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Linear Finite Element Analysis of Interlaminar Stresses in Laminated Composite Timoshenko Beams

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Composite laminates are applied in various engineering fields, e.g. in aerospace structures in which, numerical simulations are used for testing and optimization. A linear finite element analysis is considered here for the static modeling of laminated Timoshenko beams. The kinematic assumptions of Timoshenko's theory allow an excellent modeling of thick beams. However, the theory assumes that the beam's transversal stresses (vital for assessing the interlaminar stresses on composite beams) are constant in each laminate layer. This assumption is at odds with the expected parabolic form of the stresses. But integration of the equilibrium equations in strong form allows the recovery of the correct stress behavior, provided the finite element has a sufficient degree of continuity. We shall examine a cubic polynomial approximation for finite elements to recover the transversal stress state, aiming at studying the interlaminar failure of laminated composite beams. We have obtained excellent results for the interlaminar stresses, by post-processing the finite element results, via direct integration of the equilibrium equations.
Keywords: Finite Element Method, Timoshenko beam, laminated composite

INTRODUCTION

The beam element is a structural element widely used in solid mechanic analysis, which length is a lot larger than the dimensions of its cross-sectional area. This element can also resist both transversal and normal loads and experiences bending and stretching displacements. Two of the main theories in beam modeling are the Euler-Bernoulli Theory (EBT) and the Timoshenko Beam Theory (TBT). Both theories consider that a straight line in the cross sectional area remains straight and inextensible after deformation (Reddy,2015) , however the EBT assume that the line also remain normal to the mid-plane, i.e. the rotation of the section only depends of the slope of the transversal displacement ($\phi_x = dw/dx$) , neglecting shear strain ($\gamma_{xz} = 0$), while the TBT does not adopt this last assumption (Abrate, 2017), making it a more suitable approach for thick beams, where the shear effect is relevant. In terms of the stress field, the TBT results for normal stresses (σ_{xx}) agree with the linear elasticity theory giving a linear distribution across the beam section. However, the results for the transversal stresses(σ_{xz}) are constant according to the TBT, differing from the expected results of the linear elasticity(Reddy,2004).

Another application especially important for structural mechanics are composite materials, which consists of two or more materials combined in a matrix structure, that achieve better results and properties than conventional materials. Composite materials can be arranged in specific orientations to increase its resistance, which are denominated fiber-reinforced composites. Different layers of fiber-reinforced composites can be stacked to create a laminated composited. According to Reddy (2004)"Fiber orientation in each lamina and stacking sequence of the layers can be chosen to achieve desired strength and stiffness for a specific application", such materials are heavily used in aerospace the industry, for example, according to Pora (2001) around 40% of the structure of the Airbus A380 is made of carbon composites due to its excellent strength to weight ratio. Laminated beam problems are usually three dimensional, which demand plenty of time and computational effort. This analysis can be simplified by adopting the Equivalent Single Layer Theory (ESLT), in which all layers of the beam are replaced by a single equivalent layer with equivalent properties, in such a way that the problem becomes a one dimensional beam and the TBT can be used to calculate the displacements. After the derivation of the displacements, it is possible to return to the initial laminated problem and evaluate the stress field in each layer. In this paper the plate ESLT procedure presented in Reddy(2004) and Abrate (2017) will be used to calculate the stress field in beams, for that to be possible it will be assumed that the width of the plate is a lot smaller than its length and there is no change in load disposition in the width direction.

FORMULATION

Governig Equations

In this paper, we will consider the calculation of the stress field for different types of symmetric laminated composites. As mentioned before, the displacements will be obtained for a Timoshenko beam through a Finite Element Method Analysis. Considering the linear formulation, the displacement and strain fields are given by Reddy (2015)

$$u_1 = u(x) + z\phi_x(x), \quad u_2 = 0, \quad u_3 = w(x) \tag{1}$$

$$\varepsilon_{xx} = \frac{\partial u_1}{\partial x} = \frac{du}{dx} + z\frac{d\phi_x}{dx} \equiv \varepsilon^0_{xx} + z\varepsilon^1_{xx} \tag{2}$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \phi_x + \frac{dw}{dx} \equiv \gamma^0_{xz} \tag{3}$$

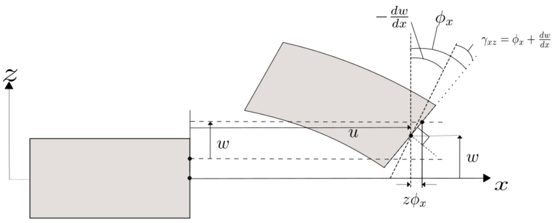

Figure 1 – Displacements of a Timoshenko beam.

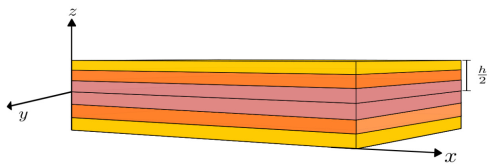

Figure 2 – Representation of a symmetric laminated beam.

where (u_1, u_2, u_3) are the displacements of a material point in the local coordinate system, u is the axial displacement, w the transversal displacement and ϕ_x the rotation. $(\varepsilon_{xx}, \gamma_{xz})$ are the one dimensional strains and the superscript 0 and 1 in Eq.(2) indicate, respectively the membrane and flexural components of the strains.

The representation of the displacements in a Timoshenko beam can be seen in Fig. 1, these results will be used in the ESLT procedure. The three dimensional analysis of the laminated composite require a complete plane stress constitutive relation. Considering a symmetric laminated composite, as shown in Fig.3, with laminae made by the same orthotropic material with different orientations, the constitutive relations, in the global system, for a plane stress-state for the kth orthotropic layer according to Reddy (2004) are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}^{(k)} \qquad \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^{(k)} \tag{4}$$

The coefficients of eqs.(4) are given by

$$\bar{Q}_{11} = Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta cos^2\theta + Q_{22}\sin^4\theta \tag{5}$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta) \tag{6}$$

$$\bar{Q}_{22} = Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta \tag{7}$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta \tag{8}$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta \tag{9}$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta) \tag{10}$$

$$\bar{Q}_{44} = Q_{44}\cos^2\theta + Q_{55}\sin^2\theta \tag{11}$$

$$\bar{Q}_{45} = (Q_{55} - Q_{44})\cos\theta\sin\theta \tag{12}$$

$$\bar{Q}_{55} = Q_{55}\cos^2\theta + Q_{44}\sin^2\theta \tag{13}$$

where the angle θ represents the fiber orientation, Fig.4 shows a schematic example of the cross-section area for the beam from Fig.2, the laminae are symmetric along the y axis and can have different orientations and heights. Fig.4 also shows that θ is the angle between the fiber, which represents the local coordinate system, and the global coordinate system, where all coefficients are calculated. The coefficients $Q_{ij}^{(k)}$ are denominated plane stress-reduced stiffnesses, and are function of the kth layer mechanical properties: the elastic modulus (E_i), the shear modulus (G_i) and Poisson's ratio (ν_{ij})

$$Q_{11}^{(k)} = \frac{E_1^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}, \quad Q_{12}^{(k)} = \frac{\nu_{12}^{(k)}E_2^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}, \quad Q_{22}^{(k)} = \frac{E_2^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}, \quad Q_{66}^{(k)} = G_{12}^{(k)}, \quad Q_{44}^{(k)} = G_{23}^{(k)}, \quad Q_{55}^{(k)} = G_{13}^{(k)} \tag{14}$$

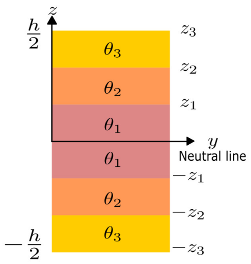

Figure 3 – Cross-section of a symmetric laminated beam with 6 layers.

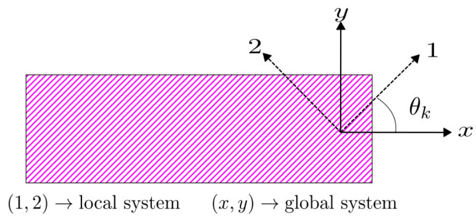

Figure 4 – Fiber orientation of the k^{th} layer of a beam.

The constitutive relations presented above between stresses and strains can be used to rewrite the relations among forces/moments and stresses in terms of the beam's strains. This change is important since the displacements and consequently the strains, can be directly obtained through the ESLT with the one dimensional TBT. The equations are given by Shafei (2020)

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{Bmatrix}^{(k)} dz \tag{15}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(0)} + z\gamma_{xy}^{(1)} \end{Bmatrix}^{(k)} dz \tag{16}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^{(k)} dz = K \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{Bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{Bmatrix}^{(k)} \begin{Bmatrix} \gamma_{xz}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix}^{(k)} dz \tag{17}$$

where K is a correction factor and according to Abrate (2017) ($\varepsilon_{xx}^{(0)}, \varepsilon_{yy}^{(0)}, \gamma_{xy}^{(0)}$) are the *membrane strains*, and ($\varepsilon_{xx}^{(1)}, \varepsilon_{yy}^{(1)}, \gamma_{xy}^{(1)}$) are the *flexural* (bending) *strains*. In terms of displacements, the linear correlations are given by

$$\{\varepsilon^{(0)}\} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \\ \phi_x + \frac{\partial w_0}{\partial x} \\ \phi_y + \frac{\partial w_0}{\partial y} \end{Bmatrix}, \quad \{\varepsilon^{(1)}\} = \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \tag{18}$$

Equations (15),(16) and(17) can also be written in the form

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \tag{19}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} \tag{20}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{Bmatrix} \tag{21}$$

The coefficients A_{ij}, B_{ij}, D_{ij} are respectively the extensional, bending-extensional coupling and bending stiffness, and are defined by Shafei (2020) in the following form

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \bar{Q}_{ij}\,(1, z, z^2)\,dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \bar{Q}_{ij}^{(k)}\,(1, z, z^2)\,dz \tag{22}$$

In a symmetric laminated composite $B_{ij} = 0$, and the the force-strain relations can be organized in a more compact form

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [0] \\ [0] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^{(0)}\} \\ \{\varepsilon^{(1)}\} \end{Bmatrix} \tag{23}$$

The systems of Eqs. (19)-(21) and the strains are written in a general form for an orthotropic plate. This formulation can be applied to a Timoshenko beam, if the magnitude of the beam length (L) is a lot greater than its width (b) and the beam is only submitted to transversal loads (Reddy, 2004). In this case the displacements are considered as function of the the x coordinate only and the moments and forces in the y direction can be neglected

$$w = w(x) \qquad \phi_x = \phi_x(x) \tag{24}$$

Adopting these considerations and isolating the displacements, eqs. (20) and (21) become

$$\begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} = \begin{Bmatrix} D_{11}^* \\ D_{12}^* \\ D_{16}^* \end{Bmatrix} \{M_{xx}\} \tag{25}$$

$$\begin{Bmatrix} \frac{\partial w}{\partial y} + \phi_y \\ \frac{\partial w}{\partial x} + \phi_x \end{Bmatrix} = \frac{1}{K} \begin{Bmatrix} A_{45}^* \\ A_{55}^* \end{Bmatrix} \{Q_x\} \tag{26}$$

where A_{ij}^* and D_{ij}^* are respectively the coefficients of the inverse of the matrices $[A]$ and $[D]$. The first two equations of the system (25) indicates that the transversal displacement w and rotation ϕ_x do depend of the y coordinate through the *Poisson effect* (D_{12}^*) and anisotropic shear coupling (D_{16}^*) and M_{xx} and Q_x are also in function of the rotation ϕ_y

$$\frac{\partial \phi_y}{\partial y} = D_{12}^* M_{xx}, \quad \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} = D_{16}^* \tag{27}$$

This dependence of y contradicts the assumption (24) of the displacements being only function of x, however for long beams, the magnitude of the y derivatives is almost negligible compared to the other derivatives (Reddy, 2004). Thereby, we can disregard these terms and eq.(24) is respected, leading to the final relations

$$\frac{\partial \phi_x}{\partial x} = D_{11}^* \cdot M_{xx} \tag{28}$$

$$\frac{\partial w}{\partial x} + \phi_x = \frac{A_{55}^*}{K} \cdot Q_x \tag{29}$$

Finally, eqs. (28) and (29) can be written in terms of the equivalent properties of a single layer beam

$$E_{xx}^b I_{yy} \frac{\partial \phi_x}{\partial x} = M(x), \quad K G_{xz}^b bh \left(\frac{\partial w}{\partial x} + \phi_x \right) = Q(x) \tag{30}$$

where I_{yy} is the moment of inercia of the cross-sectional area, $M(x)$ and $Q(x)$ are respectively the moment and force per unit length of the beam and E^b_{xx} and G^b_{xz} are the equivalent elastic modulus and shear modulus. With this properties, it is possible to link the three dimensional stress analysis with the one dimensional displacement calculation

$$M(x) = bM_{xx}, \quad Q(x) = bQ_x \quad I_{yy} = \frac{bh^3}{12}, \quad E^b_{xx} = \frac{12}{D^*_{11}h^3}, \quad G^b_{xz} = \frac{1}{A^*_{55}h} \tag{31}$$

The in-plane stresses for each layer can be obtained by solving the following expressions derived from replacing eq.(16) into eq.(6)

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{bmatrix} D^*_{11} & D^*_{12} & D^*_{16} \\ D^*_{12} & D^*_{22} & D^*_{26} \\ D^*_{16} & D^*_{26} & D^*_{66} \end{bmatrix} \begin{Bmatrix} M_{xx} \\ 0 \\ 0 \end{Bmatrix} \tag{32}$$

The interlaminar stress σ_{zz} is null when calculated through the TBT and σ_{xz} is constant through the constitutive equations (Reddy, 2004), but these results do not match with the expected behaviour and are important in the failure analysis in laminated composites, since this materials have low resistance to shear and normal strain. These stresses can be obtained by solving the equations of three dimensional elasticity given by Abrate (2017)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \tag{33}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0 \tag{34}$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{35}$$

Integrating both equations for each layer, considering that the limits for the k^{th} layer are $z_k \leq z \leq z_{k+1}$, and neglecting the derivatives in y direction, gives the final expressions

$$\sigma^{(k)}_{xz} = -\int_{z_k}^{z} \frac{\partial \sigma^{(k)}_{xx}}{\partial x} dz + G^{(k)} \tag{36}$$

$$\sigma^{(k)}_{zz} = -\int_{z_k}^{z} \frac{\partial \sigma^{(k)}_{xz}}{\partial x} dz + H^{(k)} \tag{37}$$

Where $G^{(k)}$ and $H^{(k)}$ are constants.

Finite Element Formulation

In this section, the Finite Element Method presented by Reddy (2015), adapted to the linear case, will be used to determine displacements of the beam. With a view to obtain the finite element model equations to a Timoshenko beam, the principle of virtual work will be used so that the sum of the total virtual strain energy (δW^e_I) and the total virtual work done by externals loads (δW^e_E) is zero in each beam element, with first node ate $x = x_a$ and last node at $x = x_b$

$$\delta W^e \equiv \delta W^e_I + \delta W^e_E = 0 \tag{38}$$

$$\delta W^e_I = \int_{x_a}^{x_b} \int_{A^e} (E\varepsilon_{xx}(\delta\varepsilon^0_{xx} + z\delta\varepsilon^1_{xx}) + KG\gamma_{xz}\delta\gamma_{xz})\, dAdx \tag{39}$$

$$\delta W^e_E = -\left[\int_{x_a}^{x_b} q\delta w dx + \int_{x_a}^{x_b} f\delta u dx + \sum_{i=1}^{6} Q^e_i \delta\Delta^e_i\right] \tag{40}$$

where the coefficients Q^e_i and Δ^e_i are respectively the generalized forces and displacements, q is the distributed transversal load and f is the distributed axial load. E is the elastic modulus, and G is the shear modulus. For an orthotropic laminated beam, these properties are given by eqs. (31), ensuring the coupling between the single layer one dimensional problem and the three dimensional laminated problem. Finally, $\delta\varepsilon^0_{xx}, \delta\varepsilon^1_{xx}, \delta\gamma_{xz}$ are the virtual strains and are given by

$$\delta\varepsilon^0_{xx} = \frac{d\delta u}{dx}, \quad \delta\varepsilon^1_{xx} = \frac{d\delta\phi_x}{dx}, \quad \delta\gamma^0_{xz} = \delta\phi_x + \frac{d\delta w}{dx} \tag{41}$$

Since we are analysing the linear case, the axial displacement u is uncoupled from w and ϕ_x in such a way that in the presence of only transversal loads, the beam won't experience stretching, and the u terms in the weak form can be neglected. Adopting this last assumption, eq.(38) can be written as

$$\int_{x_a}^{x_b} \frac{d\delta w}{dx}\left\{S_{xx}\left(\frac{dw}{dx}+\phi_x\right)\right\}dx - \int_{x_a}^{x_b}\delta w q dx + \left[Q_x + N_{xx}\frac{dw}{dx}\right]_{x=x_a}\delta w(x_a) - \left[Q_x + N_{xx}\frac{dw}{dx}\right]_{x=x_b}\delta w(x_b) = 0 \quad (42)$$

$$\int_{x_a}^{x_b}\left[D_{xx}\frac{d\delta\phi_x}{dx}\frac{d\phi_x}{dx} + S_{xx}\delta\phi_x\left(\frac{dw}{dx}+\phi_x\right)\right]dx + M_{xx}(x_a)\ \delta\phi_x(x_a) - M_{xx}(x_b)\ \delta\phi_x(x_b) = 0 \quad (43)$$

where A_{xx} and B_{xx} are respectively the extensional stiffness and bending stiffness and are defined exactly as in eq.22 for a homogeneous beam ($Q_{ij} = E$) and S_{xx} is the shear stiffness and is defined as

$$S_{xx} = K\int_A G dA = K\cdot G\cdot A \quad (44)$$

The beam displacements (w,ϕ_x) will be approximated as 3rd order polynomials, since this is the minimum degree necessary to fully calculate all stresses without any loss on information(σ_{xz} demands at least a C_3 function). Therefore cubic elements, as shown in Fig.7 will be used, the formulation for each element is given next

$$w(x) = \sum_{j=1}^{n} w_j^e \psi_j^{(1)}, \qquad \phi_x(x) = \sum_{j=1}^{p} \phi_{x}{}_j^e \psi_j^{(3)} \quad (45)$$

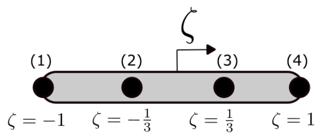

Figure 5 – Nodes and dimensionless coordinate system in a cubic element.

where ψ_i^j are the shape functions given by, with ζ being a dimensionless coordinate system

$$\begin{aligned}\psi_1 &= \frac{1}{16}(-1+\zeta+9\zeta^2-9\zeta^3), & \psi_2 &= \frac{9}{16}(-1+\zeta)(1+\zeta)(-1+3\zeta)\\ \psi_3 &= -\frac{9}{16}(-1+\zeta)(1+\zeta)(1+3\zeta), & \psi_4 &= \frac{1}{16}(1+\zeta)(-1+3\zeta)(1+3\zeta), \quad -1\leq\zeta\leq 1\end{aligned} \quad (46)$$

Finally, it is possible to arrange the equations presented in the following systems

$$\sum_{j=1}^{3} K_{ij}^{22} w_j^e + \sum_{j=1}^{3} K_{ij}^{23}\phi_{x}{}_j^e - F_i^2 = 0 \quad (47)$$

$$\sum_{j=1}^{3} K_{ij}^{32} w_j^e + \sum_{j=1}^{3} K_{ij}^{33}\phi_{x}{}_j^e - F_i^3 = 0 \quad (48)$$

Whith K_{ij}^{nm} being stiffness submatrices and F_i the components of the force vector

$$\begin{aligned}K_{ij}^{22} &= \int_{x_a}^{x_b} S_{xx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}dx \qquad K_{ij}^{23} = \int_{x_a}^{x_b} S_{xx}\frac{d\psi_i^{(2)}}{dx}\psi_j^{(3)}dx = K_{ji}^{32}\\ K_{ij}^{33} &= \int_{x_a}^{x_b}\left(D_{xx}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(3)}}{dx} + S_{xx}\psi_i^{(3)}\psi_j^{(3)}\right)dx\\ F_i^2 &= \int_{x_a}^{x_b}\psi_i^{(2)} q dx + \left[Q_x - N_{xx}\frac{dw}{dx}\right]_{x=x_a}\psi_i^{(1)} + \left[Q_x + N_{xx}\frac{dw}{dx}\right]_{x=x_b}\psi_i^{(2)}, \quad F_i^{(3)} = -M_{xx}(x_a)\psi_i^{(3)}(x_a) + M_{xx}(x_b)\psi_i^{(3)}(x_b)\end{aligned}$$

(49)

Reduced integration

In the previous section, the displacements and rotations were chosen as being approximated as polynomials of the same degree, since the calculations and the programming of the method is simplified. However, such simplification leads to inconsistent results for thin beams. For example, Reddy (1999) demonstrates the phenomenon for linear elements using first order interpolation functions. In that case the system of equations given by eqs. (47) and (48) leads to

$$\phi_x + \frac{dw}{dx} = 0, \qquad \frac{d\phi_x}{dx} = 0 \tag{50}$$

Since w and ϕ_x are approximated as 1st order polynomials, the expressions in eq.(50) are incompatible, unless the trivial solution ($\phi_x = w = 0$) , is adopted for the entire beam. This condition is called shear locking. Many approaches were developed in order to avoid such erroneous results, in this paper the reduced integration method, introduced by Zienkiewicz (1971) was adopted. It consists in, during the numerical integration of the element coefficients of eqs. (47) and (48), using one point less than necessary to fully evaluate the the terms which are multiplied by S_{xx}, since the exact integration leads to the shear locking.

Results

Firstly, the precision of the finite element analysis results for the one dimensional Timoshenko beam will be presented. Initially, results for different meshes will be compared for a pinned-pinned, as shown in Fig. 8 beam with distributed transversal load $q_0 = 100\mathrm{N/m}$, length $L = 15$ meters, and a square transversal area $0.1 \times 0.1\mathrm{m}^2$, elastic modulus $E = 200 \cdot 10^9$ Pa and shear modulus $G = 76.92 \cdot 10^9$ Pa. The displacements for meshes of 1 and 2 elements are in the form

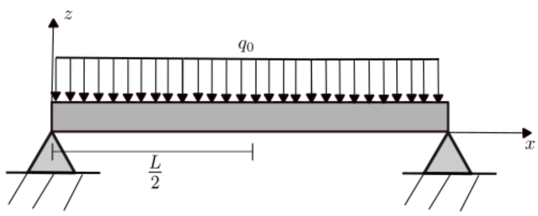

Figure 6 – Pinned-pinned beam with distributed load.

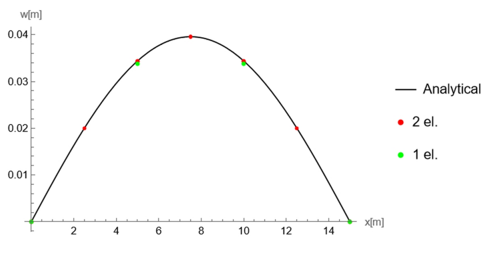

Figure 7 – Transversal displacements of a Timoshenko beam.

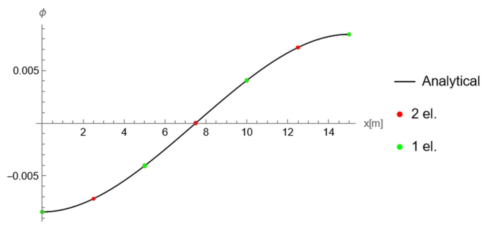

Figure 8 – Rotation displacements of a Timoshenko beam.

From Figs. (7) and (8) it is possible to infer that even for low number of elements, the finite element analysis gives displacements with a good degree of confidence, for two elements, the results almost coincide with the analytical solution. Next, the formulation presented will be tested in two different symmetric orhtotropic laminated beams, with the fiber orientation given by: $[0/45/-45/90]_s$, subject to different types of supports and external loads, both beams have a squared transversal area of 1x1 meters and length $L = 15$ meters and are made of graphite fabric-carbon which properties are given in Tab.1

Initially in order to verify the precision of the proposed method, the stresses will evaluated for two cases that have analytical solutions as e.g. in Patton (2020). The first case is a pinned-pinned beam, submitted to a concentrated force $F_0 = 1000[N]$ on its center,as shown in Fig.11, the analytical solution is symmetric and is given by Reddy(2004) for

Table 1 – Graphite fabric-carbon composite properties

E_{xx} $[GPa]$	E_{yy} $[GPa]$	E_{zz} $[GPa]$	G_{xy} $[GPa]$	G_{xz} $[GPa]$	G_{yz} $[GPa]$	ν_{xy}	ν_{yx}
173.058	33.094	5.171	9.377	8.274	3.240	0.036	0.0069

$0 \leq x \leq L/2$ by

$$w(x) = \frac{F_0 b L^3}{48 E_{xx}^b I_{yy}} \left[3\left(\frac{x}{L}\right) - 4\left(\frac{x}{L}\right)^3 \right] + \frac{F_0 b L}{2 K G_{xx}^b b h} \left(\frac{x}{a}\right) \quad (51)$$

$$\phi_x(x) = \frac{F_0 b}{4 E_{xx}^b I_{yy}} x^2 - \frac{F_0 b L^2}{16 E_{xx}^b I_{yy}} \quad (52)$$

$$\sigma_{xx}^{(k)} = \frac{F_0 L z}{4} \left(\bar{Q}_{11}^{(k)} D_{11}^* + \bar{Q}_{12}^{(k)} D_{12}^* + \bar{Q}_{16}^{(k)} D_{16}^* \right) \quad (53)$$

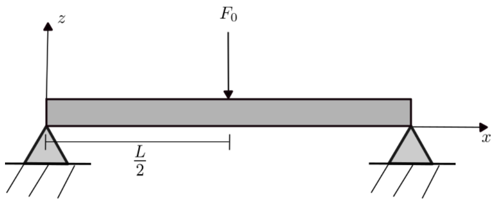

Figure 9 – Pinned-pinned beam with concentrated load at $x = L/2$.

where E_{xx}^b, G_{xx}^b are respectively the equivalent elastic and shear modulus and are given by eq.(31) and the standard value of $5/6$ for K is adopted. Since the reduced integration was adopted to avoid shear locking, it is prudent to analyse the stresses in a Gauss point which have more accurate results. Since the finite element method implemented uses only regular meshes, more elements will be necessary to match one Gauss point with a position close to the center of the beam (7.5 m), with 20 elements this condition is satisfied and both solutions agree

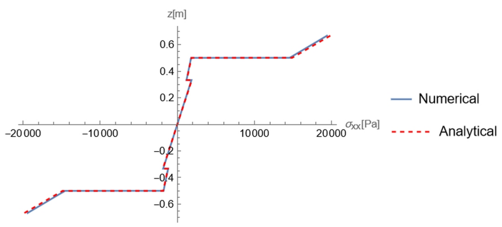

Figure 10 – Analytical and numerical solution for σ_{xx} at the nearest Gauss point: $x = 7.34$.

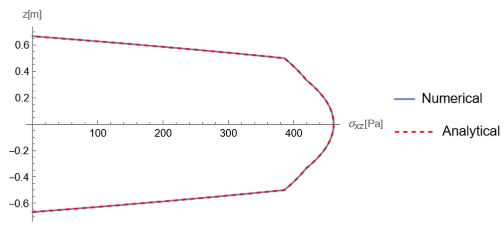

Figure 11 – Analytical and numerical solution for σ_{xz} at the nearest Gauss point: $x = 7.34$.

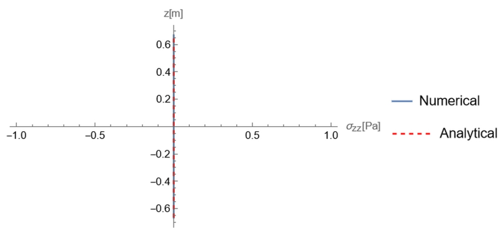

Figure 12 – Analytical and numerical solution for σ_{zz} at the nearest Gauss point: $x = 7.34$.

Figures 10 to 12 show that the relative errors between the numerical and analytical solutions are very low. Since the only dependence of the stresses in numerical results resides in the displacements obtained trough the FEM analysis and given that this method approximates with very high precision the displacements, even for coarser meshes, as shown in Fig.7 and Fig. 8. More precisely the errors are of around 2% for σ_{xx} and $6\cdot10^{-11}$% for σ_{xz}, the polynomial approximation chosen represented correctly the behaviour of σ_{zz}

The second case, is a clamped-clamped beam with a constant distributed load $q_0 = 100N/m$, as shown in Fig.16, the analytical solution is symmetric and is given by Reddy(2004) for $0 \le x \le L/2$ by

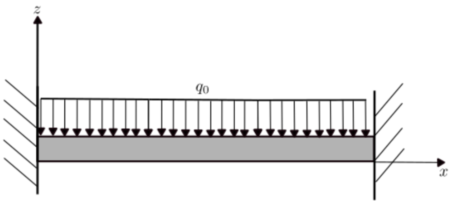

Figure 13 – Clamped-clamped beam with distributed load.

$$w(x) = \frac{q_0 bL^4}{24E^b_{xx}I_{yy}}\left[\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)\right]^2 + \frac{q_0 bL^2}{2K_s G^b_{xx} bh}\left[\left(\frac{x}{L}\right) - \left(\frac{x}{L}\right)^2\right] \tag{54}$$

$$\phi_x = \frac{q_0 bL^3}{12E^b_{xx}I_{yy}}\left[-2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)\right] \tag{55}$$

$$\sigma^{(k)}_{xx} = \left(-\frac{q_0 zx^2}{2} + \frac{q_0 xzL}{2} - \frac{q_0 zL^2}{12}\right)\left(\bar{Q}^{(k)}_{11}D^*_{11} + \bar{Q}^{(k)}_{12}D^*_{12} + \bar{Q}^{(k)}_{16}D^*_{16}\right) \tag{56}$$

Similarly to the previous case, more elements are necessary to approximate the gauss point and the point of interest. For a clamped-clamped beam, in terms of stresses the extremities are more important than the center. Thereby, the numerical and analytical solution will be compared for the position $x = 0$

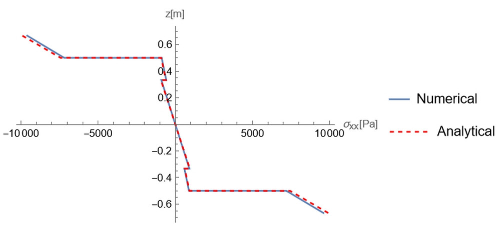

Figure 14 – Analytical and numerical solution for σ_{xx} at the nearest Gauss point: $x = 0.0792$.

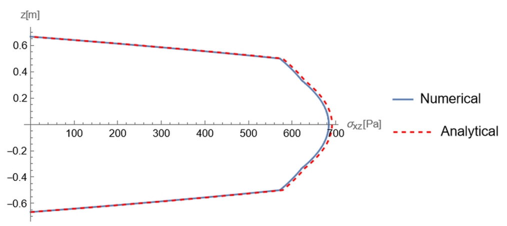

Figure 15 – Analytical and numerical solution for σ_{xz} at the nearest Gauss point: $x = 0.0792$.

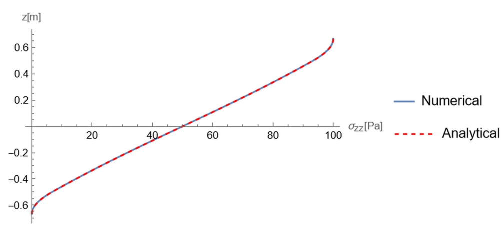

Figure 16 – Analytical and numerical solution for σ_{zz} at the nearest Gauss point: $x = 0.0792$.

Similarly to the previous case, the numerical results obtained were very close to the analytical solution, for the mesh of 40 elements considered the errors are around 3% fo σ_{xx}, 1% for σ_{xz} and $4\cdot10^{-13}$% for σ_{zz}. With more elements on the mesh, all errors in stresses stay bellow 1%, as shown in Fig.17, where the norm of the relative error was calculated for meshes going from 10 to 100 elements. In general the errors are consistent with other works dealing with similar problems, such as in Patton (2020).

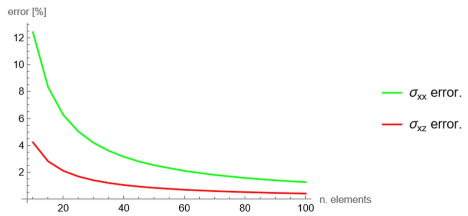

Figure 17 – Norm of the relative error for σ_{xx} and σ_{xz}.

Conclusion

The implemented method obtained good approximations for the interlaminar stresses for different types of supports and loads and was able to simplify the amount of calculations necessary if the three dimensional problem were solved directly.

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