang, "Analytical solution for transient temperature and thermal stresses within convective multilayer disks with time-dependent internal heat generation, Part II: Applications," *Journal of Thermal Stresses,* vol. 39, no. 4, pp. 414 - 436, 2016.

[22] P. K. Jain, S. Singh and Rizwan-Uddin, "Analytical Solution to Transient Asymmetric Heat Conduction in a Multilayer Annulus," *Journal of Heat Transfer,* vol. 131, no. 1, pp. 011304-1 - 011304-7, 2009.

[23] F. de Monte, "Unsteady Heat Conduction in Two-dimensional Two Slab-shaped Regions. Exact Closed-form Solution and Results," *International Journal of Heat and Mass Transfer,* vol. 46, pp. 1455 - 1469, 2003.

[24] L. W. Abshire, P. Desai, D. Mueller, W. B. Paulsen, R. D. B. Robertson and T. Solheim, "Offshore permanent well abandonment," *Oilfield Review,* vol. 24, no. 1, pp. 42 - 50, 2012.

[25] K. Campbell and R. Smith, "Permanent Well Abandonment. The Way Ahead," *Society of Petroleum Engineers: SPE-013-025-TWA,* pp. 25 - 27, 2013.

[26] R. J. Davies, S. Almond, R. S. Ward, R. B. Jackson, C. Adam, F. Worrall, L. G. Herringshaw, J. G. Gluyas and M. A. Whitehead, "Oil and gas wells and their integrity: Implications for shale and unconventional resource exploitation," *Marine Petroleum Geology,* vol. 56, pp. 239 - 254, 2014.

[27] M. J. Kaiser and R. Dodson, "Cost of Plug and Abandonment Operations in the Gulf of Mexico," *Marine Technology Society Journal,* vol. 41, no. 2, pp. 12 - 22, 2007.

[28] G. E. King and R. L. Valencia, "Environmental risk and well integrity of plugged and abandoned wells," in *SPE Annual Technical Conference and Exhibition*, Amsterdam, The Netherlands, 2014.

[29] P. Aguilar, C. R. Johnson, J. Salazar and M. Bogaerts, "Plug and Abandonment Solution for Oilfield Decomissioning in the North Sea," in *SPE Annual Technical Conference and Exhibition*, Bergen, Norway, 2016.

[30] T. E. Ferg, H. Lund, D. Mueller, M. Myhre, A. Larsen, P. Andersen, G. Lende, C. Hudson, C. Prestegaard and D. Field, "Novel approach to more effective plug and abandonment cementing techniques," in *SPE Arctic Extreme Environments Conference & Exhibition*, Moscow, Russia, 2011.

[31] R. Jordan and P. Head, "Cost effective well abandonment," in *Offshore Europe Meeting*, Aberdeen, United Kingdom, 1995.

[32] J. M. Davison, M. Salehabadi, S. De Gennaro, D. Wilkinson, H. Hogg, C. Hunter and P. Schutjens, "Pluggin and abandonment of oil and gas wells: a geomechanics perspective," in *51st US Rock Mechanics/Geomechanics Symposium*, San Francisco, California, 2017.

[33] I. Barclay, J. Pellenbarg, F. Tettero, J. Pfeiffer, H. Slater, T. Staal, D. Stiles, G. Tiling and C. Whitney, "The beginning of the end: a review of abandonment and decomissioning practices," *Oilfield Review,* pp. 28 - 41, 2001.

[34] S. Ouyang and E. Allen, "Offshore well abandonment: challenges and approach with DNV GL guideline of risk-based abandonment," *Journal of Offshore Engineering Technology,* vol. 1, no. 1, pp. 71 - 83, 2017.

[35] F. Moeinikia, K. K. Fjelde, J. a. S. Sørbø, A. Saasen and T. Vrålstad, "A Study of Possible Solutions for Cost Efficient Subsea Well Abandonment," in *International Conference on Offshore Mechanics and Arctic Engineering*, St. John's, Canada, 2015.

[36] T. Vrålstad, A. Saasen, E. Fjær, T. Øia, J. D. Ytrehus and M. Khalifeh, "Plug & abandonment of offshore wells:

Ensuring long-term well integrity and cost-efficiency," *Journal of Petroleum Science and Engineering,* vol. 173, pp. 478-491, 2019.

[37] G. S. de Andrade, M. J. S. de Lemos and D. Colombo, "A new hybrid analytical/numerical method for transient heat conduction on composite hollow cylinders applied to plug and abandonment of oil wells," *International Journal of Thermal Sciences,* vol. 168, 2021.

[38] G. F. Abdelal, A. Robotham and P. Carragher, "Numerical simulation of a patent technology for sealing of deep-sea oil wells using non linear finite element method," *Journal of Petroleum Science and Engineering,* vol. 133, pp. 192-200, 2015.

[39] H. Salt, "Transient Heat Conduction in a Two-dimensional Composite Slab - I.," *International Journal of Heat and Mass Transfer,* vol. 26, no. 11, pp. 1611 - 1616, 1983.

[40] F. de Monte, "An Analytical Approach to the Unsteady Heat Conduction Processes in One-dimensional Composite Media," *International Journal of Heat and Mass Transfer,* vol. 45, pp. 1333 - 1343, 2002.

[41] V. S. Arpaci, Conduction Heat Transfer, Massachusetts: Addison-Wesley, 1966.

[42] M. K. Özisik, Boundary Value Problems of Heat Conduction, Scranton, Pennsylvania: International Textbook Co., 1968.

[43] S. Singh and P. K. Jain, "Analytical Solution to Transient Heat Conduction in Polar Coordinates with Multiple Layers in Radial Direction," *International Journal of Thermal Sciences,* vol. 47, pp. 261 - 273, 2008.

[44] T. L. Bergman, A. S. Lavine, F. P. Incropera and D. P. Dewitt, Fundamentals of Heat and Mass Transfer (7th Edition), Hoboken, NJ: John Wiley & Sons, 2011.

[45] H. K. Versteeg and W. Malalasekera, An Introduction to Computational Fluid Dynamics (2nd Edition), Harlow, England: Pearson Education Limited, 2007.

[46] L. Durães, P. Brito, J. Campos and A. Portugal, "Modelling and simulation of Fe2O3/Aluminum thermite combustion: Experimental validation," *Computer Aided Chemical Engineering,* vol. 21, pp. 365-370, 2006.

[47] Z. Zhang, P. Farahmand and R. Kovacevic, "Laser cladding of 420 stainless steel with molybdenum on mild steel A36 by a high power direct diode laser," *Materials & Design,* vol. 109, pp. 686 - 699, 2016.

[48] Y. Chunyan, L. Cuiying and Y. Bo, "3D modeling of the hydrogen distribution in X80 pipeline steel welded joints," *Computational Materials Science,* vol. 83, pp. 158-163, 2014.

[49] Y. Xu and D. D. L. Chung, "Effect of sand addition on the specific heat and thermal conductivity of cement," *Cement and Concrete Research,* vol. 30, pp. 59-61, 2000.

[50] L. Eppelbaum, I. Kutasov and A. Pilchin, "Chapter 2 - Thermal Properties of Rocks and Density of Fluids," in *Applied Geothermics,* New York, Springer, 2014, pp. 99-149.

[51] E. J. F. Rocha, T. d. S. Antonino, P. B. Guimarães, R. A. S. Ferreira, J. M. A. Barbosa and J. Rohatgi, "Modeling of the temperature field generated by the deposition of weld bead on a steel butt joint by FEM techniques and thermographic images," *Materials Research,* vol. 21, no. 3, 2018.

[52] J. A. Da Nóbrega, D. D. S. Diniz, A. A. Silva, T. M. Maciel, V. H. C. De Albuquerque and J. M. R. S. Tavares, "Numerical Evaluation of Temperature Field and Residual Stresses in an API 5L X80 Steel Welded Joint Using the Finite Element Method," *Metals,* vol. 6, no. 2, 2016.

[53] M. A. Bouhifd, P. Besson, P. Courtial, C. Gérardin, A. Navrotsky and P. Richet, "Thermochemistry and melting properties of basalt," *Contributions to Mineralogy and Petrology* , vol. 153, pp. 689-698, 2006.

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