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ANALYTICAL SOLUTION OF ZIELKE'S UNSTEADY FRICTION MODEL FOR ONE-DIMENSIONAL PIPE FLOW IN LIGHT OF THE SECOND LAW OF THERMODYNAMICS

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Abstract. *Although different authors have been developing distinct unsteady friction models in one-dimensional pipe flow, there is no consensus about which model class better describes the water-hammer phenomenon. This disagreement happens due to the lack of analysis criteria. Over the decades, researchers usually compare the unsteady hydraulic head numerical results of distinct classes of models to the measured data of a particular experiment, pointing out the error of wave peak amplitude, or the wave shape resemblance. So, it is relevant to show the application of new criteria in order to reveal inconsistencies inside the models or even rule out physically wrong ones. The classical Second Law of Thermodynamics (SLT) must be used as a criterion since it does not admit negative rates of energy dissipation generated by the friction force for any particular case. Therefore, an analysis using the SLT needs to be carried out to identify issues and maybe suggest corrections. A classical SLT inequality to properly evaluate those models was derived. Then, two unsteady friction models were applied to it, reaching an analytical solution for both. The famous Zielke's original model failed according to the classical SLT, and the Nucci and Spena's model satisfied it.*

Keywords: *Zielke's model, Nucci and Spena's model, unsteady friction model, transient flow in pipes*

1. INTRODUCTION

The transient state analysis is crucial to safely design pipelines conveying liquids, once the internal pressures reach low and high values that cannot be predicted by stationary state analysis. This dangerous phenomenon is known as water-hammer and happens in daily typical operations, such as fast valve closure, or in unexpected circumstances, such as sudden pump failure. It can damage or even wreck the pipelines, and also contaminate potable water reservoirs (Andrade and Rachid, 2022; Wylie and Streeter, 1993; Ghidaoui et al., 2005). To avoid those issues, project engineers must predict the water-hammer effects caused by forced linear momentum variations of the fluid. Therefore, the application of accurate unsteady friction models is essential to properly calculate the pressure fluctuations.

Nowadays, in hydraulic engineering, one-dimensional (1D) friction models are used in numerical simulations to estimate the pressure and discharge oscillations inside the pipeline after external factors introduce changes in the fluid momentum (Wylie and Streeter, 1993; Andrade, 2019). The water-hammer is commonly described as a 1D phenomenon, so those models are interesting for solving engineering problems because they have a low computational time cost and great precision (Zielke, 1968; Pezzinga, 2009). In the 20th century, the quasi-steady friction model was executed in the initial simulations due to the inexistence of unsteady ones. Unfortunately, this model results did not describe precisely the phenomenon, obtaining overestimated values of pressure (Wylie and Streeter, 1993; Duan et al., 2017). This imprecision leads to an oversizing of the pipe wall thickness and, consequently, to an increase in the final product costs. Seeking to avoid such a problem, experts have been creating new unsteady friction models to reach better results.

The unsteady friction models can incorporate the effects of the time rate, which are neglected by the quasi-steady one, hence estimating more accurate results (Zielke, 1968; Vardy and Brown, 2003; Zarzycki et al., 2007; Urbanowicz and Zarzycki, 2012). The authors usually develop the models using the conservation principles of mass and linear momentum, without considering another important postulate: the classical Second Law of the Thermodynamics (SLT). According to it, negative rates of energy dissipation are inadmissible, so an analysis using the SLT is relevant to discard inconsistent models (Gonzaga Filho, 2017; Andrade, 2019; Rachid, 2021). For this reason, the objective herein is to evaluate this condition for two unsteady friction models: the well-known model originally proposed by Zielke for laminar flows, and the questionable model proposed by Nucci and Spena (Zielke, 1968; Nucci and Spena, 2013).

2. GOVERNING EQUATIONS

Taking in consideration 1D and isothermal flows with low Mach numbers of Newtonian slightly compressible liquids in conveying pipes of circular cross-section, the governing equations can be expressed, in Eulerian coordinates, as:

$$\frac{\partial H}{\partial t} + \frac{Y}{g}\frac{\partial v}{\partial x} = 0 \,, \tag{1}$$

$$\frac{\partial v}{\partial t} + g\frac{\partial H}{\partial x} + \frac{4\tau}{\rho D} = 0 \,, \tag{2}$$

$$d = \frac{4\tau v}{D} \geq 0 \,, \tag{3}$$

for $(x,t) \in [0,L] \times [0,+\infty[$, being L the pipe length. The parameter Y changes according to each model:

$$Y = \begin{cases} a^2 \,, & \text{for Zielke} \\ \dfrac{\varepsilon}{\rho} \,, & \text{for Nucci} - \text{Spena} \end{cases} . \tag{4}$$

In Eqs. (1, 2, 3), H, v and τ are the dependent variables and stand for the piezometric head, the mean flow velocity and the overall wall shear stress, respectively. They are functions of the spatial position x, measured along the pipe centerline, and the time instant t. The local gravitational acceleration is represented by g and the equivalent sand roughness is ε. The undisturbed mass density of the fluid, ρ, and the unperturbed internal diameter of the pipe, D, are constants as well as the wave front speed, a, with which disturbances propagate in the medium.

The principles of conservation of mass and linear momentum in the axial direction are presented respectively in Eqs. (1) and (2). Equation (3) stands for an alternative local version of the classical SLT known as Clausius-Duhem Inequality, and it expresses locally the overall rate of energy dissipation per unit volume, d, which in this case is solely restricted to the power dissipated by the overall wall shear stress. It is used to differentiate between impossible ($d < 0$) and possible thermodynamic processes ($d \geq 0$), in which the inequality represents the irreversible processes.

3. UNSTEADY FRICTION MODELS

Aiming more accurate models to correctly describe the water-hammer effects (pressure and flow fluctuations) in fluid transients inside conveying pipes, researchers have been proposing several distinct unsteady friction models over the years. Among them, we may cite two from different categories. A famous one: the model originally proposed by Zielke for laminar flow (1968); and a less-known one: the model proposed by Nucci and Spena (2013).

Trying to model unsteady friction in one-dimensional flows, the following additive decomposition of the overall wall shear stress, τ, is applied for the Zielke's model:

$$\tau = \tau_s + \tau_u \,, \tag{5}$$

in which τ_u represents the unsteady parcel of the wall shear stress and τ_s the steady one. The steady parcel of the wall shear stress is usually calculated by:

$$\tau_s = \frac{f\rho v|v|}{8} \,, \tag{6}$$

where $f = f(\mathrm{Re}, \epsilon)$ is the well-known Darcy-Weisbach friction factor, in which Re stands for the Reynolds number ($\mathrm{Re} = |v|D/\nu$, where ν the fluid kinematic viscosity) and ϵ the relative roughness of the pipe ($\epsilon = \varepsilon/D$).

Despite Eq. (6) holds essentially for steady state flows (both for laminar and turbulent regimes), it has been largely used for unsteady flows in the past, due to the absence of more accurate models at that time. Until the appearance of the unsteady friction models, the simulation of transient fluid flows had been executed by assuming Eq. (5) with $\tau_u = 0$. This is known as the quasi-steady model, in which f is constantly calculated for each time instant.

By adhering to the additive decomposition expressed by Eq. (5), the Zielke's original model uses the following expression to evaluate the unsteady wall shear stress for laminar flow:

$$\tau_u = \frac{4\mu}{D}\int_0^t \frac{\partial v}{\partial t}(x,t')W(t-t')\,dt' \,, \tag{7}$$

where μ is the dynamic viscosity, $W(T)$ is a positive weighting function, and t' is the dummy time used in the convolution integral. The weighting function is derived by invoking the two-dimensional transient flow in the pipe when the fluid is suddenly decelerated, is a function of the dimensionless time $T = t/\sigma$ ($\sigma = D^2/4\nu$), and is given by for $T < 0.02$:

$$W(T) = 0.282095T^{-\frac{1}{2}} - 1.250000 + 1.057855T^{\frac{1}{2}} + 0.937500T + 0.396696T^{\frac{3}{2}} - 0.351563T^2 \,, \tag{8}$$

and for $T > 0.02$:

$$W(T) = e^{-26.3744T} + e^{-70.8493T} + e^{-135.0198T} + e^{-218.9216T} + e^{-322.5544T} \,. \tag{9}$$

Due to the complex form of the convolution integral, Zielke employs a first-order approximation to estimate numerical results. Other authors propose different approximations for the integral to reduce the computational time cost (Trikha, 1975; Schohl, 1993; Vardy & Brown, 2003; Urbanowicz & Zarzycki, 2012).

The Nucci and Spena's model uses the fictitious-fluid hypothesis that considerate the fluid and the pipe as a unique continuous system. The properties of the fictitious fluid have different values from the flowing fluid and the pipe material. This model calculates the overall wall shear stress, in a low Mach unsteady flow for a Newtonian fluid, by applying the following expression:

$$\tau = c_1 v + c_2 v|v| \,, \tag{10}$$

where c_1 and c_2 are coefficients given by:

$$c_1 = \begin{cases} \dfrac{8\mu}{D} \,, & \text{for laminar flow} \\ \dfrac{f_\infty \mu}{\varepsilon} \,, & \text{for turbulent flow} \end{cases} \,, \tag{11}$$

$$c_2 = \begin{cases} 0 \,, & \text{for laminar flow} \\ \dfrac{f_\infty \rho}{8} \,, & \text{for turbulent flow} \end{cases} \,, \tag{12}$$

in which f_∞ is the friction factor for completely developed turbulent flow:

$$f_\infty = \frac{1}{4}\left[\log_{10}\left(\frac{3.71}{\epsilon}\right)\right]^{-2} \,. \tag{13}$$

4. ANALYTICAL CALCULATION

Knowing that ρ, D, f, μ and $\nu \in \mathbb{R}^+$ (all positive scalars due to their nature) and $v \in \mathbb{R}$, it is easy to observe for τ_s and consequently the quasi-steady model, by taking Eq. (6) in Eq. (3), that the classical SLT is unconditionally satisfied throughout the entire domain $(x, t) \in [0, L] \times [0, +\infty[$. Equation (14) shows locally the steady parcel of the rate of energy dissipation per unit volume:

$$d_s = \frac{f \rho v^2 |v|}{2D} \geq 0 \,. \tag{14}$$

For both unsteady models, it is possible to find an analytical solution according to the classical SLT. First for the Zielke's model, let us assume a constant value for the mean instantaneous flow acceleration ($\partial v(x, t')/\partial t = \alpha = \text{cst}$), so the convolution integral becomes easier to solve:

$$\tau_u = \frac{4\mu\alpha}{D} \int_0^t W(t - t')\, dt' \,, \tag{15}$$

where $\alpha \in \mathbb{R}$. Applying Eqs. (8) and (9) in Eq. (15) and then integrating, results in:

$$\tau_u = \begin{cases} \dfrac{4\mu\alpha}{D} \sigma w_1(T) \,, & \text{for } T < 0.02 \\ \dfrac{4\mu\alpha}{D} \sigma\big(w_1(0.02) + w_2(T)\big) \,, & \text{for } T > 0.02 \end{cases} \,, \tag{16}$$

in which $w_1(T)$ and $w_2(T)$ are positive functions for their respective domains of dimensionless time, as shown in Fig.1, given by:

$$w_1(T) = 0.564190T^{\frac{1}{2}} - 1.250000T + 0.705236T^{\frac{3}{2}} + 0.468750T^2 + 0.158678T^{\frac{5}{2}} - 0.117187T^3 \,, \tag{17}$$

$$w_2(T) = 0{,}026355 - \frac{e^{-26.3744T}}{26.3744} - \frac{e^{-70.8493T}}{70.8493} - \frac{e^{-135.0198T}}{135.0198} - \frac{e^{-218.9216T}}{218.9216} - \frac{e^{-322.5544T}}{322.5544} \,. \tag{18}$$

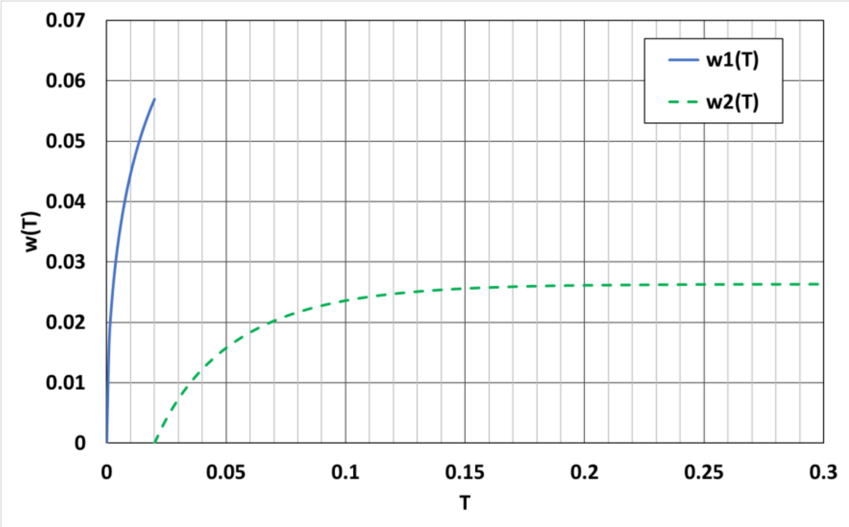

Figure 1. Graph of the functions $w_1(T)$ and $w_2(T)$ to their respective domains.

Combining Eqs. (3), (6) and (16), we find the overall rate of energy dissipation, d, for the Zielke's model:

$$d = \begin{cases} \dfrac{4v}{D}\left(\dfrac{f\rho v|v|}{8} + \dfrac{4\mu\alpha}{D}\sigma w_1(T)\right) \,, & \text{for } T < 0.02 \\ \dfrac{4v}{D}\left(\dfrac{f\rho v|v|}{8} + \dfrac{4\mu\alpha}{D}\sigma\big(w_1(0.02) + w_2(T)\big)\right) \,, & \text{for } T > 0.02 \end{cases} . \tag{19}$$

Knowing that the Zielke's original model is only for laminar flow, it is possible to use $f = 64/\mathrm{Re}$. Since $\alpha = \mathrm{cst}$, so the instantaneous acceleration is equal to its mean value. The mean flow velocity v decelerates and stops after valve closure, and it accelerates from zero to v after valve opening, meaning that $\alpha = \pm v/t$. So, Eq. (19) becomes:

$$d = \begin{cases} 16\mu\left(\dfrac{v}{D}\right)^2\left(2 \pm \dfrac{w_1(T)}{t/\sigma}\right) \,, & \text{for } T < 0.02 \\ 16\mu\left(\dfrac{v}{D}\right)^2\left(2 \pm \dfrac{w_1(0.02) + w_2(T)}{t/\sigma}\right) \,, & \text{for } T > 0.02 \end{cases} , \tag{20}$$

and changing t by the dimensionless time, $T = t/\sigma$, it finally leads to:

$$d = \begin{cases} 16\mu\left(\dfrac{v}{D}\right)^2\left(\dfrac{\Gamma_1(T)}{T}\right) \,, & \text{for } T < 0.02 \\ 16\mu\left(\dfrac{v}{D}\right)^2\left(\dfrac{\Gamma_2(T)}{T}\right) \,, & \text{for } T > 0.02 \end{cases} , \tag{21}$$

where the functions $\Gamma_1(T)$ and $\Gamma_2(T)$ are expressed by:

$$\Gamma_1(T) = 2T \pm w_1(T) \,, \tag{22}$$

$$\Gamma_2(T) = 2T \pm \big(w_1(0.02) + w_2(T)\big) \,. \tag{23}$$

It is possible to notice that $\Gamma_1(T)$ and $\Gamma_2(T)$ must be positive for all domain to satisfy the SLT because the rest is positive. As shown in Fig. 2, Zielke's model fails according the classical SLT when the valve is closed or the flow velocity decelerates. This inconsistence happens because the mean flow velocity does not assimilate the change of the velocity profile during the transient state, as so the mean instantaneous flow acceleration. In other words, the effects of 2D or 3D gradient of velocity profile are not correctly assimilated by the Zielke's model (Rachid, 2021). This same case analysis works for the first-order approximation suggested by Zielke, once the weighting function is also positive and the velocity difference determines the signal of the first-order approximation.

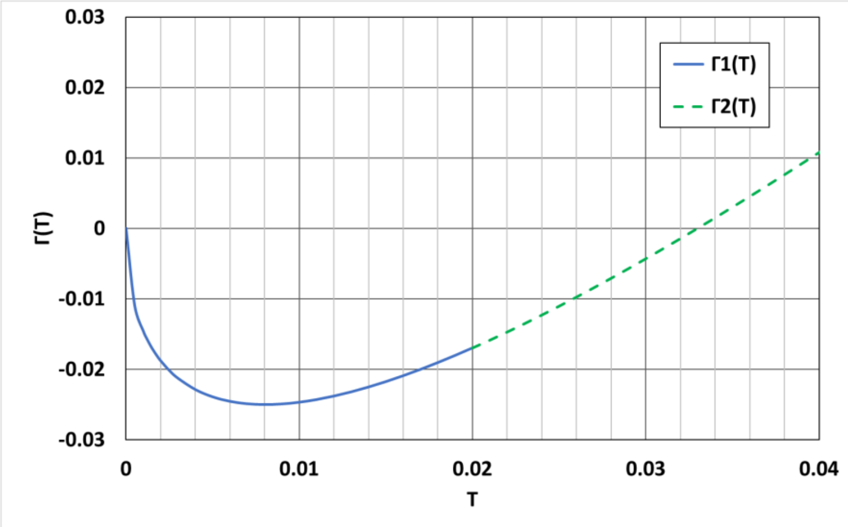

Figure 2. Negative results calculated by $\Gamma_1(T)$ and $\Gamma_2(T)$.

Now for the Nucci and Spena's model, Eqs. (10) and (3) must be combined to give rise to the overall rate of energy dissipation:

$$d = \frac{4}{D} v^2 (c_1 + c_2 |v|) \,. \tag{24}$$

Using Eqs. (11) and (12), it is possible to discern the proper expression for each type of flow:

$$d = \begin{cases} 32\mu \left(\frac{v}{D}\right)^2 \quad , & \text{for laminar flow} \\ \frac{f_\infty v^2}{D}\left(\frac{4\mu}{\varepsilon} + \frac{\rho}{2}|v|\right) , & \text{for turbulent flow} \end{cases} , \tag{25}$$

in which the friction factor $f_\infty \in \mathbb{R}^+$ since the logarithm, in Eq. (13), only gives positive values. It happens because $\epsilon < 3.71$ and $\epsilon \in \mathbb{R}^+$, consequently $3.71/\epsilon > 1$. Remembering that ρ, D, μ and $\varepsilon \in \mathbb{R}^+$ and $v \in \mathbb{R}$, it is possible to conclude that $d \geq 0$ for laminar and turbulent flow throughout the entire domain $(x, t) \in [0, L] \times [0, +\infty[$. This means that the Nucci and Spena's model does satisfy the classical SLT.

5. CONCLUSIONS

In order to develop coherent constitutive model for the wall shear stress in an unsteady flow, it is fundamental to invoke the classical Second Law of Thermodynamics (SLT). According to this law, a constitutive model is physically correct as long as no particular problem produces negative rates of energy dissipation during the whole thermomechanical process, regardless of the initial and boundary conditions. In this paper, the analytical investigation of two unsteady state models was done. Those are the model proposed originally by Zielke for laminar flow, and the one proposed by Nucci and Spena. It was analytically proved that the quasi-steady model and the Nucci and Spena's model do respect the classical SLT for any and all cases, although the fact that Nucci and Spena use a questionable hypothesis to theoretically base their model. Unfortunately, it was analytically shown, for a simple case which assumes constant mean instantaneous flow acceleration, that the well-known Zielke's model violates this important law. This happens because the mean instantaneous acceleration and the mean velocity of the flow fails to properly assimilate the changes of the velocity profile during the unsteady state, thus resulting in unacceptable negative rates of energy dissipation.

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