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Exact modal analysis of time-dependent problems: a proof of concept in application to one-dimensional truss structures

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Abstract. We proposed about two decades ago to solve transient problems of potential and elasticity using an advanced mode superposition technique that applies to equilibrium-based finite elements. The developments combine and extend Pian's hybrid finite element formulation and Przemieniecki's suggestion of displacement-based, frequency-dependent elements, thus arriving at a hybrid finite element method (actually initially conceived in the frame of a variationally-based boundary element method) for the general analysis of transient problems. Starting from a frequency-domain formulation, we have shown that the traditional structural dynamics taught in the textbooks is just a first-order truncation of a power series for which there is an underlying complex-symmetric (if viscous damping is included), non-linear eigenvalue problem related to the lambda-matrices of a free-vibration analysis, with an effective stiffness matrix expressed as a frequency power series. In the present contribution, we show that whenever such effective stiffness matrix can be represented (preferably) as an analytical function of frequencies we can formulate the exact – not just an improved or generalized – modal analysis of a given structural problem. This exact modal analysis may also be carried out if the problem's generalized stiffness and mass matrices can only be expressed numerically, albeit exactly within machine precision, for a given frequency number (which only in passing resembles a Laplace-transform analysis), although this may become computationally intensive. The analytical developments apply directly to some families of two- and three-dimensional finite elements. Still, we restrict our numerical applications to the simple truss problem including viscous damping as just a proof of concept. With this novel formulation an engineering structure – given its geometric and discretizing simplifications – can ultimately have its time response represented exactly.
Keywords: structural dynamics, generalized modal analysis, exact modal analysis, generalized eigenvalues and eigenvectors

INTRODUCTION

An established technique to solve time-dependent problems – other than directly tackling the time-domain problem – is the formulation of a complete frequency-domain analysis via Laplace or Fourier transforms, with subsequent ad hoc expression of results by numerical inversion. Although usually easy to implement, such a transform inversion is computationally intensive, if accurate results are desired, and is not void of numerical instabilities. Moreover, the inclusion of non-homogeneous initial conditions and general body forces may become troublesome. For diffusion-type problems, the Gaver-Stehfest algorithm – or some of its improvements – seems well suited (Aguilar Marón, 2008). General dynamics problems demand more robust algorithms usually based on Fourier series expansions, which descend from a proposition firstly made by Dubner and Abate (1968). Algorithms of either kind are already implemented in mathematical languages such as Matlab and Mathematica. The Fortran IMSL library has in the subroutine INLAP a Fourier-series algorithm proposed by Crump (1976) and improved by de Hoog et al (1982).

Our research group at PUC-Rio managed to develop a means to solve transient problems of potential and elasticity using an advanced mode superposition technique that applies to equilibrium-based finite element and boundary element models (Dumont and Oliveira, 2001; Dumont, 2005). The developments combine and extend Pian's (1964) hybrid finite element formulation and Przemieniecki's (1968) suggestion of displacement-based, frequency-dependent elements, thus arriving at a hybrid finite/boundary element method for the general analysis of transient problems. Actually, we are dealing with generalized finite element families, as the interpolation functions that satisfy domain equilibrium may be singular (fundamental solutions or Green's functions) or not [non-singular transcendental functions (Dumont and Prazeres, 2004)]. Transient problems of potential and elasticity are modeled, also including viscous damping. Starting from a frequency-domain formulation, we managed to show that the traditional structural dynamics taught in the text books (Przemieniecki, 1968; Bathe, 1976) is just a first-order truncation of a power series for which there is an underlying complex-symmetric (if viscous damping is included), nonlinear eigenvalue problem related to the λ-matrices of a free-vibration analysis

(λ-matrix is a technical mathematical terminology but we use the notation ω to substitute for λ in this paper), with the *effective* stiffness matrix – function of frequency ω and viscous factor ζ – expressed as a frequency power series: $\mathbf{K}_{eff}(\omega,\zeta) \approx \mathbf{K}_0 - i\omega\mathbf{C}_1 - \omega^2\mathbf{M}_1 - i\omega^3\mathbf{C}_2 - \omega^4\mathbf{M}_2 - i\omega^5\mathbf{C}_3 - \omega^6\mathbf{M}_3 - \cdots$, where $\mathbf{K}_0, \mathbf{C}_j, \mathbf{M}_j \in \Re^{n\times n}$ are generalized stiffness, damping and mass matrices (Dumont, 2005). The eigenvectors of this problem fulfill generalized orthogonality properties (Dumont, 2007) that enable the implementation of an advanced mode superposition technique. This leads to the solution in the time domain and the immediate expression of all results of interest. Owing to the equilibrium-based formulation, general domain actions (including body forces and moving loads) as well as boundary and initial conditions are dealt with in a straightforward way.

In fact, it is shown in Dumont (2007) that the eigenvalue problem $\mathbf{K}_{eff}(\omega,\zeta)\phi = \mathbf{0}$ can be restated as $[\mathbf{K}(\omega,\zeta) - \omega\mathbf{M}(\omega,\zeta)]\phi = \mathbf{0}$, where $\mathbf{K}^{\mathrm{T}}(\omega,\zeta) = \mathbf{K}(\omega,\zeta)$ and $\mathbf{M}^{\mathrm{T}}(\omega,\zeta) = \mathbf{M}(\omega,\zeta)$ are complex-symmetric matrices given as power series of the complex eigenfrequencies ω, where, if (ω,ϕ) is a solution eigenpair, ϕ may be normalized in such a way that $\phi^{\mathrm{T}}\mathbf{M}(\omega,\zeta)\phi = 1$ and $\phi^{\mathrm{T}}\mathbf{K}(\omega,\zeta)\phi = \omega$. The traditional Rayleigh quotient iteration and the modified Jacobi-Davidson method are outlined in Dumont (2007) and shown to be mathematically equivalent, both with asymptotically cubic convergence for complex-symmetric nonlinear problems.

Quite recently, it has been concluded by this author (paper in preparation) that a slight generalization of the modified Jacobi-Davidson method proposed in Dumont (2007) also presents cubic convergence in the solution of completely general, complex nonlinear eigenvalue problems given in terms of λ-matrices. Moreover, we show in the present paper – as a proof of concept in the application to simple truss elements – that nonlinear eigenvalue problems can be solved for a general problem $\mathbf{K}_{eff}(\omega,\zeta)\phi = \mathbf{0}$ in which the *effective* stiffness matrix has an exact expression rather than the power series expansion showed above. To complement the proof of concept we also show that an exact modal analysis is applicable as the most possible generalization of the developments proposed in Dumont (2007).

A simple numerical example shows the advantages of applying the proposed, exact technique.

BASICS FOR THE SIMPLE TRUSS ELEMENT IN THE FREQUENCY DOMAIN

The following developments for the simplest case of a truss element are a sequel of works by Dumont and Oliveira (2001), Dumont (2005, 2006, 2007), Dumont and Chaves (2003), Chaves (2003), Dumont and Prazeres (2004), Prazeres (2005), Oliveira (2006), Aguilar Marón (2008), Dumont and Aguilar Marón (2009), Dumont et al. (2019), which also present analytical developments for Euler-Bernoulli and Timoshenko beam elements including viscous damping. It is not our scope to presently review third parties' technical literature on the subject of such analytical developments. However, the conceptually paramount expressions of this Section are to the author's best knowledge displayed for the first time.

The general frequency-domain, small displacement solution $u(x,\omega)$, $0 \le x \le \ell$, for a given frequency ω (s^{-1}), of a two-degree of freedom straight truss element of constant length ℓ, cross section A, elasticity modulus E, specific mass ρ (total mass $\rho A\ell$), and longitudinal viscous damping $2\zeta\rho$ (force by volume per unit velocity) is, for applied frequency-dependent displacements $d_1(\omega)$ and $d_2(\omega)$ at its extremities,

$$u(x,\omega) = \frac{1}{\sin k\ell}\left\langle \sin k(\ell - x) \quad \sin kx \right\rangle \begin{Bmatrix} d_1(\omega) \\ d_2(\omega) \end{Bmatrix} \equiv \mathbf{N}(x,\omega)\mathbf{d}(\omega) \tag{1}$$

where $k^2 = \frac{\rho}{E}\left(\omega^2 + 2i\zeta\omega\right)$, which includes the case of a damping-free problem. Observe that $\omega \to 0$ implies $k \to 0$, and $\mathbf{N}(\omega \to 0) = \left\langle 1 - x/\ell \quad x/\ell \right\rangle$, the static case, but the reverse is not necessarily true, as for critical damping, $\omega \to -2i\zeta$ (Dumont, 2007), a subject of no interest presently. For this very simple problem, the shape functions of Eq. (1) are also the fundamental solutions of the truss's governing differential equation

$$\frac{\partial^2 u(x,\omega)}{\partial x^2} + k^2 u(x,\omega) = 0 \tag{2}$$

The following developments have been carried out in the frame of the hybrid finite element method, which is based on the Hellinger-Reissner potential, as laid down in Dumont and Oliveira (2001), for instance. However, since in this very particular case of a truss element – as well as for beam elements and for a few more general finite elements (Sales, 2022) – the fundamental solutions may be directly used as displacement shape functions, we are entitled to think also in terms of the plain displacement variational formulation.

The *effective* stiffness matrix for the truss element is obtained as (Dumont, 2007)

$$\mathbf{K}_{eff}(\omega,\zeta) = \frac{EA}{\ell}\frac{k\ell}{\sin k\ell}\begin{bmatrix} \cos k\ell & -1 \\ -1 & \cos k\ell \end{bmatrix} \tag{3}$$

which is valid whether or not viscous damping is considered, being in the latter case equivalent to $\mathbf{K}_{eff}(\omega^2)$.

Problem without damping $(\zeta = 0)$

The displacement variational formulation was the approach used by Przemieniecky (1968) for the case of the damping-free problem, $c = \sqrt{E/\rho}$, $k = \omega/c$, coming up with the two autonomous expressions of the frequency-dependent mass and stiffness matrices (arguments sometimes dropped for the sake of notation simplicity)

$$\mathbf{M}(\omega^2) = \int_0^\ell \mathbf{N}^{\mathrm{T}}\rho\mathbf{N}A\mathrm{d}x = \frac{\rho A\ell}{2k\ell\sin^2 k\ell}\begin{bmatrix} k\ell - \sin k\ell\cos k\ell & \sin k\ell - k\ell\cos k\ell \\ \sin k\ell - k\ell\cos k\ell & k\ell - \sin k\ell\cos k\ell \end{bmatrix} \tag{4}$$

$$\mathbf{K}(\omega^2) = \int_0^\ell \frac{\partial\mathbf{N}^{\mathrm{T}}}{\partial x}E\frac{\partial\mathbf{N}}{\partial x}A\mathrm{d}x = \frac{EA}{\ell}\frac{k\ell}{2\sin^2 k\ell}\begin{bmatrix} k\ell + \sin k\ell\cos k\ell & -\sin k\ell - k\ell\cos k\ell \\ -\sin k\ell - k\ell\cos k\ell & k\ell + \sin k\ell\cos k\ell \end{bmatrix} \tag{5}$$

which add up to build the same *effective* stiffness matrix of Eq. (3) as

$$\mathbf{K}_{eff}(\omega^2) = \mathbf{K}(\omega^2) - \omega^2\mathbf{M}(\omega^2) \tag{6}$$

By the way, Dumont (2007) remarked that these analytical expressions "can by no straightforward means be adapted for viscous damping" – an assertion that no longer holds, as we lay out in sequence. Before advancing with the present developments, it is worth writing down the analytical mass and stiffness matrix expressions that relate two different frequencies ω_r, ω_s, as this will be in part required in the generalized eigenvalue problem to be dealt with next:

$$\mathbf{M}(\omega_r^2,\omega_s^2) = \int_0^\ell \mathbf{N}^{\mathrm{T}}(\omega_r)\rho\mathbf{N}(\omega_s)A\mathrm{d}x = \frac{\rho A\ell}{\ell\,(k_r^2-k_s^2)}\begin{bmatrix} k_s\cot k_s\ell - k_r\cot k_r\ell & k_r\csc k_r\ell - k_s\csc k_s\ell \\ k_r\csc k_r\ell - k_s\csc k_s\ell & k_s\cot k_s\ell - k_r\cot k_r\ell \end{bmatrix} \tag{7}$$

$$\mathbf{K}(\omega_r^2,\omega_s^2) = \int_0^\ell \frac{\partial\mathbf{N}^{\mathrm{T}}(\omega_r)}{\partial x}E\frac{\partial\mathbf{N}(\omega_s)}{\partial x}A\mathrm{d}x = \frac{EA}{\ell}\frac{k_rk_s\ell}{k_r^2-k_s^2}\begin{bmatrix} k_r\cot k_s\ell - k_s\cot k_r\ell & k_s\csc k_r\ell - k_r\csc k_s\ell \\ k_s\csc k_r\ell - k_r\csc k_s\ell & k_r\cot k_s\ell - k_s\cot k_r\ell \end{bmatrix} \tag{8}$$

$$\mathbf{K}_{eff}(\omega_r^2,\omega_s^2) = \mathbf{K}(\omega_r^2,\omega_s^2) - \omega_r\omega_s\mathbf{M}(\omega_r^2,\omega_s^2) = \frac{EA}{\ell}\frac{\ell k_rk_s}{k_r+k_s}\begin{bmatrix} \cot k_s\ell + \cot k_r\ell & -\csc k_r\ell - \csc k_s\ell \\ -\csc k_r\ell - \csc k_s\ell & \cot k_s\ell + \cot k_r\ell \end{bmatrix} \tag{9}$$

where $k_r = \omega_r/c$ and $k_s = \omega_s/c$, with Eqs. (4), (5) and (3) obtained in the limit $\omega_r = \omega_s = \omega$.

Problem with viscous damping $(\zeta > 0)$

If viscous damping is considered, the same Eq. (3) is valid (Dumont, 2007), with $k^2 = \frac{\rho}{E}\left(\omega^2 + 2i\zeta\omega\right)$. However, the frequency-dependent mass and stiffness matrices now enter the equation

$$\mathbf{K}_{eff}(\omega,\zeta) = \mathbf{K}(\omega,\zeta) - \omega\mathbf{M}(\omega,\zeta) \tag{10}$$

where the argument ζ is added to indicate that we are dealing with a different problem than Eq. (3). The above equation has been up to now expressed only in terms of frequency power series (Dumont, 2007), that is, not analytically – as one should expect for this very simple case, which was the reason of the remark after Eq. (6). In fact, the frequency power-series expansions of $\mathbf{M}(\omega,\zeta)$ and $\mathbf{K}(\omega,\zeta)$ in the latter composition of the *effective* stiffness matrix are of different nature than the ones of Eq. (6) and – except for Eqs. (1)-(3), which are structurally unchanged, – the equations of the previous Section cannot be obtained as the limit case for $\zeta \to 0$.

In the frame of the variational, hybrid finite (and boundary) element method proposed by Dumont and Chaves (2003) and Dumont (2007), Eqs. (3)-(9) are in general obtainable only as frequency power series – in some cases even Eq. (1) cannot be expressed analytically. Then, dealing or not with viscous damping does not involve different concepts. On the other hand, the present sheer analytical developments require a deeper knowledge of the way the mass and stiffness matrices $\mathbf{M}(\omega,\zeta)$ and $\mathbf{K}(\omega,\zeta)$ would be obtained. As a matter of fact, back to the damping-free problem of the previous Section, the analytical expression of Eq. (4) may also be obtained as

$$\mathbf{M}(\omega^2) = \int_0^\ell \mathbf{N}^{\mathrm{T}}E\frac{\partial\,(k^2)}{\partial\,(\omega^2)}\mathbf{N}A\mathrm{d}x = -\frac{\partial}{\partial\,(\omega^2)}\mathbf{K}_{eff}(\omega^2) = -\frac{\partial}{\partial k}\mathbf{K}_{eff}(\omega^2)\frac{\partial k}{\partial\,(\omega^2)} \tag{11}$$

since, if there is no damping,

$$k^2 = \frac{\rho}{E}\left(\omega^2\right) \Rightarrow 2k\frac{\partial k}{\partial\left(\omega^2\right)} = \frac{\rho}{E} \Rightarrow \frac{\partial k}{\partial\left(\omega^2\right)} = \frac{\rho}{2kE} \tag{12}$$

where the parenthesis notation (ω^2) and (k^2) indicates that ω^2 and k^2 are to be dealt with as single variables.

By the same token, we may evaluate the frequency- and viscosity-dependent mass matrix of Eq. (10) as

$$\begin{aligned}\mathbf{M}(\omega,\zeta) &= \int_0^\ell \mathbf{N}^{\mathrm{T}} E\frac{\partial k^2}{\partial\omega}\mathbf{N}A\mathrm{d}x = -\frac{\partial}{\partial\omega}\mathbf{K}_{eff}(\omega,\zeta) = -\frac{\partial}{\partial k}\mathbf{K}_{eff}(\omega,\zeta)\frac{\partial k}{\partial\omega}\\ &= \frac{\rho A\ell(\omega+i\zeta)}{k\ell\sin^2 k\ell}\begin{bmatrix} k\ell-\sin k\ell\cos k\ell & \sin k\ell - k\ell\cos k\ell\\ \sin k\ell-k\ell\cos k\ell & k\ell-\sin k\ell\cos k\ell\end{bmatrix}\end{aligned} \tag{13}$$

(The author does not claim originality of the present analytical developments, as the technical literature on the subject could not be exhaustively screened, but very much stresses this paper's autonomous developments.) The validity of this expression is checked in terms of the corresponding power series developments, as given by Dumont (2007) and in the Appendix, with substantial simplification obtained by replacing the damping term ζ with $\alpha c/\ell$, where α is a nondimensional viscosity parameter. An independent expression of the stiffness matrix $\mathbf{K}(\omega,\zeta)$ – as the counterpart of $\mathbf{K}(\omega^2)$ obtained in Eq. (5) for the damping-free problem – does not seem of as simple consecution as for the mass matrix. However, this is absolutely unnecessary, as, if required, we can simply and economically write the reverse of Eq. (10):

$$\mathbf{K}(\omega,\zeta) = \mathbf{K}_{eff}(\omega,\zeta) + \omega\mathbf{M}(\omega,\zeta) \tag{14}$$

for $\mathbf{K}_{eff}(\omega,\zeta)$ and $\mathbf{M}(\omega,\zeta)$ given in Eqs. (3) and (13).

On the other hand, the analytical, mixed mass matrix counterpart of Eq. (7) that relates two different frequencies ω_r, ω_s, paramount in the solution of the generalized eigenvalue problem of the next Section, may be simply expressed as

$$\begin{aligned}\mathbf{M}(\omega_r,\omega_s,\zeta_r,\zeta_s) &= \int_0^\ell \mathbf{N}^{\mathrm{T}}\left(2\rho\sqrt{\omega_r+i\zeta_r}\sqrt{\omega_s+i\zeta_s}\right)\mathbf{N}A\mathrm{d}x\\ &\equiv \frac{2\rho A\ell}{\ell}\frac{\sqrt{\omega_r+i\zeta_r}\sqrt{\omega_s+i\zeta_s}}{k_r^2-k_s^2}\begin{bmatrix} k_s\cot k_s\ell - k_r\cot k_r\ell & k_r\csc k_r\ell - k_s\csc k_s\ell\\ k_r\csc k_r\ell - k_s\csc k_s\ell & k_s\cot k_s\ell - k_r\cot k_r\ell\end{bmatrix}\end{aligned} \tag{15}$$

where $k_r^2 = \frac{\rho}{E}\left(\omega_r^2+2i\zeta_r\omega_r\right)$, $k_s^2 = \frac{\rho}{E}\left(\omega_s^2+2i\zeta_s\omega_s\right)$. The different values ζ_r and ζ_s are of rather conceptual relevancy by assuming the viscosity to be proportional to the frequency. Maybe we just use $\zeta_r = \zeta_s = \zeta$ in a practical application. We retrieve Eq. (13) from the above equation as the limit for $\omega_r = \omega_s = \omega$. Although expanding the terms of $\mathbf{M}(\omega_r,\omega_s,\zeta_r,\zeta_s)$ as power series of ω_r, ω_s requires some tedious work and ingenuity (even if resorting to the symbolic mathematics package Maple), we have been able to demonstrate the exactness of this expression when compared with previous developments based on the Hellinger-Reissner potential that deal from the beginning with the power series expansion of $\mathbf{K}_{eff}(\omega,\zeta)$, as required in the generalized nonlinear eigenvalue problem proposed by Dumont (2007).

EXACT MODAL ANALYSIS OF TIME-DEPENDENT PROBLEMS

Associated eigenvalue problems without or with viscous damping

The modal analysis of problems without damping, with underdamping and with overdamping cannot be dealt with as just one single case, as laid out in Dumont (2007) for the generalized eigenvalue problem. We consider in the following the cases of no damping and underdamping side by side for the sake of conceptual completion – and refer the reader to Dumont (2007) for the problem of overdamping – since our focus herein is in the *exact modal analysis*. We are looking for the solution of the general, real-symmetric or complex-symmetric, nonlinear eigenvalue problem

$$[\mathbf{K}(\omega^2) - \omega^2\mathbf{M}(\omega^2)]\phi = \mathbf{0} \tag{16}$$

$$[\mathbf{K}(\omega,\zeta) - \omega\mathbf{M}(\omega,\zeta)]\phi = \mathbf{0} \quad \Leftrightarrow \quad \phi^{\mathrm{T}}[\mathbf{K}(\omega,\zeta) - \omega\mathbf{M}(\omega,\zeta)] = \mathbf{0} \tag{17}$$

for $\mathbf{M}(\omega^2)$ and $\mathbf{K}(\omega^2)$ given in Eqs. (4) and (5), $\mathbf{M}(\omega,\zeta)$ defined in Eq. (13) and $\mathbf{K}(\omega,\zeta)$ obtainable in principle from Eq. (14). According to Eqs. (6) and (10), the terms in brackets above add up to the *effective* stiffness matrix of the problem, Eq. (3), whether or not viscous damping is considered. Moreover, given two eigenpairs (ω_r,ϕ_r) and (ω_s,ϕ_s) with normalized eigenvectors, they satisfy either orthogonality condition

$$\langle\phi_r^{\mathrm{T}},\phi_s\rangle \equiv \phi_r^{\mathrm{T}}\mathbf{M}(\omega_r^2,\omega_s^2)\phi_s = \delta_{rs} \quad\Leftrightarrow\quad \phi_r^{\mathrm{T}}\mathbf{K}(\omega_r^2,\omega_s^2)\phi_s = \delta_{rs}\omega_r^2 \equiv \delta_{rs}\omega_s^2 \tag{18}$$

$$\langle\phi_r^{\mathrm{T}},\phi_s\rangle \equiv \phi_r^{\mathrm{T}}\mathbf{M}(\omega_r,\omega_s,\zeta_r,\zeta_s)\phi_s = \delta_{rs} \quad\Leftrightarrow\quad \phi_r^{\mathrm{T}}\mathbf{K}(\omega_r,\omega_s,\zeta_r,\zeta_s)\phi_s = \delta_{rs}\omega_r \equiv \delta_{rs}\omega_s \tag{19}$$

where δ_{rs} is the Kronecker delta (no summation implied above by repeated indices). Neither $\mathbf{K}(\omega,\zeta)$ nor $\mathbf{K}(\omega_r,\omega_s,\zeta_r,\zeta_s)$ are actually explicitly required in the algorithm to be outlined. It is worth remarking that the developments comprised by the latter equations have been first proposed in terms of direct expansion of the *effective* stiffness matrix of the problem:

$$\mathbf{K}_{eff}(\omega^2) = \mathbf{K}_0 - \sum_{j=1}^{n}\omega^{2j}\mathbf{M}_j + O(\omega^{2n+2}) \approx \mathbf{K}_0 - \omega^2\mathbf{M}_1 - \omega^4\mathbf{M}_2 - \omega^6\mathbf{M}_3 - \dots \tag{20}$$

$$\begin{aligned}\mathbf{K}_{eff}(\omega,\zeta) &= \mathbf{K}_0 - \sum_{j=1}^{n}\left(i\omega^{2j-1}\mathbf{C}_j(\zeta) + \omega^{2j}\mathbf{M}_j(\zeta)\right) + O(\omega^{2n+1})\\ &\approx \mathbf{K}_0 - i\omega\mathbf{C}_1(\zeta) - \omega^2\mathbf{M}_1(\zeta) - i\omega^3\mathbf{C}_2(\zeta) - \omega^4\mathbf{M}_2(\zeta) - \dots\end{aligned} \tag{21}$$

where $\mathbf{K}_0$ is the problem's static, stiffness matrix, and $\mathbf{C}_j(\zeta)$ and $\mathbf{M}_j(\zeta)$ are generalized damping and mass matrices with components of inertia and viscosity terms. We have also shown that, for sheer linear algebra consistency, the matrix coefficients above enter the expanded expression of either Eq. (16) or Eq. (17) with separate approximations of matrices $\mathbf{M}(\omega^2)$ and $\mathbf{K}(\omega^2)$ of Eqs. (4) and (5) as well as of matrices $\mathbf{M}(\omega,\zeta)$ and $\mathbf{K}(\omega,\zeta)$ of Eqs. (13) and (14) as

$$\mathbf{M}(\omega^2) = \sum_{j=1}^{n} j\omega^{2j-2}\mathbf{M}_j + O(\omega^{2n+2}) \approx \mathbf{M}_1 + 2\omega^2\mathbf{M}_2 + 3\omega^4\mathbf{M}_3 + \dots \tag{22}$$

$$\begin{aligned}\mathbf{M}(\omega,\zeta) &= \sum_{j=1}^{n}\left(i(2j-1)\omega^{2j-2}\mathbf{C}_j(\zeta) + 2j\omega^{2j-1}\mathbf{M}_j(\zeta)\right) + O(\omega^{2n+1})\\ &\approx i\mathbf{C}_1(\zeta) + 2\omega\mathbf{M}_1(\zeta) + 3i\omega^2\mathbf{C}_2(\zeta) + 4\omega^3\mathbf{M}_2(\zeta) + \dots\end{aligned} \tag{23}$$

$$\mathbf{K}(\omega^2) = \mathbf{K}_0 + \sum_{j=2}^{n}(j-1)\omega^{2j}\mathbf{M}_j \approx \mathbf{K}_0 + \omega^4\mathbf{M}_2 + 2\omega^6\mathbf{M}_3 + \dots \tag{24}$$

$$\begin{aligned}\mathbf{K}(\omega,\zeta) &= \mathbf{K}_0 + \sum_{j=1}^{n}\left(i(2j-2)\omega^{2j-1}\mathbf{C}_j(\zeta) + (2j-1)\omega^{2j}\mathbf{M}_j(\zeta)\right) + O(\omega^{2n+1})\\ &\approx \mathbf{K}_0 + \omega^2\mathbf{M}_1(\zeta) + 2i\omega^3\mathbf{C}_2(\zeta) + 3\omega^4\mathbf{M}_2(\zeta) + \dots\end{aligned} \tag{25}$$

Observe that we know the expanded expression of $\mathbf{K}(\omega,\zeta)$, as above, but its analytical expression is rather expressed in terms of Eq. (14). The frequency-power series of Eqs. (20)-(25) are given in Dumont (2007) for the simple truss element and reproduced in the Appendix in an edited version.

By the way, we have in the meantime been able to show, and it is part of a paper in preparation, that all eigenvalue developments previously proposed (Dumont, 2007) as well as the ones in the present outline may be generalized for a completely complex, non-symmetric problem – and mathematically solved in terms of a further modified Jacobi-Davidson algorithm –, even though its mechanical meaning and applicability may just lack.

Solution of the proposed generalized eigenvalue problem either as a power series or exact

The solution of the generalized eigenvalue problem proposed above for potential or elasticity time-dependent problems in one, two or three dimensions and for any possible initial conditions in terms of frequency power series has been almost thoroughly outlined and solved in Dumont (2007). We shall now address how to complete this outline and extend it to the present subject of the exact, generalized modal analysis of time-dependent problems. In fact, we had been able to extend to nonlinear eigenvalue problems the so-called Jacobi-Davidson algorithm, with a comprehensive literature review and the development of existence and convergence theorems for the case of damping-free problems, as particularly presented here in Eqs. (16), (18), (20), (22) and (24). Its was shown that convergence occurs and is cubic as long as the generalized mass matrix given in Eq. (22) remains positive definite. Moreover, it was shown on the basis of a perturbation theorem that a generalized problem with damping, as represented here by Eqs. (17), (19), (21), (23) and (25), can be solved starting from the corresponding damping-free solution with a small number of additional iterations for the successive eigenpairs. The solution is very fast when eigenpairs are close and, as given by a theorem, in the case of geometric multiplicity, moving from the previously found eigenpairs to the next one only requires only one iteration step. Dumont (2007) also shows how to apply the whole modal analysis scheme to problems with overdamping.

There are only two aspects of the developed algorithm that have prevented it from being universally applicable.

(1) Although the first solution eigenpair (ω_1^2, ϕ_1) usually corresponds to the smallest ω^2, no theorem in the frame of the Jacobi-Davidson algorithm shows how to find the eigenpairs for $\omega_j^2, j = 1, \ldots$ in strictly increasing order.

(2) The convergence characteristic of the algorithm relies on the fact that a search eigenpair (ω_j^2, ϕ_j) is orthogonal to all previously found eigenpairs $(\omega_k^2, \phi_k), k = 1, \ldots, j-1$ in terms of either Eq. (16) or (17): round-off errors, particularly for large-scale problems, lead to eigenpairs of interest being missed as there is no longer guarantee that the generalized mass matrix of Eq. (22) preserves its positive-definiteness along the iteration steps.

Solution strategies

As an attempt to solve these difficulties we have first tried to deflate the evaluated eigenpairs – and this has led to an interesting academic paper in preparation, as a by-product of Carvalho's (2017) Ph.D. thesis, namely the development of deflation techniques for general nonlinear eigenvalue problems. However, they share the same weakness reported above: avalanching round-off errors that end up disfiguring the original problem.

We now envisage two possible strategies to guide the robust solution of the proposed nonlinear eigenvalue problem.

(1) The first one is to add a zero step to the strategy developed by Dumont (2007), in which the underlying linear, damping-free problem with just one stiffness and one mass matrix – the first truncation of Eqs. (20) and (22), thus just the classical structural dynamics problem, – is solved by some built-in solver that enables the evaluation of the eigenpairs for a given number of the problem's smallest eigenvalues. Since the case of geometric multiplicity is in general (maybe almost exclusively) due to the structure's geometry, such zero-step solution would adequately give eigenpair estimates to start the modified Jacobi-Davidson algorithm (Dumont, 2007) for damping-free problems and subsequently, if required, for the final problem with damping. Since the frequency power series outlined above must have converging properties for the mechanical problem to be a feasible one, the modified Jacobi-Davidson schemes shall converge cubically and precisely, with no sensible round-off errors.

(2) Another possibility is the use of a robust solver for a certain number of roots of interest of the highly nonlinear characteristic polynomial given by the determinant of either Eq. (16) or (17). Then, once an eigenvalue is known, the evaluation of the corresponding eigenvector takes only one iteration step, as shown in Dumont (2007). This procedure works fine for not too large problems, when all eigenvalues can be just evaluated and the ones of interest selected. There may be optimal strategies to find the roots of such nonlinear characteristic polynomials in the case of very large problems. However, this goes beyond this paper's scope.

Algorithm description

For not too large problems, the Fortran execution module developed in Dumont (2007) works fine – with no round-off errors – for the eigenvalue problem stated in terms of frequency power series, as given in Eqs. (20)-(25). For the case of an exact modal analysis, the structure's *effective* stiffness matrix must be assembled such as using the one of Eq. (3) for a truss element, and we may solve the characteristic polynomial using some built-in command for the whole set – or a subset – of eigenvalues, then selecting the ones whose real part are the smallest. In a second and final step, we call the procedure given by the pseudocode below as Algorithm 1 for the problem with damping, to find in just one iteration step the corresponding eigenvectors, which come out normalized. In this one-step algorithm we only need the expressions of $\mathbf{M}(\omega_r, \zeta)$ and $\mathbf{K}_{eff}(\omega_r, \zeta)$, which are explicitly given in the particular case of truss elements by Eqs. (13) and (3): the explicit expression of $\mathbf{K}(\omega, \zeta)$ is not required as well as we do not need the expression of $\mathbf{M}(\omega_r, \omega_s, \zeta_r, \zeta_s)$ in Eq. (15).

General modal time solution

For the sake of simplicity, we are not considering the effect of body forces in the following developments. Let the eigenpairs (ω_r, ϕ_r) of interest that come out of the solution of the eigenvalue problem outlined above be grouped as (Ω, Φ), where Ω is diagonal, the problem's spectrum, and Φ is the modal matrix. Whether these eigenpairs come from a truncated or an exact solution, they enter in the final outline of the time-dependent nodal solution of the problem in the same way as obtained by Dumont (2007). The time-dependent nodal displacement solution of the discretized structure is given as

$$\mathbf{d}(t) = \Phi \eta(t) \tag{26}$$

where $\eta(t)$ is a vector with the solutions $\eta_r(t)$ of the uncoupled differential equations, as for the eigenpair (ω_r, ϕ_r).

Algorithm 1 One-step general nonlinear complex-symmetric eigenvalue algorithm for pre-evaluated eigenvalues

Define a constant γ of the same order of a typical element of the stiffness matrix (to minimize round-off errors)
for r from 1 to up to N (or less) **do**
(1.1) Input the already evaluated eigenvalue ω_r and the eigenvector estimate $\phi_r = \langle 1 \quad 1 \quad \ldots \quad 1 \rangle^{\mathrm{T}}$
(1.2) Evaluate $\tilde{\mathbf{M}} \leftarrow \mathbf{M}(\omega_r, \zeta)$ after the element mass matrix such as in Eq. (13) assembled for the whole structure
(1.3) Create the biorthogonal base vector $\mathbf{b} =$ and normalize: $\mathbf{b} \leftarrow \mathbf{b}/\sqrt{\mathbf{b}^{\mathrm{T}}\mathbf{b}}$
(2.1) Evaluate $\tilde{\mathbf{K}}_{eff} \leftarrow \mathbf{K}_{eff}(\omega_r, \zeta)$ after assembling the element *effective* stiffness matrix such as in Eq. (3)
(2.2) Evaluate the residual vector $\mathbf{r} = \tilde{\mathbf{K}}_{eff}\phi_r$
(2.3) Evaluate the orthogonal eigenvector increment $\delta\phi$, that is, such that $\mathbf{b}\mathbf{b}^{\mathrm{T}}\delta\phi = \mathbf{0}$:
Project $\tilde{\mathbf{K}}_{eff} \leftarrow \tilde{\mathbf{K}}_{eff} + \left(\gamma - \mathbf{b}^{\mathrm{T}}\tilde{\mathbf{K}}_{eff}\mathbf{b}\right)\mathbf{b}\mathbf{b}^{\mathrm{T}}$
Solve $\tilde{\mathbf{K}}_{eff}\delta\phi = -\mathbf{r}$
Project $\delta\phi \leftarrow \delta\phi - \mathbf{b}\left(\mathbf{b}^{\mathrm{T}}\delta\phi\right)$
(3.1) Update $\phi_r \leftarrow \phi_r + \delta\phi$
(3.2) Normalize $\phi_r \leftarrow \phi_r/\sqrt{\phi_r^{\mathrm{T}}\tilde{\mathbf{M}}\phi_r}$
end for

Damping-free problems

For a damping-free problem,

$$\omega_r^2\eta_r(t) + \ddot{\eta}_r(t) = \phi_r^{\mathrm{T}}\mathbf{p}(t) \quad \Rightarrow \quad \eta_r(t) = \eta_r(0)\cos\omega_r t + \dot{\eta}(0)\frac{\sin\omega_r t}{\omega_r} + \frac{1}{\omega_r}\phi_r^{\mathrm{T}}\int_0^t \mathbf{p}(\tau)\sin\omega_r(t-\tau)\mathrm{d}\tau \tag{27}$$

In the case of nonhomogeneous initial conditions, the vectors of initial amplitudes $\eta(0)$ and $\dot{\eta}(0)$ are related to the nodal initial conditions $\mathbf{d}(0)$ and $\dot{\mathbf{d}}(0)$ by (Dumont, 2007,2009)

$$\eta_{(el)} = \left[\Phi_{(el)}^{\mathrm{T}}\mathbf{K}_0\Phi_{(el)}\right]^{-1}\Phi_{(el)}^{\mathrm{T}}\mathbf{K}_0\mathbf{d}, \quad \eta_{(rig)} = \Phi_{(rig)}^{\mathrm{T}}\mathbf{M}_1\mathbf{d} \tag{28}$$

where eigenpair subsets $(\Omega_{(el)}, \Phi_{(el)})$ related to elastic deformation are dealt with differently than subsets $(\Omega_{(rig)}, \Phi_{(rig)})$ related to rigid body displacements. The expression given in the literature only applies to the one-term truncated eigenvalue developments of Eq. (20).

Solution considering damping

When damping is included,

$$\omega_r\eta_r(t) - i\dot{\eta}_r(t) = \phi_r^{\mathrm{T}}\mathbf{p}(t) \quad \Rightarrow \quad \eta_r(t) = \mathrm{e}^{-i\omega_r t}\left(\eta_r(0) + i\phi_r^{\mathrm{T}}\int_0^t \mathbf{p}(\tau)\mathrm{e}^{i\omega_r\tau}\mathrm{d}\tau\right) \tag{29}$$

In the case of underdamping, the existence of an eigenpair (ω_r, ϕ_r) implies a second eigenpair $(-\overline{\omega}_r, \pm i\overline{\phi}_r)$. Then, a solution η_r of Eq. (29) implies a solution $\mp i\overline{\eta}_r$ of the associated equation and we obtain that the whole set of associated eigenpairs of interest corresponding to underdamping contributes to the nodal displacements as

$$\mathbf{d} = \Phi\eta + \bar{\Phi}\bar{\eta} \equiv 2\Re(\Phi\eta) \tag{30}$$

In the case of nonhomogeneous initial conditions, the vector of initial amplitudes $\eta(0)$ of Eq. (29) is obtained from the following expression, which is equivalent to, although simpler than the one proposed in Dumont (2009):

$$\begin{bmatrix} \Phi^{\mathrm{T}}\Phi & \Phi^{\mathrm{T}}\bar{\Phi} \\ \Omega\Phi^{\mathrm{T}}\Phi\Omega & -\Omega\Phi^{\mathrm{T}}\bar{\Phi}\bar{\Omega} \end{bmatrix}\begin{Bmatrix} \eta \\ \bar{\eta} \end{Bmatrix} = \begin{bmatrix} \Phi^{\mathrm{T}}\mathbf{d} \\ i\Omega\Phi^{\mathrm{T}}\dot{\mathbf{d}} \end{bmatrix} \tag{31}$$

Although not shown, the case of overdamping can also be addressed in the present general framework.

A SIMPLE NUMERICAL ILLUSTRATION

We show a very simple problem – to serve as just a proof of concept. On the top of Fig. 1 is illustrated a fixed-free bar submitted to a step force $\mathcal{H}(t)P$, $t>0$ at the free extremity, where $\mathcal{H}(t)$ is the Heaviside function of time t and $P=1000$ (unity of force). In consistent unities, the bar has length $L=1$, cross-section area $A=1$, elasticity modulus $E=1000$, specific mass $\rho=1$ and longitudinal viscous damping $\zeta=5$. We obtain from the exact expression of Eq. (3) that the eigenfrequency solutions ω_n of this problem are

$$\cos(k_nL)=0 \quad\Rightarrow\quad k_n=(2n+1)\pi/L \quad\Rightarrow\quad \omega_n=\pm\sqrt{(ck_n/2)^2-\zeta^2}-i\zeta,\ n=0,1,2,3,4 \tag{32}$$

where $c=\sqrt{E/\rho}$ is the wave propagation speed. They are shown in the first and fourth columns of Tab. 1 for the solutions ω_n of interest in the present illustration with five degrees of freedom – which are also the solutions for the assembled exact *effective* stiffness matrix independently from number and length of elements with which the structure may be discretized.

As implemented for a general problem, they are found from the characteristic polynomial of the assembled exact *effective* stiffness matrix, according to the second strategy proposed in the last Section. For the damping-free problem in terms of frequency-series developments, we do not need to find ω_n beforehand, as it is much simpler to just use the Fortran package made available by Dumont (2007) – provided that round-off errors do not occur, as for this simple example.

Percentage error results are also given in the Table for the series developments of Eqs. (20) and (21) with $n=3$ and $n=1$ (damping $\mathbf{C}_j$ and mass $\mathbf{M}_j$) terms – the latter case corresponding to the classical structural dynamics approach. Both real and imaginary parts of the approximate solutions are always greater than in the exact solutions.

Table 1 – Eigenfrequencies obtained exactly for the five degree-of-freedom truss on the top of Fig. 1 considering either no damping (left) or damping with $\zeta=5$ (right), and percentage errors when using truncated series expansions.

Exact	% error (3 terms)	% error (1 term)	Exact	% error (3 terms)	% error (1 term)
49.67294	0.00004	0.41173	$\pm 49.42065-5i$	0.00005	0.43021
149.01882	0.02585	3.72947	$\pm 148.93492-5i$	0.02650	3.74766
248.36471	0.43797	10.26578	$\pm 248.31437-5i$	0.44132	10.28236
347.71059	2.34747	18.10634	$\pm 347.67464-5i$	2.35439	18.11876
447.05647	5.13814	18.15228	$\pm 447.02851-5i$	5.14265	18.15829

The reference displacement solution for this fixed-free bar problem is quite accurately given as the 80-term series

$$u(x,t)=\frac{8P}{\rho LA}\sum_{n=0}^{80}\frac{(-1)^n}{(k_nc)^2}\left[1-\mathrm{e}^{-\zeta t}\left(\cos\sqrt{(ck_n/2)^2-\zeta^2}t+\zeta\frac{\sin\sqrt{(ck_n/2)^2-\zeta^2}t}{\sqrt{(ck_n/2)^2-\zeta^2}}\right)\right]\sin\frac{k_nx}{2} \tag{33}$$

and the corresponding nodal solutions of Eq. (30) are given for amplitudes η_r in Eq. (29) evaluated as

$$\eta_r(t)=\frac{\phi_r^{\mathrm{T}}\mathbf{p}}{\omega_r}(1-\mathrm{e}^{-i\omega_rt})+\mathrm{e}^{-i\omega_rt}\eta_r(0),\quad \eta_r(0)=0 \tag{34}$$

where $\mathbf{p}$ is a vector of applied nodal forces with zero entries except for the node with the prescribed force $P=1000$. Both reference and modal analysis solutions above are implied for initial homogeneous conditions $u(x,0)=0$ and $\mathbf{d}(0)=\mathbf{0}$.

All three graphs of Fig. 1 show as horizontal lines the displacement value $d_2=0.2$ of node 2 corresponding to the applied load considered as static, a value to which $d_2(t)\equiv u(0.2,t)$ must converge when the time tends to infinite. The solid lines in all three graphs are the time-dependent displacement at node 2 for $0<t\le 0.6s$ obtained with Eq. (33), here used as the target solution. The dash line in the top graph is the result of the modal analysis according to Eq. (30) for the eigenvalue problem solved for Eq. (21) truncated after $n=1$, as in the classical structural dynamics expression $\mathbf{K}_0-i\omega\mathbf{C}_1(\zeta)-\omega^2\mathbf{M}_1(\zeta)$: since this cannot be deemed accurate enough, one‘s first impulse might be to increase the mesh discretization in the attempt of getting some numerical improvement. On the other hand, the dash line in the middle graph, for $n=3$ in Eq. (21), shows that results can be significantly improved even with the proposed coarse mesh. Finally, the dash line in the bottom graph is the result with the eigenvalue problem solved exactly, which is the best possible solution we may achieve with this mesh discretization. We daresay that no other frequency or time-domain method can deliver better results than the latter ones, which are in fact just exact for the considered mesh discretization and initial conditions.

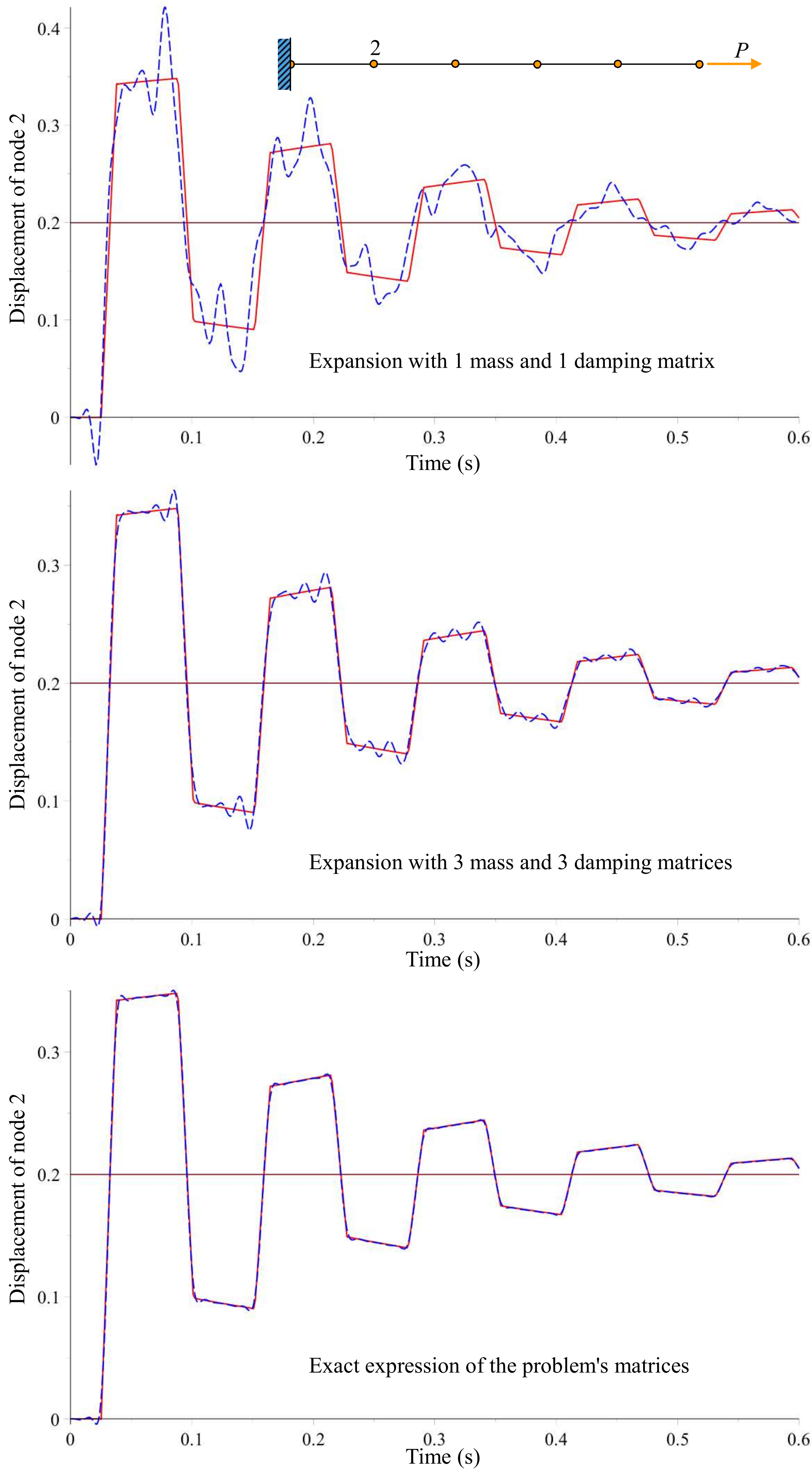

Figure 1 – Three numerical cases of axial displacements of node 2 for the fixed-free five-degree of freedom truss structure given on the top submitted to a constant force at the free extremity.

How to still improve the numerical solution

We have just asserted that the above modal analysis is exact. However, there is still room for improvement, as the problem is based on a spatial discretization and the evaluated modes – exact for the considered mesh discretization – are possibly unnecessarily applied to represent in a substantial way the linear, steady solution $u(x,t \to \infty) = Px/EA$ of the problem we are dealing with, indicated in Fig. 1 as $u(0.2,t \to \infty) = 0.2$ for node 2.

If we are interested in results for a short period of time after load application, the previous developments represented in Fig. 1 are just fine. However, if we are concerned with results for large values of time, we may obtain errors that are in principle not expected. To illustrate that, Fig. 2 presents the same developments of Fig. 1, as already described, but for a relatively long time $1.5s \le t \le 2.0s$ after load application. We observe that the results with one mass and one damping matrix are adequate whereas the results with more expansion terms as well the ones related to the exact modal analysis do not seem to converge well to the static solution. In fact, such errors become more perceptible if we measure displacements at the truss's free extremity.

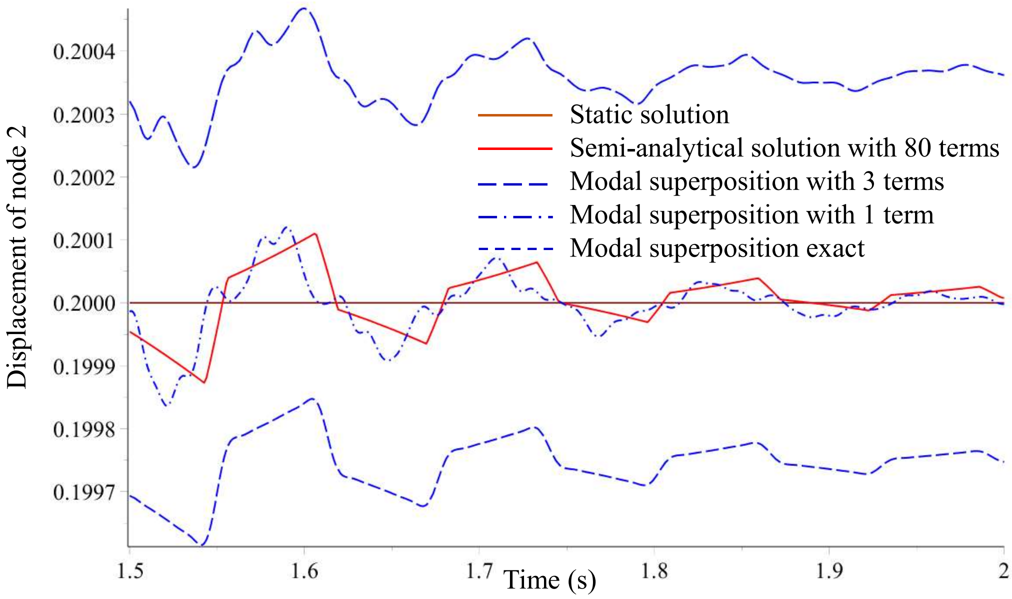

Figure 2 – Numerical cases of Fig. 1 grouped to show that the high-order and exact modal analyses with homogeneous initial conditions do not strictly converge to the static solution as $t \to 0$.

A means to overcome this inconvenience is to use the modal analysis to represent only the transient part of the mechanical phenomenon

$$u(x,t,p)|_{u_0=0} = u_{st}(x,p) - u(x,t,0)|_{u_0=u_{st}(x,p)} \tag{35}$$

where $u_{st}(x,p)$ is the static response, in the case $u(x,t \to \infty) = Px/EA$, and $u(x,t,0)|_{u_0=u_{st}(x,p)}$ is the response – to be subtracted – obtained by imposing the initial condition $\mathbf{d}(t=0)$ given as $u(x,t,0) = Px/EA$ evaluated at the nodes. Then, Eq. (34) is solved for the (five) modes of interest using $\mathbf{p}(t=0)$ and the vector of initial mode amplitudes $\eta(0)$ given by Eq. (31), where, in this case, $\dot{\mathbf{d}}(t=0) = \mathbf{0}$. The response for such a transient analysis is illustrated in Fig. 3 for $2.0s \le t \le 2.5s$, which is on purpose shifted in relation to the time span of Fig. 2 in order to make evident that the target solution given by Eq. (33) with 80 terms is now the most inaccurate one – albeit slightly. As a matter of fact, the traditional structural dynamics approach based on one term of the series given by Eq. (21) is shown to be sufficiently accurate for large values of t.

The solutions obtained in terms of the superposition indicated in Eq. (35) are checked in Fig. 4 against the solutions according to Eq. (34), which had been previously shown in Fig. 1 for a larger time span after load application. Whereas there is no sensible difference for the solutions with one term of the series in Eq. (21), maybe because they are already sufficiently inaccurate for $t \to 0$, we see by the indicated dot lines in Fig. 4 that Eq. (35) leads to less accurate results immediately after load application, which is understandable as in the latter approach the modes are used to simulate initially large oscillations about the static response.

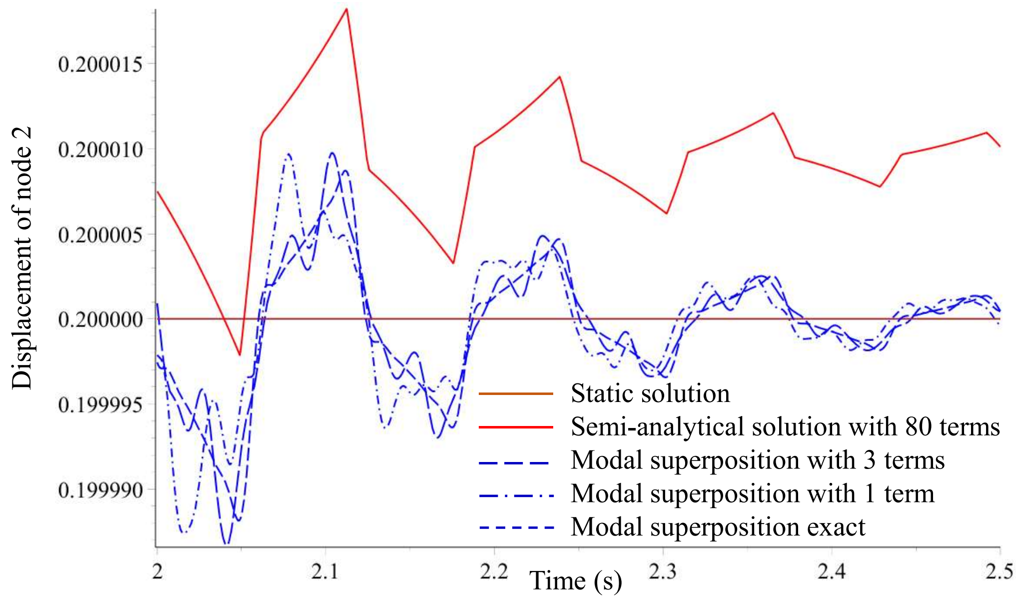

Figure 3 – Numerical cases of Fig. 1 grouped to show that all modal analysis simulations converge to the static solution as $t \to 0$ if the superposition of Eq. (35) is used.

CONCLUSIONS

Our main goal was to present a proof of concept, namely, that an *exact modal analysis of time-dependent problems* is possible. We have chosen as illustration the one-dimensional, simple truss element in order not to deviate from the relevant mechanical and linear algebra aspects. As a matter of fact, Dumont et al. (2019) show that two- or three-dimensional truss and frame structures must be dealt with exactly in the same way, as lateral, thus bending, inertia effect takes place and plays an important role, in general, whether or not the structural elements are connected by hinges. Our developments have evolved from the classical structural dynamics modal analysis, which has been shown in the author's most representative contribution (Dumont, 2007) to be just the first-order truncation of a more general and – for some applications – by far more accurate approach, as shown in this paper's simple numerical illustration: just compare the eigenfrequencies listed in Table 1 for the different approaches. We have recently realized that the existence, convergence and perturbation theorems developed by Dumont (2007) for the frequency power series approach are equally applicable to a formulation in terms of the exact expression $[\mathbf{K}(\omega,\zeta) - \omega\mathbf{M}(\omega,\zeta)]\phi = \mathbf{0}$ of Eq. (17) whenever it is available. This is the case for truss and beam elements, among other simple models (Sales, 2022), although it seems possible to apply the formulation also when these matrices can be obtained only numerically – albeit exactly within machine precision. The adequate application of non-homogeneous initial conditions – particularly for problems with viscous damping – has shown to be an important resource depending on our analysis' purpose. The power series expansion is relatively precise and economical in terms of computational effort but suffers from round-off errors, since the developed matrices tend to become singular, as shown in the Appendix for this simple truss example. Not least, it should be remarked that such exact modal analysis comes with the price of more computational effort, which must be better assessed. We have discussed that more advanced solvers are required and this is the subject of ongoing developments.

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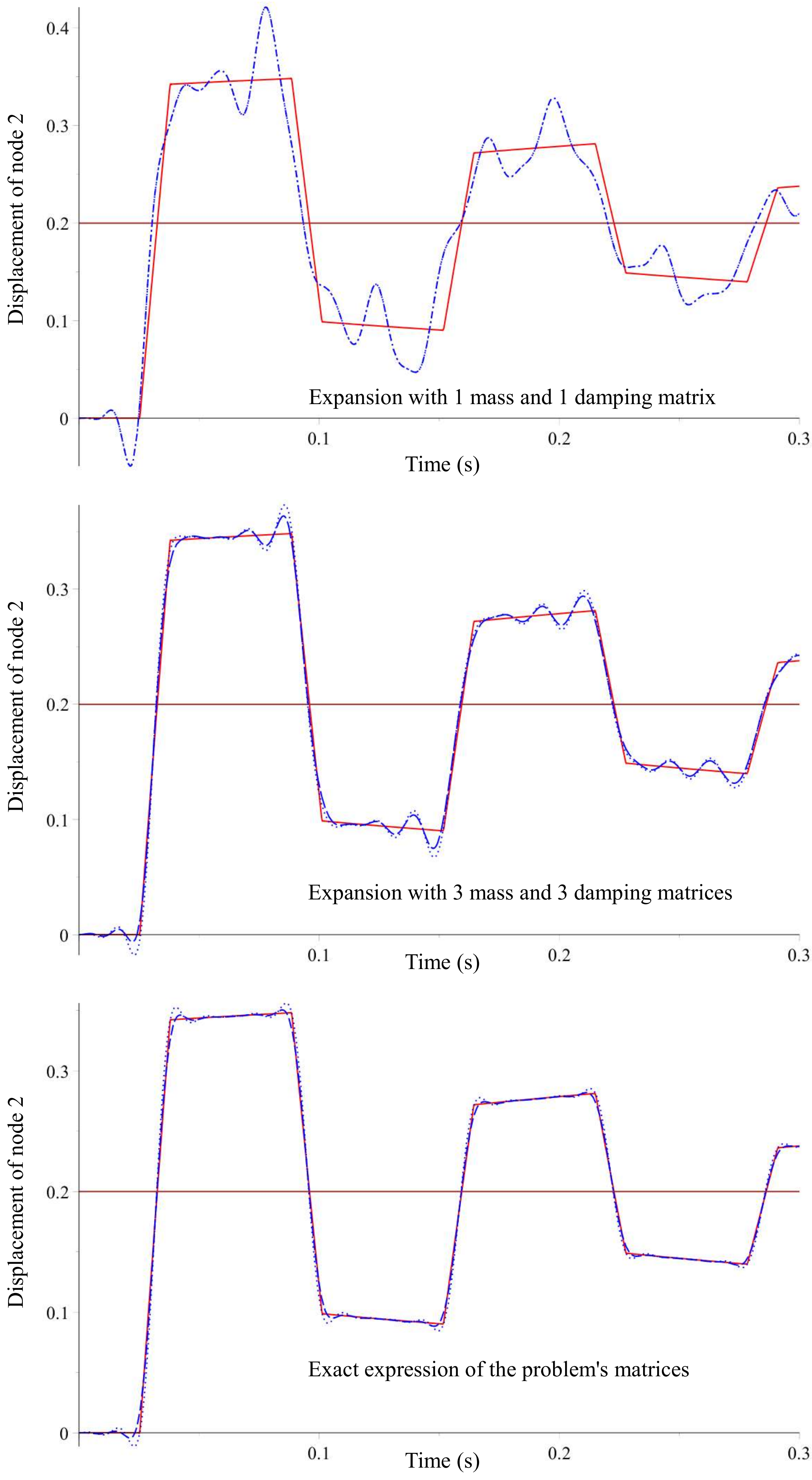

Figure 4 – Numerical cases of Fig. 1 compared with results (blue dots) according to Eq. (35) to show that high-order and exact modal analyses for non-homogeneous initial conditions are slightly less accurate close to $t = 0$.

APPENDIX

The following developments are an adapted and partially expanded version of similar developments available in Dumont (2007).

For a damping-free problem, the frequency power series expansion of the effective stiffness matrix of Eq. (3) is

$$\mathbf{K}_{eff}\left(\omega^2\right) = \frac{EA}{\ell}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\omega^2\ell^2}{6c^2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \frac{\omega^4\ell^4}{360c^4}\begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} - \frac{\omega^6\ell^6}{15120c^6}\begin{bmatrix} 32 & 31 \\ 31 & 32 \end{bmatrix}\right) + O(\omega^8) \quad (36)$$

where $c = \sqrt{E/\rho}$ is the wave propagation velocity through the elastic medium. The corresponding frequency-dependent mass and stiffness matrices, as developed in Eqs. (4) and (5) – and coinciding with the expansions of Przemieniecki's equations – are

$$\mathbf{M}\left(\omega^2\right) = \frac{EA}{\ell}\left(\frac{\ell^2}{6c^2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{\omega^2\ell^4}{180c^4}\begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} + \frac{\omega^4\ell^6}{5040c^6}\begin{bmatrix} 32 & 31 \\ 31 & 32 \end{bmatrix}\right) + O(\omega^6) \quad (37)$$

$$\mathbf{K}\left(\omega^2\right) = \frac{EA}{\ell}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{\omega^4\ell^4}{360c^4}\begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} + \frac{\omega^6\ell^6}{7560c^6}\begin{bmatrix} 32 & 31 \\ 31 & 32 \end{bmatrix}\right) + O(\omega^8) \quad (38)$$

In the case of viscous damping, we obtain the frequency power series expansion of the effective stiffness matrix of Eq. (3):

$$\begin{aligned}\mathbf{K}_{eff}(\omega,\zeta) = \tfrac{EA}{\ell}\Bigg(&\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \tfrac{i\omega\ell\alpha}{c}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} - \tfrac{\omega^2\ell^2}{c^2}\begin{bmatrix} \frac{15-4\alpha^2}{45} & \frac{15-7\alpha^2}{90} \\ \frac{15-7\alpha^2}{90} & \frac{15-4\alpha^2}{45} \end{bmatrix} \\ &- \tfrac{i\omega^3\ell^3\alpha}{c^3}\begin{bmatrix} \frac{4(21-4\alpha^2)}{945} & \frac{147-31\alpha^2}{1890} \\ \frac{147-31\alpha^2}{1890} & \frac{4(21-4\alpha^2)}{945} \end{bmatrix} - \tfrac{\omega^4\ell^4}{c^4}\begin{bmatrix} \frac{16\alpha^4-120\alpha^2+105}{4725} & \frac{127\alpha^4-930\alpha^2+735}{37800} \\ \frac{127\alpha^4-930\alpha^2+735}{37800} & \frac{16\alpha^4-120\alpha^2+105}{4725} \end{bmatrix}\Bigg) + O(\omega^5)\end{aligned} \quad (39)$$

where ζ is replaced with $\alpha c/\ell$, in terms of a nondimensional viscosity parameter α, to simplify notation. The corresponding expansions of the frequency-dependent mass and stiffness matrices, as introduced in Eqs. (13) and (14), are

$$\begin{aligned}\mathbf{M}(\omega,\zeta) = \tfrac{EA}{\ell}\Bigg(&\tfrac{i\ell\alpha}{c}\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} + \tfrac{2\omega\ell^2}{c^2}\begin{bmatrix} \frac{15-4\alpha^2}{45} & \frac{15-7\alpha^2}{90} \\ \frac{15-7\alpha^2}{90} & \frac{15-4\alpha^2}{45} \end{bmatrix} + \tfrac{i\omega^2\ell^3\alpha}{c^3}\begin{bmatrix} \frac{4(21-4\alpha^2)}{315} & \frac{147-31\alpha^2}{630} \\ \frac{147-31\alpha^2}{630} & \frac{4(21-4\alpha^2)}{315} \end{bmatrix} \\ &+ \tfrac{\omega^3\ell^4}{c^4}\begin{bmatrix} \frac{64\alpha^4-480\alpha^2+420}{4725} & \frac{127\alpha^4-930\alpha^2+735}{9450} \\ \frac{127\alpha^4-930\alpha^2+735}{9450} & \frac{64\alpha^4-480\alpha^2+420}{4725} \end{bmatrix}\Bigg) + O(\omega^5)\end{aligned} \quad (40)$$

$$\begin{aligned}\mathbf{K}(\omega,\zeta) = \tfrac{EA}{\ell}\Bigg(&\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \tfrac{\omega^2\ell^2}{c^2}\begin{bmatrix} \frac{15-4\alpha^2}{45} & \frac{15-7\alpha^2}{90} \\ \frac{15-7\alpha^2}{90} & \frac{15-4\alpha^2}{45} \end{bmatrix} + \tfrac{i\omega^3\ell^3\alpha}{c^3}\begin{bmatrix} \frac{8(21-4\alpha^2)}{945} & \frac{147-31\alpha^2}{945} \\ \frac{147-31\alpha^2}{945} & \frac{8(21-4\alpha^2)}{945} \end{bmatrix} \\ &+ \tfrac{\omega^4\ell^4}{c^4}\begin{bmatrix} \frac{16\alpha^4-120\alpha^2+105}{1575} & \frac{127\alpha^4-930\alpha^2+735}{12600} \\ \frac{127\alpha^4-930\alpha^2+735}{12600} & \frac{16\alpha^4-120\alpha^2+105}{1575} \end{bmatrix}\Bigg) + O(\omega^5)\end{aligned} \quad (41)$$

All matrix terms in the expansions for the undamped problem are positive definite, although they tend to become singular as the power of ω increases. In the case of damping, however, only the matrix $\mathbf{C}_1$ – see Eq. (21) – is unconditionally positive definite, since the remaining matrices' condition depends strictly speaking on the magnitude of α an they also tend to become singular as the power of ω goes to infinite. The above expansions also show that whereas $\mathbf{K}_{eff}(\omega,\zeta) \to \mathbf{K}_{eff}(\omega^2)$ as $\zeta \to 0$ the same transition does not happen to $\mathbf{M}(\omega,\zeta)$ and $\mathbf{K}(\omega,\zeta)$, which we already infer from the general expressions of Eqs. (22)-(25).

The convergence of the series expansions above depend on the magnitude of $\ell\omega/c \equiv \ell/\lambda$, where λ is the wave length. Such convergence issue does not occur when the modal analysis is based on the exact expressions of either Eqs. (4) and (5) or (13) and (14), the latter ones when viscous damping is considered.