

## MECSOL 2022 - The arc-length method utilized for nonlinear frequency response: an improved radius control

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*Abstract: One of the most difficult challenges of computational structural mechanics is solving nonlinear problems. This is the reason why several methods have been developed over the last few decades to deal with these more complex mathematical models. Most of these approaches are based on the Newton-Raphson method. However, some of the nonlinear problems present inflection points, resulting in difficult paths in the load/displacement space, which such algorithms fail to determine precisely. The arc-length method is capable of solving problems with limit points, which is a common occurrence in snap-back and snap-through phenomena. This method imposes an additional constraint to the system of nonlinear equations of a given problem to favor convergence and overcome the limit points. One of the challenges of the method is the radius control of the additional constraint. It represents a key factor for the algorithm to be successful and efficient, therefore different approaches have been studied over the years. Although the arc-length method has been mainly used to determine load/displacement paths, it can also be used to determine frequency response functions (FRFs). This study aims to determine nonlinear FRFs using the arc-length method and propose modifications to the radius control in order to increase performance. Initially, the traditional arc-length algorithm is used to determine FRFs with classic nonlinearities such as cubic stiffness and gap nonlinearity. Later the proposed modifications to the radius control are introduced. The changes proved to be relevant, since the number of solution points to correctly determine the FRF is greatly reduced, increasing computing efficiency.*

**Keywords:** frequency response function, nonlinearity, arc-length

### INTRODUCTION

Solving nonlinear problems has continually been subject of research regarding structural analysis. One of the most used approaches to solve these problems is the Newton-Raphson method, notably due to its fast convergence. However, this method fails to follow the equilibrium path for unstable structures that present critical points, such as buckling analysis, for example. The arc-length method is a technique capable of correctly following the equilibrium path when these difficulties are present.

Many works have been developed using the arc-length method in the time domain. However, given that frequency response functions of nonlinear dynamic systems under harmonic excitation also present critical points with a severe change of direction, the arc-length method is also suited to solve these problems. Lewandowski (1992) used the arc-length method to determine the steady-state vibration response of nonlinear beams.

This study applied the classic arc-length method to determine FRFs of systems with common nonlinearities, such as cubic stiffness and gap function. The describing function method was used to compute the effects of the nonlinearities in the frequency domain, assuming that the response of the system is almost entirely dominated by its fundamental harmonic.

Later, the radius control aspect of the arc-length method was studied. The radius of the additional constraint imposed by the method is crucial for its accomplishment. A value set excessively high might lead to failure in determining the right equilibrium path, while the opposite might be too resource consuming.

The proposed radius control induces subtle changes to the arc-length radius from one solution point to another, when compared to classical radius control. In the context of nonlinear FRFs, the proposed control proved to be overall more efficient, since the algorithm needs less solution points than traditional methods to determine the same FRF, decreasing computing time.

### The arc-length method applied in nonlinear static structural analysis

The first concept of the arc-length method was developed by Riks (1979). It was later modified by Crisfield (1981), resulting in the classical arc-length method. The method adds a new constraint to the system of equations and doing so limits the next solution point to be calculated in the vicinity of the last converged point. This new constraint is responsible

for the method to be able to overcome critical points near instability zones. The geometric interpretation of the arc-length method is shown in Fig. 1.

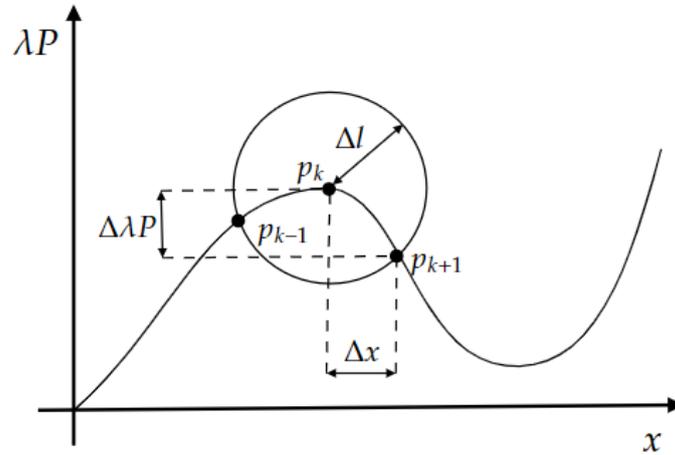


Figure 1 – Geometric interpretation of the arc-length method

Figure 1 shows the force-displacement solution path of a nonlinear static structural problem. The point  $p_k$  represents the last converged point. The circle around  $p_k$  is the additional constraint imposed by the arc-length method. The point  $p_{k+1}$  is the next solution point to be calculated. One of the main drawbacks of the method is the fact that two points,  $p_{k-1}$  and  $p_{k+1}$ , are valid solutions to the system of equations and the additional constraint. Determination of the right solution point is a key factor for the method to be successful.

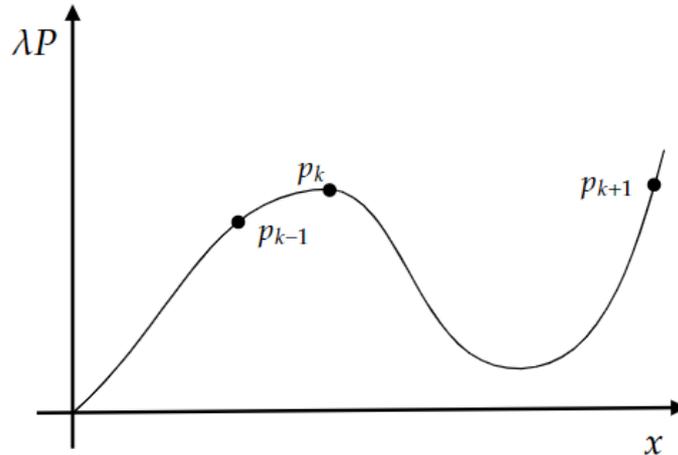


Figure 2 – Snap-Through phenomenon

Figure 2 shows the importance of the additional constraint. When using a more traditional approach like the Newton-Raphson method, it is common for the algorithm to neglect critical points of the equilibrium path. In this case, from point  $p_k$  the algorithm would then calculate point  $p_{k+1}$ , ignoring the instability zone between these two points. This phenomenon is known as Snap-Through.

A nonlinear static structural problem can be summarized by Eq. (1):

$$\mathbf{f}_i(\mathbf{x}) = \lambda \mathbf{P} \quad (1)$$

where the external load vector is represented by  $\mathbf{P}$ , while  $\lambda$  is the load parameter and  $\mathbf{f}_i$  is the vector of internal forces, which is a function of the displacement vector  $\mathbf{x}$ .

The constraint equation of the arc-length method is given by Eq. (2):

$$\Delta \mathbf{x}^T \Delta \mathbf{x} + \Delta \lambda^2 \psi^2 \mathbf{P}^T \mathbf{P} = \Delta l^2 \quad (2)$$

Where  $\Delta l$  is the arc-length radius and  $\psi$  is a scaling parameter to adjust the scaling between load and displacement terms. Unlike in the Newton-Raphson method, in the arc-length method the load parameter  $\lambda$  is a variable, which is its main essence to overcome critical points.

The arc-length method consists of two phases. The first one is the prediction phase which determines the first estimation of the next solution point  $p_{k+1}$  and its direction regarding the last converged point  $p_k$ . The second one is the correction phase which uses the estimation of the prediction phase as input to an iterative procedure. After a few iterations this procedure converges to point  $p_{k+1}$ , which is the solution to Eq. (1) and Eq. (2).

## The arc-length method applied in nonlinear frequency response

Similar to some nonlinear static structural problems, nonlinear FRFs also present critical points that are difficult to determine. Ferreira and Serpa (2005) used the arc-length method to obtain the frequency response of some nonlinear problems adapting the existing method developed for structural analysis to the frequency domain problem.

### General equation

The equation of motion of a nonlinear oscillatory system subject to external harmonic forces can be described by Eq. (3):

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) + \mathbf{v}(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{F}_e(t) \quad (3)$$

where  $\mathbf{M}$  is the mass matrix,  $\mathbf{C}$  is the viscous damping matrix,  $\mathbf{K}$  is the stiffness matrix,  $\mathbf{v}$  is the vector of nonlinear internal forces,  $\mathbf{F}_e$  is the external harmonic force vector and  $\mathbf{x}$  is the displacement vector.

In the frequency domain, Eq. (3) can be written as Eq. (4):

$$[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K} + \boldsymbol{\Theta}(\mathbf{X})]\mathbf{X} = \mathbf{F} \quad (4)$$

where  $i$  is the imaginary unit,  $\omega$  is the excitation frequency,  $\mathbf{X}$  is the complex displacement amplitude,  $\mathbf{F}$  is the complex excitation amplitude, and  $\boldsymbol{\Theta}$  is the describing function matrix associated to  $\mathbf{v}(\mathbf{x}, \dot{\mathbf{x}})$  that takes into consideration the nonlinear internal forces.

When it comes to frequency response analysis, the excitation frequency is equivalent to the external load in static structural analysis. Therefore, to apply the arc-length method for frequency response analysis, the load parameter  $\lambda$  must be associated with  $\omega$ . This leads to Eq. (5):

$$[-\lambda^2\omega^2\mathbf{M} + i\lambda\omega\mathbf{C} + \mathbf{K} + \boldsymbol{\Theta}(\mathbf{X})]\mathbf{X} = \mathbf{F} \quad (5)$$

The arc-length constraint from Eq. (2) becomes Eq. (6):

$$\Delta\mathbf{X}^T\Delta\mathbf{X} + \Delta\lambda^2\psi^2\omega^2 = \Delta l^2 \quad (6)$$

In order to solve Eq. (5) and Eq. (6) the same technique from the classical nonlinear static structural analysis is applied. The prediction phase estimates the solution point, and the corrective phase converges to the desired solution point after a few iterations.

### Radius control

The radius  $\Delta l$  of the arc-length constraint plays a crucial role in the method. If the radius is too large, the constraint may intersect the equilibrium path at more than two points, and the method will fail to solve the problem. However, the computational cost might be very expensive if the radius is too small. Crisfield (1981) proposed the following technique to adapt the arc-length radius depending on the difficulty to achieve convergence:

$$\Delta l_{(k+1)} = \Delta l_{(k)} \frac{I_d}{I_{(k)}} \quad (7)$$

where  $\Delta l_{(k+1)}$  is the radius on the solution point  $k + 1$ ,  $\Delta l_{(k)}$  is the radius on the solution point  $k$ ,  $I_d$  is the desired number of iterations for convergence and  $I_{(k)}$  is the necessary number of iterations to achieve convergence on point  $k$ . When more than  $I_d$  iterations are needed to achieve convergence, the radius will be decreased for the next solution point, facilitating convergence. If fewer iterations are required, the radius will be increased, improving computational time.

### Describing functions

The describing function method is commonly used to solve nonlinear problems based on the assumption that the system response will be dominated by its fundamental frequency. The method developed by Krylov and Bogoliubov (1949) is based on the van der Pol (1927) method of slowly varying coefficients and the method of equivalent linearization proposed by Bogoliubov and Mitroposky (1944).

For a single-degree-of-freedom system, the describing function,  $\Theta$ , of a nonlinear function  $v(x, \dot{x})$  is obtained based on the assumption that the system response  $x(t)$  is close to a pure sinusoid as in Eq. (8):

$$x \approx X \sin(\omega t + \phi) = X \sin(\tau) \quad (8)$$

where  $X$  is the complex amplitude,  $\omega$  is the excitation frequency,  $\phi$  is the phase angle and  $\tau = \omega t + \phi$ . The nonlinear function can then be rewritten in terms of  $X$  and  $\tau$ , as it follows:

$$v(x, \dot{x}) \approx v(X, \tau) \quad (9)$$

The describing function  $\Theta$  is obtained from the first two terms of the Fourier series:

$$\Theta(X) = \frac{1}{\pi X} \int_0^{2\pi} v(X, \tau) \sin(\tau) d\tau + i \frac{1}{\pi X} \int_0^{2\pi} v(X, \tau) \cos(\tau) d\tau \quad (10)$$

## NUMERICAL TESTS

In this section, the arc-length method will be used to determine the frequency response function  $H_{ij}(\omega)$  of systems with nonlinearities. The frequency response function is determined by Eq. (11):

$$H_{ij}(\omega) = \frac{X_i}{F_j} \quad (11)$$

where  $X_i$  is the complex displacement amplitude of the  $i$ -th degree of freedom, and  $F_j$  is the complex excitation amplitude of the  $j$ -th degree of freedom.

For both tests, the scaling parameter  $\psi$  was set to 1 while the arc-length radius was controlled by Eq. (7) with  $I_d = 2$ .

### Cubic stiffness

The cubic stiffness function is represented by Eq. (12) in the time domain:

$$v(x, \dot{x}) = \beta x^3 \quad (12)$$

The describing function of the cubic stiffness function can be obtained using Eq. (10) and is given by Eq. (13):

$$\Theta(X) = \frac{3}{4} \beta X^2 \quad (13)$$

A multiple-degree-of-freedom system with cubic stiffness is shown in Fig. 3:

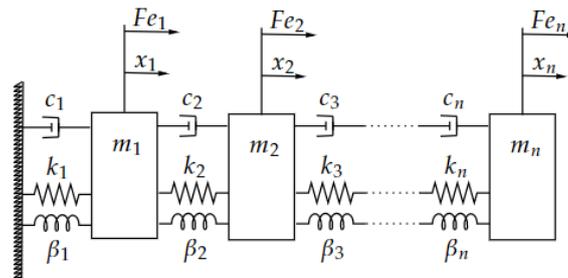


Figure 3 – Multiple-degree-of-freedom system with cubic stiffness

The arc-length method was used to solve the system shown in Fig. 3 with 6 degrees of freedom. The initial radius  $\Delta l_{(1)}$  was set to 1.9 mm. The physical parameters of the system are:  $m_1 = 1.5$  kg,  $m_2 = 1$  kg,  $m_3 = 1.5$  kg,  $m_4 = 1$  kg,  $m_5 = 2$  kg,  $m_6 = 1.5$  kg,  $c_1 = 2$  Ns/m,  $c_2 = 0.3$  Ns/m,  $c_3 = 0.08$  Ns/m,  $c_4 = 0.1$  Ns/m,  $c_5 = 0.1$  Ns/m,  $c_6 = 0.2$  Ns/m,  $k_1 = 3000$  N/m,  $k_2 = 2000$  N/m,  $k_3 = 4000$  N/m,  $k_4 = 3000$  N/m,  $k_5 = 2000$  N/m,  $k_6 = 3000$  N/m,  $\beta_1 = 2000$  N/m<sup>3</sup>,  $\beta_2 = 2000$  N/m<sup>3</sup>,  $\beta_3 = 2000$  N/m<sup>3</sup>,  $\beta_4 = 2000$  N/m<sup>3</sup>,  $\beta_5 = 2000$  N/m<sup>3</sup> and  $\beta_6 = 2000$  N/m<sup>3</sup>.

The external force is applied only in the first degree of freedom at three different levels:  $F_1 = 1$  N,  $F_1 = 50$  N and  $F_1 = 150$  N.  $F_2 = F_3 = F_4 = F_5 = F_6 = 0$  N.

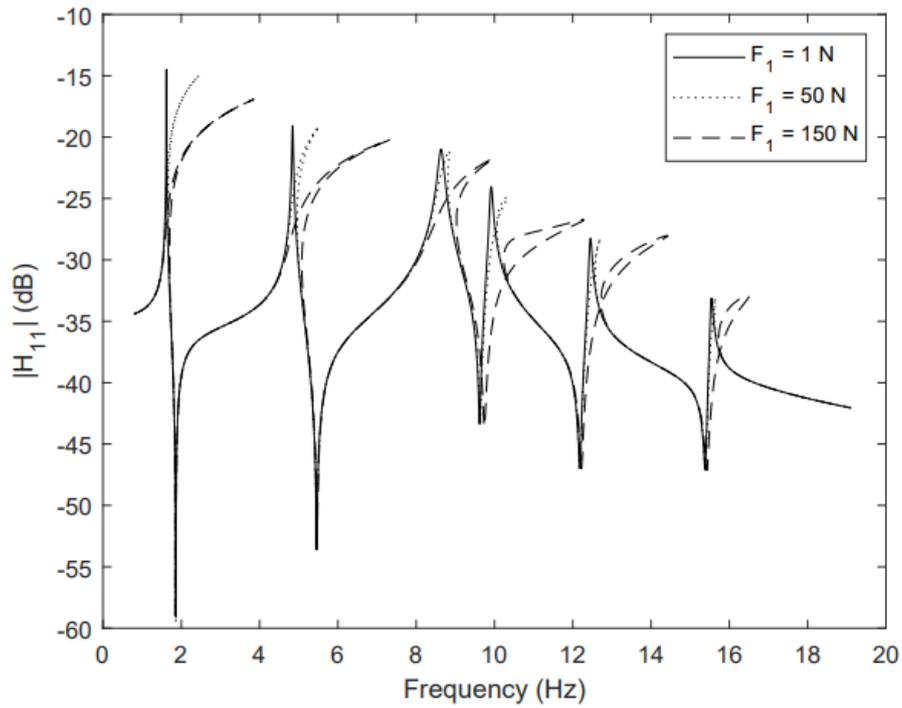


Figure 4 – Frequency response magnitude of the system with cubic stiffness

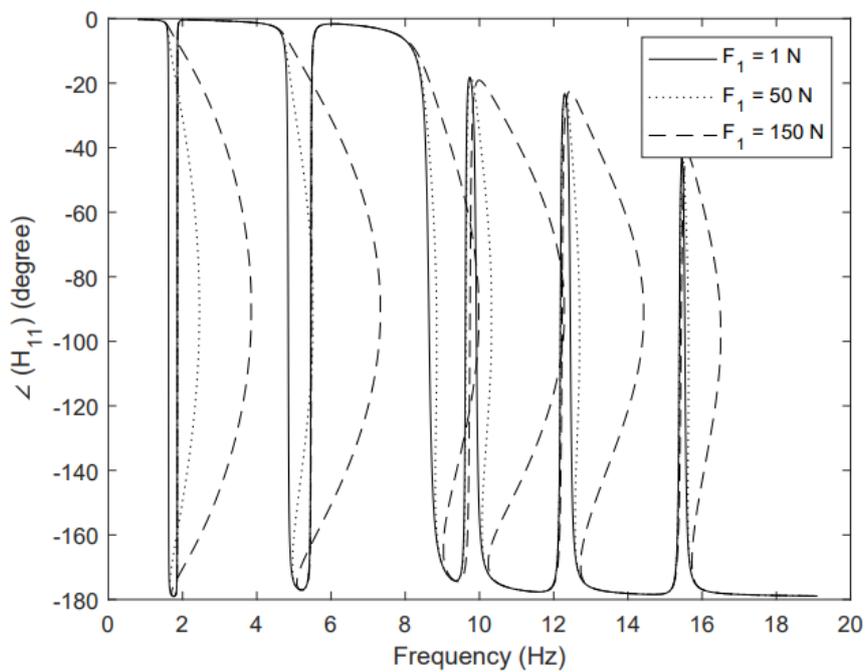


Figure 5 – Frequency response phase of the system with cubic stiffness

## Gap

The gap function is represented by Eq. (14) in the time domain:

$$v(x, \dot{x}) = \begin{cases} 0 & \text{if } |x| < b \\ k_g(x - b) & \text{if } x \geq b \\ k_g(x + b) & \text{if } x \leq -b \end{cases} \quad (14)$$

Figure 4 shows the plot of the respective function:

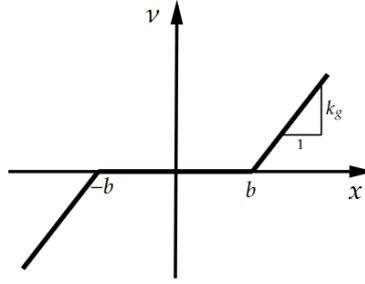


Figure 6 – Gap function

The describing function of the gap function can be obtained using Eq. (10) and is given by Eq. (15):

$$\Theta(X) = k_g \left[ 1 - \frac{2}{\pi} \left( \sin^{-1} \frac{b}{X} + \frac{b}{X} \sqrt{1 - \frac{b^2}{X^2}} \right) \right] \quad (15)$$

A multiple-degree-of-freedom system with gap nonlinearity is shown in Fig. 7:

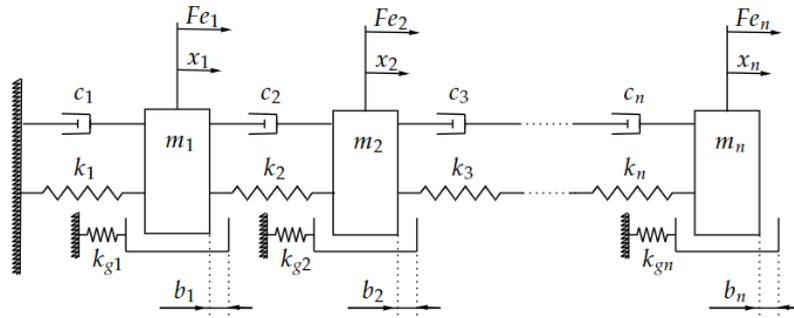


Figure 7 – Multiple-degree-of-freedom system with gaps

The arc-length method was used to solve the system shown in Fig. 7 with 6 degrees of freedom. The initial radius  $\Delta l_{(1)}$  was set to 0.93 mm. The physical parameters of the system are:  $m_1 = 2$  kg,  $m_2 = 1$  kg,  $m_3 = 1.5$  kg,  $m_4 = 1$  kg,  $m_5 = 2$  kg,  $m_6 = 1.5$  kg,  $c_1 = 2.5$  Ns/m,  $c_2 = 0.3$  Ns/m,  $c_3 = 0.08$  Ns/m,  $c_4 = 0.1$  Ns/m,  $c_5 = 0.1$  Ns/m,  $c_6 = 0.2$  Ns/m,  $k_1 = 5000$  N/m,  $k_2 = 2000$  N/m,  $k_3 = 4000$  N/m,  $k_4 = 3000$  N/m,  $k_5 = 2000$  N/m,  $k_6 = 5000$  N/m,  $k_{g1} = k_{g2} = k_{g3} = k_{g4} = k_{g5} = k_{g6} = 3000$  N/m,  $b_1 = 0.2$  m,  $b_2 = 0.4$  m,  $b_3 = 0.6$  m,  $b_4 = 0.3$  m,  $b_5 = 0.2$  m and  $b_6 = 0.2$  m.

The external force is applied only in the first degree of freedom at three different levels:  $F_1 = 1$  N,  $F_1 = 50$  N and  $F_1 = 150$  N.  $F_2 = F_3 = F_4 = F_5 = F_6 = 0$  N.

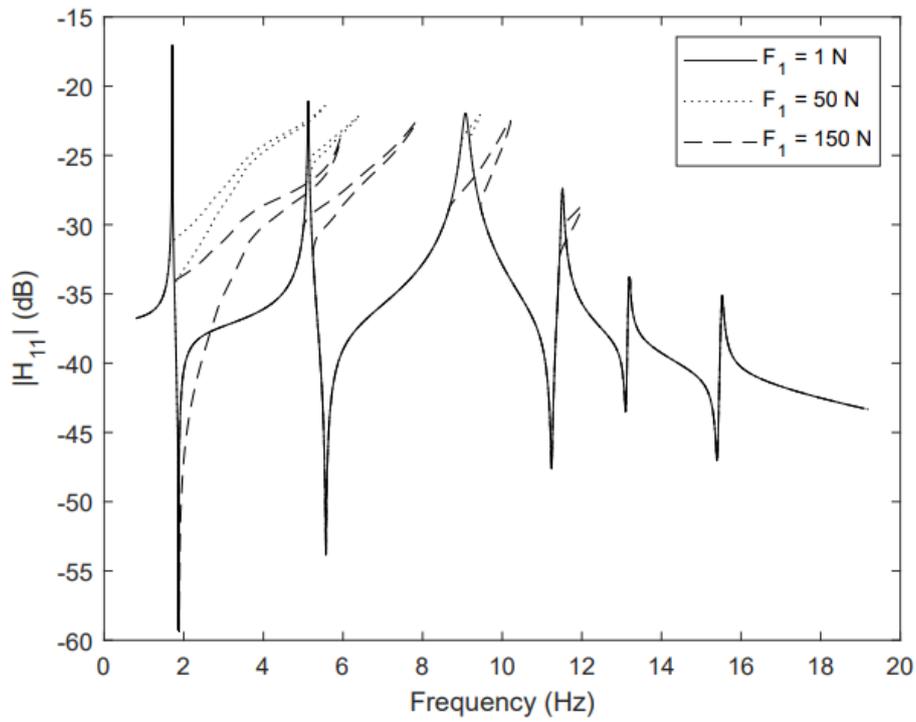


Figure 8 – Frequency response magnitude of the system with gaps

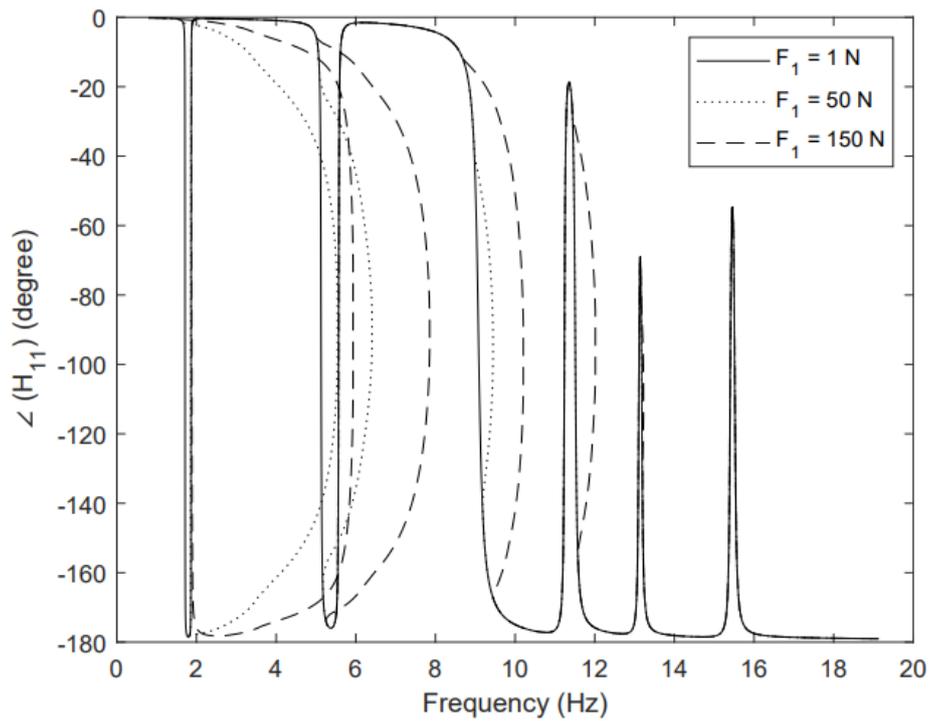


Figure 9 – Frequency response magnitude of the system with gaps

## IMPROVED RADIUS CONTROL

Even though the radius control from Eq. (7) works well for most cases, the way the arc-length radius is changed in the algorithm has always been studied. Ramm (1981) proposed a modification to Eq. (7), defining Eq. (16):

$$\Delta l_{(k+1)} = \Delta l_{(k)} \sqrt{\frac{I_d}{I_{(k)}}} \quad (16)$$

Similarly, Bellini and Chulya (1987) proposed another modification to Eq. (7), given by Eq. (17):

$$\Delta l_{(k+1)} = \Delta l_{(k)} \sqrt[4]{\frac{I_d}{I_{(k)}}} \quad (17)$$

In this study, another modification of Eq. (7) is proposed as follows:

$$\Delta l_{(k+1)} = \Delta l_{(k)} \left( \frac{I_d}{I_{(k)}} \right)^{\frac{1}{\varphi}} \quad (18)$$

It is important to note that the case where  $\varphi = 1$  corresponds to Eq. (7) originally proposed by Crisfield (1981). In the same way,  $\varphi = 2$  corresponds to Eq. (16) and  $\varphi = 4$  corresponds to Eq. (17). If  $\varphi$  tends to infinity, the arc-length radius tends to remain constant.

Different values to the parameter  $\varphi$  will be attributed in order to verify which one performs the best, when using the arc-length method to determine nonlinear frequency response functions. The results for the cubic stiffness example with  $F_1 = 150$  N are shown in Tab. 1. The parameter  $I_d$  was set to 2. The same curves from Fig. 4 and Fig. 5 were obtained for every value of  $\varphi$ .

**Table 1 – Influence of the parameter  $\varphi$  on the computational cost for the cubic stiffness example**

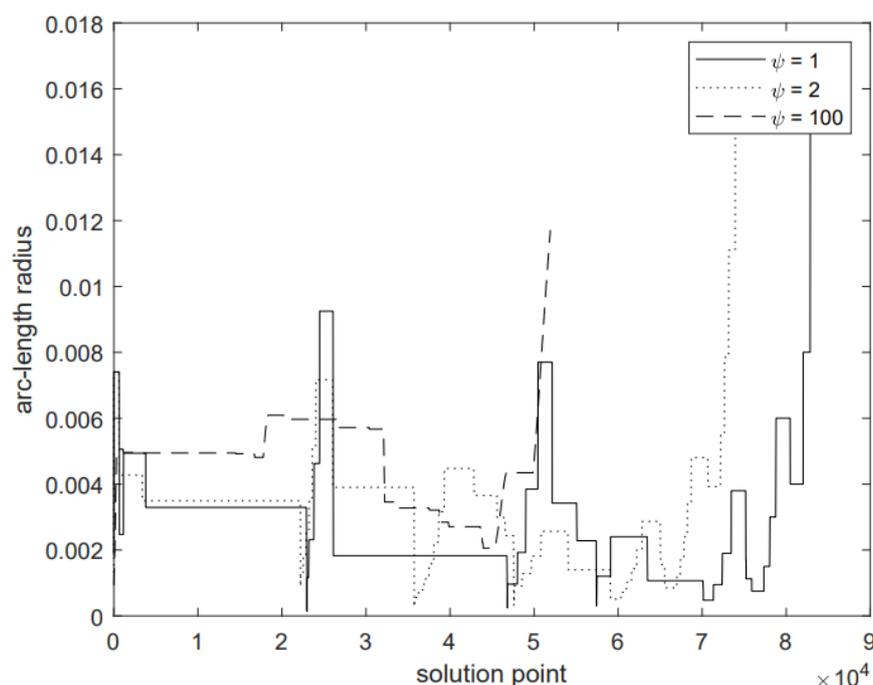
$\varphi$	Number of solution points	Total number of iterations	Computational time (s)
1	355 852	711 716	786
2	374 350	748 718	820
4	371 103	742 230	800
10	363 843	727 841	799
100	294 416	589 177	646
1 000	162 211	324 891	313
10 000	142 377	276 488	273
50 000	159 648	303 160	295
100 000	163 832	309 357	319

The results for the gap example with  $F_1 = 150$  N are shown in Tab. 2. The parameter  $I_d$  was set to 2. The same curves from Fig. 8 and Fig. 9 were obtained for every value of  $\varphi$ .

**Table 2 – Influence of the parameter  $\varphi$  on the computational cost for the gap example**

$\varphi$	Number of solution points	Total number of iterations	Computational time (s)
1	83 511	167 034	160
2	74 076	148 167	141
4	69 789	139 593	134
10	62 096	124 193	120
100	51 949	103 650	100
500	54 400	107 175	105
1 000	60 731	118 232	114
10 000	114 287	207 135	199

These results show that the choice of the parameter  $\varphi$  may greatly affect the performance of the arc-length method while dealing with nonlinear FRFs. In the first example, it is possible to decrease the total number of iterations (and computational time) by more than 50% using  $\varphi = 10\ 000$  instead of  $\varphi = 1$  or  $\varphi = 2$  as it is commonly used. In the second example, similar results are obtained when using  $\varphi = 100$ . However, care must be taken when choosing  $\varphi$ , since a value too high might have the unwanted effect of slowing the program. Figure 10 shows the evolution of the arc-length radius for the gap example for  $\varphi = 1$ ,  $\varphi = 2$  and  $\varphi = 100$ .



**Figure 10 – Evolution of the arc-length radius for the gap example**

A higher value of  $\varphi$  adapts slowly to the strong nonlinearities. However, because the radius changes are very subtle in strong nonlinear vicinity, the radius is decreased by just the correct factor to converge in  $I_d$  iterations. When a smaller value of  $\varphi$  is used the radius changes are very abrupt. Therefore, in nonlinear regions the radius is decreased by an excessive amount to achieve convergence in  $I_d$  iterations, making the algorithm overall slower.

## CONCLUDING REMARKS

Although generally used to solve nonlinear static structural problems, the arc-length method can also be used to determine frequency response functions of nonlinear dynamic systems. These functions usually present severe inflection points, creating an extremely difficult solution path to be determined.

The method was capable of correctly determining FRFs with common nonlinearities without any significant problem. However, care must be taken when choosing some parameters, such as initial radius or expected number of iterations to achieve convergence. The program may fail if these parameters are set inconsistently.

One of the main challenges of the method is the determination of the radius of the imposed constraint. If the radius is inappropriate, it may lead to convergence failure or high processing time. In this work, the automatic radius control previously used in static structural problems was modified to reduce the number of points required to determine the FRFs, decreasing computing cost. The new parameter  $\varphi$ , when set to an appropriate value, is able to increase the efficiency of the method drastically.

Some refinements can still be made to the method, such as predicting which value of the parameter  $\varphi$  will yield the best results in terms of computational cost.

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