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CORRECTIONS ON THE RECOVERED VALUES FOR GFEM RESULTS ON DYNAMIC LAMINATED PLATE BENDING

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Abstract: This paper describes the development of the procedures for the recovery of transverse stress, through post-processing, in dynamic problems of laminated composite plates, based on First-order Shear Deformation Theory (FSDT), modeled using the Generalized Finite Element Method (GFEM). A correction process was added to the post-processing for the integrated stresses in order to fix the boundary conditions on the top surface of the laminate. The objective of this work is to evaluate the influence that this correction procedure has on both integrated stresses and displacements. The post-process makes use of the results obtained directly from the GFEM models, using constitutive equations, in order to apply them to the general local equations. Using Chaudhuri Method, with the addition of body and inertial forces, the transverse stresses are obtained, through integration of the local differential equilibrium equations. Displacements are then acquired using stress-strain relations on the recovered transverse stresses. Numerical and analytical solutions are created for comparison of the final results of this procedure.
Keywords: Shear stress and displacement recovery, Dynamic analysis, Generalized finite element model, Layered composite structures, First-order shear deformation theory

INTRODUCTION

Due to the increased application of composite materials on industries different than the aerospace and aviation industries, the study of this kind of materials has felt an increase as well. A popular type of composite that is seen in many applications is the laminated plates due to its versatility and easy customization. The analysis of these still poses complications because of its complexity. Problems like delamination, meaning the separation of the layers, or the several randomly occurring failure modes difficult the correct design of structures that use these materials. A correct and accurate estimate of the distribution of stresses and displacements are imperative to ensure safety.

The usual approach is to use a Finite Element Model (FEM), commonly trough a commercial code like ANSYS or Abaqus, to obtain a numerical three dimensional approximation rooted on solid elements. This solution, although very effective and versatile, is not used often because of the high computational cost associated with it. A popular alternative to this solution is the use of simplified models that reduce the problem to a two dimensional one resulting in a much lower computational cost while, in many cases, still obtaining adequate stress and displacement values.

The Reissner-Mindlin model, also known as the First-Order Shear Deformation Theory, is one of the most popular simplified models found in many commercial codes. It considers uniform shear strain deformations across the thickness and discontinuous transverse shear stresses constant in every layer and in most situations it doesn't meet the necessary boundary conditions. Using a three dimensional exact solution we can observe that in fact the shear deformation are supposed to be discontinuous and the shear stress distribution should show some variability across the thickness of the laminate.

The problems shown with the Reissner-Mindlin model are nothing new and, still today, studies in order to correct or improve its results are made. One of the most popular procedure to better the results is presented in Chaudhuri and Seide (1987). The method applies the results obtained directly from a numerical solution using FEM or the Generalized Finite Element Method (GFEM) in the in-plane local equilibrium equations. These are then integrated usually prescribing a null or a nonzero value for the stress at the bottom of the laminate. In the end of the procedure it is common to see that the force boundary conditions are not satisfied.

The classic Chaudhuri method doesn't consider the inertial and body forces. These are added to the post-processing alongside a correction procedure meant to fix the value of the stress at the top of the laminate in order for it to satisfy the boundary condition. The main objective of this study is to analyze how the correction procedure affects the transverse stresses and the displacements results of a laminate plate under a harmonic load. The stresses applied to this post-processing are acquired from GFEM, chosen due to its low cost and easy adaptability. Using in-plane enrichment of the function basis leads to an expansion of the solution space that results in better values for the in-plane stresses. Another great aspect of this method is its easy and direct way to obtain the required differentiation while maintaining a fixed mesh and a low cost. This is not so easy in most commercial codes due to the use of low order elements. In those cases, the common approach is the creation of a patch with higher order elements that result in an increase of the cost.

LITERATURE REVIEW

This section will cover a broad view on the main topics necessary to understand the post-processing. These are the GFEM, which is used in order to obtain the in-plane stresses used in post-processing, the Reissner-Mindlin model that describes the behavior of the laminated plate, and lastly the stress recovery procedure based on the Chaudhuri method.

Generalized Finite Element Method

GFEM is a method based on the classic FEM that uses the concepts of Partition of Unity (PoU) and enrichment functions to get better results at a low computational cost. PoU is any set a functions that obey the following properties:

$$\sum_{\alpha=1}^{N} \varphi_\alpha(\mathbf{x}) = 1, \quad \varphi_\alpha \in C_0^\infty, \quad \varphi_\alpha \geq \forall \mathbf{x} \in \Omega \tag{1}$$

where Ω is the plate domain in $\mathbb{R}^n$.

Enrichment functions are chosen à priori and, through multiplication, are meant to enhance the PoU functions. They can have a polynomial, harmonic or any other format that contains part of the solution to the boundary value problem. The chosen format for this paper has the following format:

- 1. Linear enrichment $p = 1 : L_{i\alpha} = [1, \bar{x}, \bar{y}]$;
- 2. Quadratic enrichment $p = 2 : L_{i\alpha} = [1, \bar{x}, \bar{y}, \bar{x}^2, \overline{xy}, \bar{y}^2]$;

where

$$\bar{x} = \frac{x - x_\alpha}{h_\alpha}, \quad \bar{y} = \frac{y - y_\alpha}{h_\alpha} \tag{2}$$

where x_α and y_α are the coordinates of the node $\mathbf{x}_\alpha$ and h_α is a representative radius of the cloud ω_α.

By enriching the PoU functions, the solution space is expanded, improving the results, and it is possible to easily control the degree of the functions in order to guaranty the necessary differentiations for the post-processing stage. For these reasons the popularity of this method continues to rise, especially in problems where discontinuity is present, such as crack opening and crack movement studies.

Reissner-Mindlin model

The Reissner-Mindlin model or the First-order Shear Deformation Theory (FSDT) is a simplified 2D alternative to the solid three-dimensional solution. This model was initialy proposed in the works of Reissner (1945) and Mindlin (1951), later expanded by Yang (1966) to include composite materials and modified for the laminated plates by Whitney (1969). One of the most popular theories found in many commercial codes, this model is widely used because of its requirement of only C^0 approximation functions resulting in reduced computational implementation and cost.

This model is an adaptation of the Classic Lamination Theory (CLT). The adaptation is done on the hypothesis that the segment which is initially normal and straight to the non-deformed reference surface will remain straight and non-stretched, however no longer necessarily normal to the deformed reference surface, allowing the estimation of the transverse shear stresses (Mendonça, 2019). The Reissner-Mindlin kinematic model's formulation is described by the displacement relations, the strain-displacement relations and the local equilibrium equations for infinitesimal deformations presented in Eq. (3).

$$\begin{aligned}
&u(\mathbf{x},z,t) = u^0(\mathbf{x},t) + z\theta_x(\mathbf{x},t), && \varepsilon_x = \frac{\partial u}{\partial x}, && \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, && \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho^k b_x = \rho^k \frac{\partial^2 u}{\partial t^2}, \\
&v(\mathbf{x},z,t) = v^0(\mathbf{x},t) + z\theta_y(\mathbf{x},t), && \varepsilon_y = \frac{\partial v}{\partial y}, && \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, && \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho^k b_y = \rho^k \frac{\partial^2 v}{\partial t^2}, \\
&w(\mathbf{x},z,t) = w(\mathbf{x},t), && \varepsilon_z = \frac{\partial w}{\partial z}, && \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, && \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \rho^k b_z = \rho^k \frac{\partial^2 w}{\partial t^2},
\end{aligned} \tag{3}$$

Stress recovery post-processing

As stated before the changes made to the CLT allows the Reissner-Mindlin model to estimate a result for the transverse stress. This is done directly by calculating the transverse stresses from the constitutive equations. However, this results in incorrect distributions across the thickness making it necessary to use stress recovery techniques in post-processing.

The Chaudhuri method, applied in this work, uses the first two equations calculate the transverse shear stresses distribution. This is done by firstly employing the in-plane stress obtained, in this study, through GFEM. Isolating the transverse stress on all local equilibrium equations and integrating them results in:

$$\begin{aligned}
\tau_{xz}^i(z) - \tau_{xz}^i(-H/2) &= -\int_{-H/2}^{z} \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho^k b_x - \rho^k \frac{\partial^2 u}{\partial t^2} \right) dz, \\
\tau_{yz}^i(z) - \tau_{yz}^i(-H/2) &= -\int_{-H/2}^{z} \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \rho^k b_y - \rho^k \frac{\partial^2 v}{\partial t^2} \right) dz, \\
\sigma_z^i(z) - \sigma_z^i(-H/2) &= -\int_{-H/2}^{z} \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \rho^k b_z - \rho^k \frac{\partial^2 w}{\partial t^2} \right) dz.
\end{aligned} \tag{4}$$

Analyzing the equations it is important to note that all of transverse stresses are functions of the in-plane stresses. This means that the transverse normal stress can be calculated using transverse shear stresses obtained from the FEM model or alternatively we can use the integrated transverse shear stresses proposed by the Chaudhuri method.

METHODOLOGY

Due to using an approximating method like FEM or GFEM in most problems the results obtained commonly don't comply with the force boundary conditions in the superior surface of the laminate. A method then is proposed in this paper to correct these and potentially better the results.

The method consists in creating a function for each of the integrated transverse stresses and adding it to them. First it is considered a cubic polynomial function of z such as $f(z) = a + bz + cz2 + dz3$ as shown in Fig. 1.

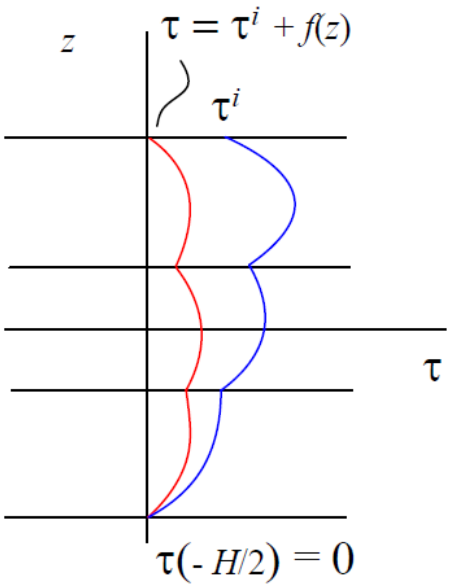

Figure 1 – Correction of the integrated transverse shear stresses.

By applying the $f(-H/2) = 0$ and $f(H/2) = A$ restrictions, it is possible to correct the one boundary condition seen in the top of the laminate. This transforms the cubic function to:

$$f(\mathbf{x}, z) = A(\mathbf{x}) \left(\frac{1}{2} + \frac{z}{H} \right) + c \left(z^2 + \frac{H^2}{4} \right) \tag{5}$$

where $A(\mathbf{x})$ is the term dependent on the $\mathbf{x} = (x, y)$ position on the surface.

Using the in-plane stresses, obtained by GFEM, with the transverse stresses calculate through integration of the local equilibrium equations find each of the correcting equations that adjusts the integrated transverse stress:

$$
\begin{aligned}
f_x &= A_x(\mathbf{x}) \left(\frac{1}{2} + \frac{z}{H} \right) + c_x \left(z^2 + \frac{H^2}{4} \right), \\
f_y &= A_y(\mathbf{x}) \left(\frac{1}{2} + \frac{z}{H} \right) + c_y \left(z^2 + \frac{H^2}{4} \right), \\
f_z &= A_z(\mathbf{x}) \left(\frac{1}{2} + \frac{z}{H} \right).
\end{aligned}
\tag{6}
$$

To each of the correcting functions the following boundary conditions are applied:

$$
\begin{cases}
\tau_{xz}(H/2) = q_x^s, \\
\tau_{yz}(H/2) = q_y^s, \\
\sigma_z(H/2) = q_z^s, \\
\int_{-H/2}^{H/2} \{\tau_{xz}, \tau_{yz}\} dz = \{Q_x, Q_y\}, \\
\left. \frac{\partial \tau_{xz}}{\partial x} \right|_{H/2} = \frac{\partial q_x^s}{\partial x}, \\
\left. \frac{\partial \tau_{yz}}{\partial y} \right|_{H/2} = \frac{\partial q_y^s}{\partial y}.
\end{cases}
\tag{7}
$$

where q_x^i, q_y^i e q_z^i are known values of the distributed loads at the Cartesian directions applied on the upper surface of the laminate. The presented shear forces Q_x and Q_y are calculated using the GFEM results with the constitutive relations. This means that the adjusted functions are defined by only five constants, A_x, A_y, A_z, c_x e c_y.

The last step of the post-processing is to estimate the in-plane and transverse displacements. To calculate displacements firstly the deformations are needed by applying the stress vector in the 3D stress-strain relations for anisotropic layer as shown in Eq. (8). The stress vector is constructed with the integrated transverse stresses and the in-plane stresses calculated with GFEM.

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \overline{\mathbf{S}} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz}^i \\ \tau_{xz}^i \\ \tau_{xy} \end{Bmatrix}
\tag{8}
$$

where the matrix $\overline{\mathbf{S}}$, is the material flexibility matrix, which for orthotropic material is $\overline{\mathbf{S}} = \mathbf{T}^T \mathbf{S} \mathbf{T}$. The displacements are then calculated by integrating the strain-displacement resulting in:

$$
u^i(z) = \int_{-H/2}^{z} \left(\gamma_{xz}^i + \frac{\partial w^i}{\partial x} \right) dz, \quad v^i(z) = \int_{-H/2}^{z} \left(\gamma_{yz}^i + \frac{\partial v^i}{\partial x} \right) dz, \quad w^i(z) = \int_{-H/2}^{z} \varepsilon_z dz.
\tag{9}
$$

After the integration to the top surface, the next step consists in the translation of each integrated displacement function by taking each respective value at reference surface $u_0 = u^i(0)$, $v_0 = v^i(0)$ e $w_0 = w^i(0)$, as seen in Eq. (10).

$$
\begin{aligned}
&\delta u = u_0 - u_{GFEM}, \quad \delta v = v_0 - v_{GFEM}, \quad \delta w = w_0 - w_{GFEM}, \\
&u^c(z) = u^i(z) - \delta u, \quad v^c(z) = v^i(z) - \delta v, \quad w^c(z) = w^i(z) - \delta w,
\end{aligned}
\tag{10}
$$

where u_{GFEM}, v_{GFEM} and w_{GFEM} are the GFEM value of displacement associated to the reference surface.

RESULTS

The problem chosen to present the results is a dynamic problem of a simply supported anisotropic plate with length and width $a = b = 1m$, and the thickness $H = 0.25m$, made with layers of orthotropic materials with the following material constants: $E_1 = 172.5GPa$, $E_2 = E_3 = 6.89GPa$, $G_{12} = G_{13} = 0.5E2$, $G_{23} = 0.2E_2$ and $\nu_{12} = \nu_{23} = \nu_{31} = 0.25$. A transverse harmonic load with $q_0 = 1N/m^2$ is applied with a double sinusoidal variation in the x plane:

$$q(x,y) = q_0 \sin\left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi y}{b}\right) e^{i\Lambda t} \tag{11}$$

where Λ is the load frequency.

The type of plate used is a three layer symmetric laminate with the $[0°, 90°, 0°]$ configuration. The mesh used in the GFEM analysis is a M=8 mesh composed of triangular elements enriched with p=4 functions. Figure 2 illustrates the mesh used alongside the boundary conditions of the simply supported plate.

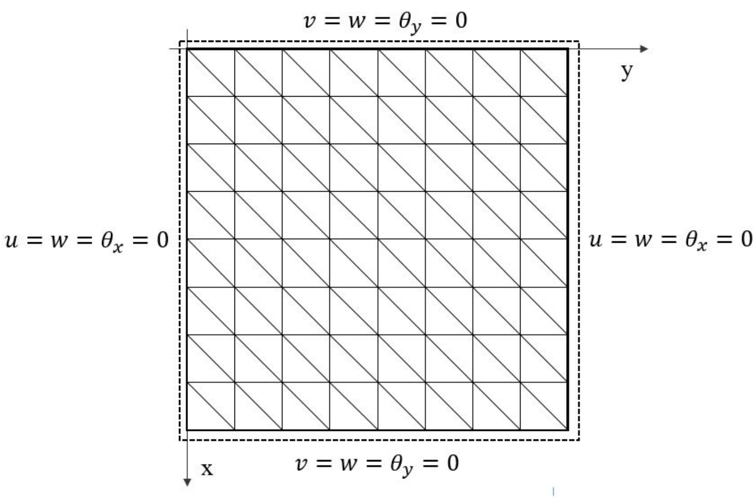

Figure 2 – Mesh arrangement example with boundary conditions.

Four different approaches were made of the same analysis in order to better understand the effects of the correction procedure. The first is an analytical solution made using the Reissner-Mindlin model, the harmonic solution was adapted from the static solution presented by Dobyns (1981) and uses the presented post-processing correcting stress recovery method, displayed with the FSDT marker. A university program was developed using FORTRAN language that uses the in-plane stresses, obtained from GFEM, on the same correcting stress recovery post-procedure applied to the first approach, presented with and without the inertial components in the GFEM and GFEMNI markers respectively. The two last approaches were made through ANSYS using a solid and a shell element to acquire a 3D and a 2D solution respectively, shown with the ANSYS 3D and ANSYS 2D markers.

The results for the graphs across the thickness are obtained by applying a frequency load equal to 95% of the first natural frequency calculated analytically, $\Lambda = 0.95\Lambda_1$. The spectral graph was made in an interval of 50% and 95% of the first natural frequency. The first natural frequency is $\Lambda_1 = 577.0786Hz$.

Figure 3 shows the results obtained for the transverse normal stress. This stress shows the influence that the application of the inertial and body forces has when applied in the post-processing. Observing the GFEM results with and without these forces it is clear to see that the absence of them greatly impacts the results when comparing to the ANSYS 3D result. The spectral graph shows also that this influence increases when approaching a natural frequency.

Figure 4 showcases the differences between the application of the correction procedure and the integrated results of the transverse normal stress. Analyzing the results we observed that the correction procedure greatly mitigates the effects of the omission of the inertial and body forces in post-processing. This leads to better stress distribution and a more accurate approximation when compared with the ANSYS 3D solution.

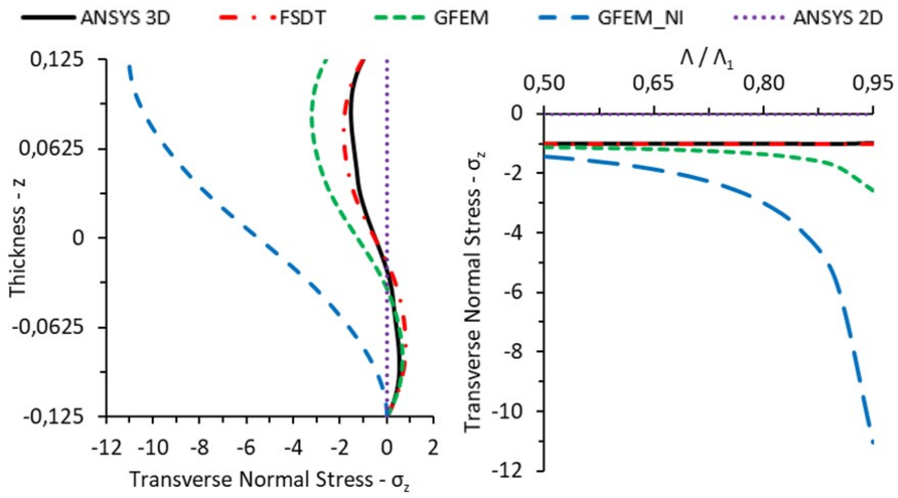

Figure 3 – Results of the transverse normal stress for symmetric plate, across the thickness in the left, spectral results in the right.

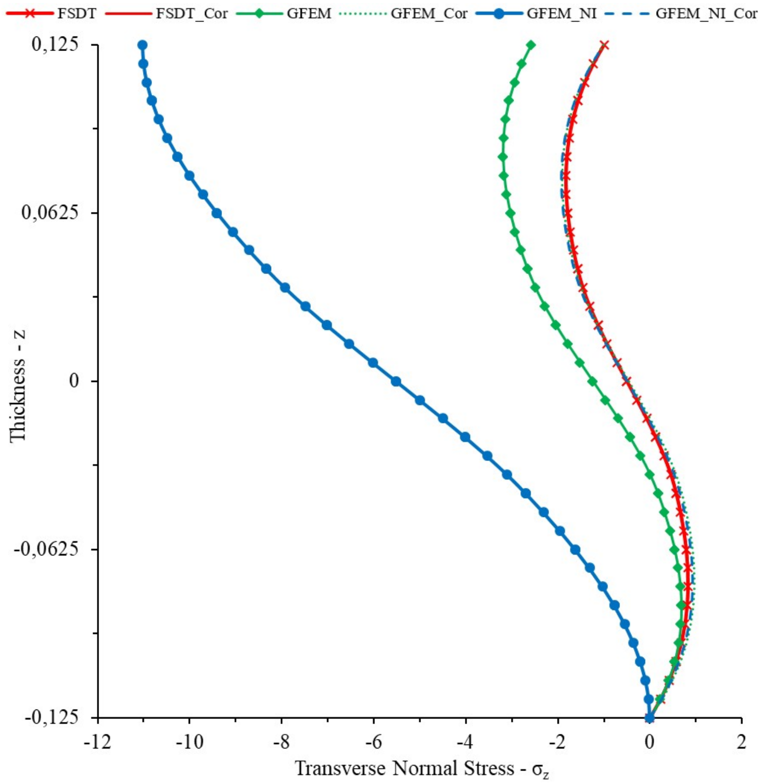

Figure 4 – Comparison between integrated results and corrected results.

CONCLUSIONS

Figures and results shows that the absence of the inertial and body forces in the post-processing stage greatly affects the both the distribution and the values of the integrated stresses. The correction procedure presents as an alternative to the application of these forces as it not only corrected the boundary condition in the top laminate but also the effects of not applying the inertial components in post-processing. The correction method presented produced better distributions and better estimates that can be used for a safe and correct structure design.

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