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Stable computation of mode shapes of uniform Euler-Bernoulli beams subject to classical and non-classical boundary conditions via Lie symmetries

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Studying structural elements undergoing transverse vibration is crucial for scientific and industrial applications' design, safety, and efficiency. The Euler-Bernoulli beam model addresses a broad class of slender beam-like structures. The underlying assumption of neglecting shearing and rotary inertia effects at the beam's cross-section is reasonable. Displacement solutions of free vibrating uniform Euler-Bernoulli beams are usually defined using trigonometric and hyperbolic terms, prone to numerical instabilities at high frequencies as the wavelength becomes small. This conference paper proposes a Lie symmetry approach for determining numerically stable expressions systematically for the mode shape functions of Euler-Bernoulli beams subject to classical and non-classical boundary conditions. These mode shape solutions are computed by considering the invariance properties under a Lie group of transformations of the differential equation of a vibrating beam and its boundary conditions. The mode shapes are written as functions of constant parameters that do not exhibit instabilities when numerically assessed. Such stability provides appropriate mode shape solutions for high-frequency analysis, relevant for enhancing the accuracy of vibrating-beam solutions through the superimposition of different mode shapes within the range of validity of the Euler-Bernoulli beam model or building symbolic approximate mode shape solutions for rectangular plates.

Keywords: Euler-Bernoulli beams, mode shape functions, numerically stable solutions, Lie symmetries

1 INTRODUCTION

Elastic body vibration models play a crucial role in the dynamic analysis of continuous structures, whose geometrical and material complexity and the level of accuracy sought to guide the choice of an appropriate modeling theory (Doyle, 1989; Rayleigh, 1945). The classical Euler-Bernoulli beam theory addresses a broad class of slender structures in which the underlying assumption of neglecting shearing and rotary inertia effects in the cross-section is reasonable, e.g., vibrating beams whose cross-section dimensions are small compared to the wavelength (Meirovitch, 2010; Weaver; Timoshenko; Young, 1990). Remarkably, the mode shape study, i.e., how vibrating structures behave spatially, assists in building solutions for beams subject to various boundary conditions (BCs). For uniform Euler-Bernoulli beams, the numerical assessment of the classical analytical mode shape expressions available in the literature is usually prone to instabilities in high-frequency analysis and high-order modes (Bishop, 2011; Grover, 1972). Mathematical identities have been recently used for providing new mode shape functions from the classical ones, carefully rewritten to avoid errors at high frequencies (Gonçalves et al., 2019; Khasawneh and Segalman, 2019; Gonçalves; Peplow; Brennan, 2018). Another strategy for obtaining corresponding functions is seeking mathematical transformations for expressing boundary value problems (BVPs) in suitable subspaces, e.g., considering the Euler-Bernoulli beam problem via symmetry-based methods (Zhang and Li, 2020; Yang and Hua, 2014).

Symmetry methods encompass changing coordinate procedures for differential equations (DEs) with noteworthy contributions to modern research fields (Wilson, 2020; Einstein, 2015; Winternitz, 2003). Introduced by the mathematician Sophus Lie, the Lie symmetry method provided the first well-founded mathematical theory aimed at revealing symmetries of DEs systematically. Lie's approach allows the straightforward construction of exact solutions through analytical simplifications and order reductions for DEs, reaching new spaces of variables in which the study of BVPs may be advantageous or new solutions found from known ones (Bluman, 2002; Olver, 1986). Within the framework of symmetry method applications for the study of BVPs, the fundamental idea behind this conference paper is to compute naturally numerically stable mode shape functions of uniform Euler-Bernoulli beams directly from the well-founded Lie symmetry method, hence without requiring rewriting them to avoid numerical instabilities at high frequencies. This approach investigates the invariance of both governing DE for the mode shapes of the beam and its BCs, considering classical and non-classical ones such as boundary impedance terms like added masses and springs.

The succeeding sections of the paper are organized as follows. Sec. 2 presents the problem statement, the classical way to determine mode shapes of Euler-Bernoulli beams and their related numerical instability issues at high frequencies. Sec. 3 introduces the Lie symmetry method for ordinary differential equations (ODEs). In the Sec. 4, Lie's method is applied to compute numerically stable expressions for the mode shapes of Euler-Bernoulli beams subject to classical and non-classical BCs. Sec. 5 considers a qualitative comparison between the proposed approach and alternative methods issued from the literature. Concluding remarks are finally brought in Sec. 6.

2 PROBLEM DESCRIPTION

A uniform Euler-Bernoulli beam undergoing free transverse vibration is modeled by (Inman, 2013; Wang; Reddy; Lee, 2000):

$$EI\frac{\partial^4 \tilde{u}}{\partial x^4} + \rho S\frac{\partial^2 \tilde{u}}{\partial t^2} = 0, \tag{1}$$

where $\tilde{u} = \tilde{u}(x,t)$ is the transverse displacement, $x \in [l_1\,, l_2]$ is the position along the beam of length $l_2 - l_1 = l$, $t \geq 0$ denotes time, EI is the bending stiffness, ρ is the density, and S is the cross-section area.

Vibration modes represent harmonic non-trivial solutions of the form $\tilde{u}(x,t) = u(x)\,\mathrm{e}^{\mathrm{i}\omega t}$, for ω being the angular frequency and i the imaginary unit. With this assumption, Eq. (1) yields the mode shape equation (Geradin, 2015):

$$u^{(4)} - k^4 u = 0, \tag{2}$$

where $u = u(x)$ is the mode shape function, $u^{(4)}$ is the fourth derivative of u with respect to x from the Lagrange's notation for differentiation, and $k = (\omega^2 \rho S/EI)^{\frac{1}{4}}$ is the wavenumber. The classical solution of Eq. (2) is given by (Blevins, 2001; Weaver; Timoshenko; Young, 1990):

$$u = A_1 \sinh(kx) + A_2 \cosh(kx) + A_3 \sin(kx) + A_4 \cos(kx), \tag{3}$$

for the real constants A_1, A_2, A_3 and A_4. The jth mode can be characterized by the following mode shape function:

$$u_j = A_1(\omega_j)\sinh(k_j x) + A_2(\omega_j)\cosh(k_j x) + A_3(\omega_j)\sin(k_j x) + \cos(k_j x), \tag{4}$$

associated with the natural angular frequency ω_j and related wavenumber defined by $k_j = (\omega_j^2 \rho S/EI)^{\frac{1}{4}}$, where, without loss of generality, the assumption has been made that $A_4 = 1$. Equation (4) involves the four unknowns: (i) the natural angular frequency ω_j (and the related wavenumber k_j) and (ii-iv) the three constants $A_1(\omega_j)$, $A_2(\omega_j)$ and $A_3(\omega_j)$, whose determination follows from the analysis of the BCs of the beam. The numerical evaluation of such constants at high frequencies, e.g., by plotting high-order mode shapes, is usually prone to round-off errors. This might be explained since the functions $\sinh(k_j l_i)$ and $\cosh(k_j l_i)$ for $l_i = l_1, l_2$ are becoming to be closely similar with the same asymptotic behavior as their exponential growth follows the frequency increasing, and operations between them cannot be accurately assessed by floating-point arithmetic (Wilkinson, 1994; Goldberg, 1991). As such, the analytical mode shape solution is likely to yield meaningless results due to the complete cancellation of significant digits for ω_j, A_1, A_2, and A_3.

Table 1 presents general expressions of BCs for Euler-Bernoulli beams. Here, $u^{(1)}$, $EIu^{(2)}$ and $-EIu^{(3)}$ denote, respectively, slope, bending moment, and shearing force. For added impedances at l_i, translational and rotational stiffnesses are denoted by $K_{i,T}$ and $K_{i,R}$, respectively, translational masses are given by m_i and rotational inertias by J_i, $\mathcal{M}_i = (EI/\rho S)\,J_i k^4 - K_{i,R}$ and $\mathcal{V}_i = (EI/\rho S)\,m_i k^4 - K_{i,T}$ for $i = 1, 2$. $b_1 = -1$ and $b_2 = 1$ represent the convention for positive and negative signs at the left and the right ends of the beam, respectively (Karnovsky and Lebed, 2004; Wang; Reddy; Lee, 2000).

3 LIE SYMMETRIES

This section briefly introduces the Lie symmetry method for ODEs. The reader is referred to (Bluman, 2002; Ibragimov, 2001; Olver, 1986; Ovsiannikov, 1982) for broader mathematical development and contributions to the field of symmetry analysis.

Table 1 – Classical and non-classical BCs for Euler-Bernoulli beams

BC	Expression at $x = l_i$
clamped	$u\|_{x=l_i} = 0$ and $u^{(1)}\|_{x=l_i} = 0$
free	$EIu^{(2)}\|_{x=l_i} = 0$ and $-EIu^{(3)}\|_{x=l_i} = 0$
pinned	$u\|_{x=l_i} = 0$ and $EIu^{(2)}\|_{x=l_i} = 0$
sliding	$u^{(1)}\|_{x=l_i} = 0$ and $-EIu^{(3)}\|_{x=l_i} = 0$
general non-classical BC	$EIu^{(2)}\|_{x=l_i} = b_i \mathcal{M}_i u^{(1)}\|_{x=l_i}$ and $-EIu^{(3)}\|_{x=l_i} = b_i \mathcal{V}_i u\|_{x=l_i}$

Consider the nth-order ODE written in the general form:

$$\mathcal{F}\big(x, u, u^{(1)}, \ldots, u^{(n)}\big) = 0, \tag{5}$$

for $\mathcal{F} = \mathcal{F}\big(x, u, u^{(1)}, \ldots, u^{(n)}\big)$, which may admit the following infinitesimal generator, i.e., the vector field acting on the space of variables (x, u) (Olver, 1986):

$$\mathcal{X} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}, \tag{6}$$

for the infinitesimal functions $\xi = \xi(x, u)$ and $\eta = \eta(x, u)$, components of the infinitesimal transformations that map each point of (x, u) into a new space of variables. These infinitesimals are elements of an admitted Lie group and can be extended to act on the complete space of variables of Eq. (5). In this sense, the jth extended infinitesimal generator is given by:

$$\mathcal{X}^j = \mathcal{X} + \eta_1 \frac{\partial}{\partial u^{(1)}} + \cdots + \eta_j \frac{\partial}{\partial u^{(j)}}, \tag{7}$$

where the prolonged infinitesimals follows as:

$$\eta_i = \mathcal{D}(\eta_{i-1}) - u^{(i)} \mathcal{D}(\xi), \quad i = 1, \ldots, j, \tag{8}$$

and

$$\eta_0 = \eta. \tag{9}$$

Also, $\mathcal{D}$ is the total derivative operator defined as:

$$\mathcal{D} = \frac{\partial}{\partial x} + u^{(1)} \frac{\partial}{\partial u} + \cdots + u^{(n+1)} \frac{\partial}{\partial u^{(n)}}. \tag{10}$$

The ODE from Eq. (5) is said to be invariant under the Lie group if and only if (Ovsiannikov, 1982):

$$\mathcal{X}^n \mathcal{F} = 0. \tag{11}$$

The evaluation of Eq. (11) yields the so-called determining equations, i.e., a system of linear partial differential equations for the unknown infinitesimal functions ξ and η. Introducing $u^{(n)} = f(x, u, u^{(1)}, \ldots, u^{(n-1)})$ in Eq. (11), isolated from Eq. (5), allows some derivative terms to be removed; then, the above-mentioned determining equations are obtained by equating the coefficients of the remaining derivatives of u to zeros. The integration of these differential equations gives the infinitesimal functions and, hence, the admitted infinitesimal transformations from the Lie group. The determining equation resolution is usually assisted by symbolic manipulation packages (Butcher; Carminati; Vu, 2003; Baumann, 2000; Hereman, 1994).

Canonical coordinates are sets of variables in which groups of translations express Lie groups. This property represents the possibility of integrating the ODE under consideration directly when written in their canonical space of variables. Here, two canonical coordinates $r = r(x, u)$ and $s = s(x, u)$ are introduced, whose determination involves solving (Oliveri, 2010):

$$\mathcal{X} r = \xi \frac{\partial r}{\partial x} + \eta \frac{\partial r}{\partial u} = 0, \tag{12}$$

and

$$\mathcal{X}s = \xi \frac{\partial s}{\partial x} + \eta \frac{\partial s}{\partial u} = 1. \tag{13}$$

Symmetry transformations can also be applied to BCs to investigate their invariance. For a 1D space domain, a BC can be expressed in a general way as $\mathcal{B}_{j,i} = \mathcal{B}_{j,i}(x, u, u^{(1)}, \ldots, u^{(j)})$ when $x = l_i$ for $i = 1, 2$. In this case, the invariance conditions are written as (Gai; Li; Sudao, 2020; Bluman, 2002):

$$[\mathcal{X}(x - l_i)]|_{x=l_i} = 0, \tag{14}$$

and

$$(\mathcal{X}^j \mathcal{B}_{j,i})|_{x=l_i} = 0 \quad \text{when } \mathcal{B}_{j,i}|_{x=l_i} = 0. \tag{15}$$

For zero-order BCs such as $\mathcal{B}_{0,i}|_{x=l_i} = u|_{x=l_i} = 0$, the non-extended symmetry generator $\mathcal{X}$ is considered in Eq. (15).

4 MODE SHAPES OF UNIFORM EULER-BERNOULLI BEAMS VIA LIE SYMMETRIES

Modes shapes of uniform Euler-Bernoulli beams subject to classical and non-classical BCs are computed by considering the procedure presented in Sec. 3. A well-posed BVP is obtained by exploiting the invariance properties of the ODE and the BCs, leading to numerically stable solutions.

4.1 Lie symmetries of the mode shape equation

From the criterion of Eq. (11), the invariance of Eq. (2) simply follows:

$$\mathcal{X}^4 (u^{(4)} - k^4 u) = 0, \tag{16}$$

which leads to the system of determining equations:

$$\frac{\partial^4 \eta}{\partial x^4} = \left(\eta - u \frac{\partial \eta}{\partial u}\right) k^4, \quad \frac{\partial^2 \eta}{\partial x \partial u} = 0, \quad \frac{\partial^2 \eta}{\partial u^2} = 0, \quad \frac{\partial \xi}{\partial x} = 0, \quad \frac{\partial \xi}{\partial u} = 0. \tag{17}$$

The resolution of such determining equations gives the admitted infinitesimal functions:

$$\eta = u + \alpha_1 \mathrm{e}^{kx} + \alpha_2 \mathrm{e}^{-kx} + \alpha_3 \sin(kx) + \alpha_4 \cos(kx), \quad \xi = \alpha_5, \tag{18}$$

where the constants α_1, α_2, α_3, α_4 and α_5 are the so-called transformation parameters.

4.2 Invariance of boundary conditions

A fully invariant BVP under a symmetry transformation admitted by the corresponding DE must obey the invariance conditions of Eqs. (14) and (15). Eq. (14) yields for the boundaries of the domain:

$$[\mathcal{X}(x - l_i)]|_{x=l_i} = \xi \frac{\partial}{\partial x}(x - l_i)|_{x=l_i} + \eta \frac{\partial}{\partial u}(x - l_i)|_{x=l_i} = 0, \tag{19}$$

which gives $\alpha_5 = 0$ and, therefore, $\xi = 0$. The remaining condition directly involves the BCs from the BVP. For instance, zero-order BCs such as a clamped end under Eq. (15) follows as:

$$(\mathcal{X}u)|_{x=l_i} = \eta \frac{\partial}{\partial u} u|_{x=l_i} = 0 \quad \text{when } u|_{x=l_i} = 0, \tag{20}$$

from where the resulting restriction equation is:

$$\alpha_1 \mathrm{e}^{kl_i} + \alpha_2 \mathrm{e}^{-kl_i} + \alpha_3 \sin(kl_i) + \alpha_4 \cos(kl_i) = 0. \tag{21}$$

Table 2 displays the invariance conditions for the BCs of Tab. 1 and their related restriction equations.

Table 2 – Invariance conditions for BCs

Invariance condition at $x = l_i$	Restriction equation
$(Xu)\vert_{x=l_i}=0$	$\alpha_1 e^{kl_i}+\alpha_2 e^{-kl_i}+\alpha_3 \sin(kl_i)+\alpha_4 \cos(kl_i)=0$
$\left(X^1 u^{(1)}\right)\Big\vert_{x=l_i}=0$	$-\alpha_1 e^{kl_i}+\alpha_2 e^{-kl_i}-\alpha_3 \cos(kl_i)+\alpha_4 \sin(kl_i)=0$
$\left[X^2\left(EI u^{(2)}-b_i \mathcal{M}_i u^{(1)}\right)\right]\Big\vert_{x=l_i}=0$	$\alpha_1\left(k-b_i\frac{\mathcal{M}_i}{EI}\right)e^{kl_i}+\alpha_2\left(k+b_i\frac{\mathcal{M}_i}{EI}\right)e^{-kl_i}$ $-\left(\alpha_3 k-\alpha_4 b_i\frac{\mathcal{M}_i}{EI}\right)\sin(kl_i)-\left(\alpha_4 k+\alpha_3 b_i\frac{\mathcal{M}_i}{EI}\right)\cos(kl_i)=0$
$\left[X^3\left(EI u^{(3)}+b_i \mathcal{V}_i u\right)\right]\Big\vert_{x=l_i}=0$	$\alpha_1\left(k^3+b_i\frac{\mathcal{V}_i}{EI}\right)e^{kl_i}-\alpha_2\left(k^3-b_i\frac{\mathcal{V}_i}{EI}\right)e^{-kl_i}$ $+\left(\alpha_4 k^3+\alpha_3 b_i\frac{\mathcal{V}_i}{EI}\right)\sin(kl_i)-\left(\alpha_3 k^3-\alpha_4 b_i\frac{\mathcal{V}_i}{EI}\right)\cos(kl_i)=0$

4.3 Canonical coordinates from the Lie group

The canonical coordinates (r,s) which result from the resolution of Eqs. (12) and (13) by taking the admitted infinitesimal functions from the invariance of the boundary conditions follow as:

$$r = x, \quad s = kr + \ln\left(u + \alpha_1 e^{kr} + \alpha_2 e^{-kr} + \alpha_3 \sin(kr) + \alpha_4 \cos(kr)\right),$$

which gives, through inverse mapping, the following function:

$$u = e^{s-kr} - \left(\alpha_1 e^{kr} + \alpha_2 e^{-kr} + \alpha_3 \sin(kr) + \alpha_4 \cos(kr)\right). \tag{22}$$

Substituting Eq. (22) into Eq. (2) with $x = r$ and solving the resulting equation:

$$s = \ln\left[\frac{1}{4k^3}\left(-C_1 e^{2kr} + C_2\right) + \frac{e^{kr}}{2k^3}\left(C_3 \sin(kr) - C_4 \cos(kr)\right)\right].$$

Finally, the inverse mapped solution is obtained as:

$$u = -\left(\frac{C_1}{4k^3} + \alpha_1\right)e^{kx} + \left(\frac{C_2}{4k^3} - \alpha_2\right)e^{-kx} + \left(\frac{C_3}{2k^3} - \alpha_3\right)\sin(kx) - \left(\frac{C_4}{2k^3} + \alpha_4\right)\cos(kx). \tag{23}$$

A relevant solution can be obtained by setting constants C_1, C_2, C_3 and C_4 to zero. Then, the general mode shape function of Euler-Bernoulli beams subject to classical or non-classical BCs follows as:

$$u = -\alpha_1 e^{kx} - \alpha_2 e^{-kx} - \alpha_3 \sin(kx) - \alpha_4 \cos(kx). \tag{24}$$

Equation (24) together with the expressions of the restriction equations listed in Tab. 2 yield exact expressions for the mode shape functions.

4.4 Example 1: mode shapes of a clamped-free beam

Consider an Euler-Bernoulli beam clamped at $l_1 = 0$ and free at $l_2 = l$ as shown in Fig. 1. The related restriction equations are obtained from Tab. 2, where $\mathcal{M}_i = \mathcal{V}_i = 0$ for $i = 1, 2$, i.e.:

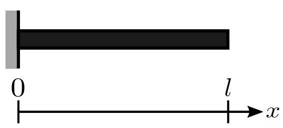


Figure 1 – Clamped-free beam

$$\alpha_1 + \alpha_2 + \alpha_4 = 0, \tag{25}$$

$$\alpha_1 - \alpha_2 + \alpha_3 = 0, \tag{26}$$

$$\alpha_1 \mathrm{e}^{kl} + \alpha_2 \mathrm{e}^{-kl} - \alpha_3 \sin(kl) - \alpha_4 \cos(kl) = 0, \tag{27}$$

$$\alpha_1 \mathrm{e}^{kl} - \alpha_2 \mathrm{e}^{-kl} - \alpha_3 \cos(kl) + \alpha_4 \sin(kl) = 0. \tag{28}$$

Assuming, without loss of generality, that $\alpha_4 = 1$, Eqs. (25-28) lead to the characteristic equation:

$$\cos(kl)\left(\mathrm{e}^{kl} + \mathrm{e}^{-kl}\right) + 2 = 0, \tag{29}$$

which can be multiplied by e^{-kl} to get a numerically stable frequency equation (Gonçalves; Peplow; Brennan, 2018):

$$\cos(kl)(1 + \mathrm{e}^{-2kl}) + 2\mathrm{e}^{-kl} = 0. \tag{30}$$

The roots $kl = k_j l$ of Eq. (30) are displayed in Tab. 3. The transformation parameters α_1, α_2 and α_3 follow as:

$$\alpha_1 = -\frac{\sin(kl) - \cos(kl) - \mathrm{e}^{-kl}}{2\sin(kl) + \mathrm{e}^{kl} - \mathrm{e}^{-kl}}, \tag{31}$$

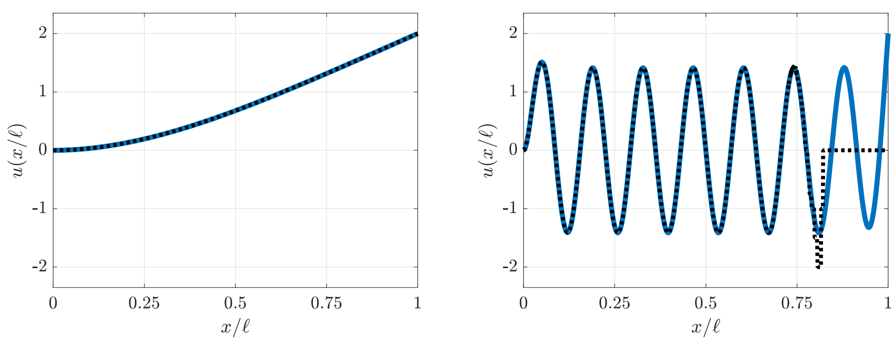
$$\alpha_2 = -\frac{\sin(kl) + \cos(kl) + \mathrm{e}^{kl}}{2\sin(kl) + \mathrm{e}^{kl} - \mathrm{e}^{-kl}}, \tag{32}$$

$$\alpha_3 = -\frac{2\cos(kl) + \mathrm{e}^{kl} + \mathrm{e}^{-kl}}{2\sin(kl) + \mathrm{e}^{kl} - \mathrm{e}^{-kl}}. \tag{33}$$

By considering Eqs. (31-33) and $\alpha_4 = 1$ in Eq. (24), this yields the mode shapes u_j of the clamped-free beam. One may easily check that these expressions for α_1, α_2, α_3 and α_4 produce $\mathcal{C}_{1\text{-}4} = 0$ in Eq. (23). At very high frequencies (above the 225th mode), positive exponential values cannot be represented by double precision floating-point arithmetic. To solve this issue, both the numerator and the denominator in the expressions of α_1, α_2 and α_3 can be multiplied by e^{-kl}. The resulting expression of the mode shape is given by:

$$u = \frac{\sin(kl) - \cos(kl) - \mathrm{e}^{-kl}}{1 - \mathrm{e}^{-2kl} + 2\mathrm{e}^{-kl}\sin(kl)}\mathrm{e}^{k(x-l)} + \frac{1 + (\sin(kl) + \cos(kl))\,\mathrm{e}^{-kl}}{1 - \mathrm{e}^{-2kl} + 2\mathrm{e}^{-kl}\sin(kl)}\mathrm{e}^{-kx} + \frac{1 + \mathrm{e}^{-2kl} + 2\mathrm{e}^{-kl}\cos(kl)}{1 - \mathrm{e}^{-2kl} + 2\mathrm{e}^{-kl}\sin(kl)}\sin(kx) - \cos(kx). \tag{34}$$

Figure 2 shows the 1st and 15th mode shapes computed by the Lie symmetry approach (solid line) and the corresponding classical solution from Eq. (3) (dashed line). In this case, a beam of length $l = 1$ m having the following properties has been considered: cross-sectional area $S = 2.5 \times 10^{-4}$ m^2, Young's modulus $E = 70$ GPa, inertia moment $I = 5.2 \times 10^{-10}$ m^4 and density $\rho = 2.7 \times 10^3$ kg/m^3. In Fig. 2, it is shown that the proposed approach is able to describe the mode shapes of the beam without round-off errors, even at high frequencies.



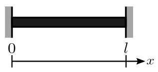
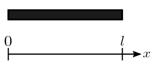
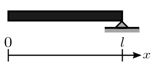
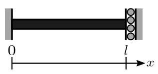
(a) 1st mode shape 15th mode shape

Figure 2 – 1st and 15th mode shapes of a clamped-free beam computed via the Lie symmetry approach (blue solid line) and the classical approach (black dashed line)

4.5 Mode shapes for beams with classical and non-classical boundary conditions

An overview of numerically stable mode shapes, computed via the proposed approach, is brought for classical and non-classical BCs of Euler-Bernoulli beams. Table 3 presents the expressions of $\alpha_1 e^{kx}$, $\alpha_2 e^{-kx}$, $\alpha_3 \sin(kx)$ and $\alpha_4 \cos(kx)$ for mode shapes involving classical BCs, while Tab. 4 considers non-classical BCs from general boundary impedance terms, including added masses and springs. The related frequency equations for the addressed cases are given in Appendix. Also, the procedure for analyzing classical BCs from non-classical ones is given in Appendix.

Table 3 – Parameters for classical BCs

	$\alpha_1 e^{kx}$	$\alpha_2 e^{-kx}$	$\alpha_3 \sin(kx)$	$\alpha_4 \cos(kx)$	kl
clamped-clamped 	$\frac{sc^- + e^{-kl}}{1-es^+} e^{k(x-l)}$	$-\frac{1-sc^+ e^{-kl}}{1-es^+} e^{-kx}$	$-\frac{1+ec^-}{1-es^+}\sin(kx)$	$\cos(kx)$	$k_1l=4.7300407$ $k_2l=7.8532046$ $k_3l=10.9956078$
free-free 	$-\frac{sc^- + e^{-kl}}{1-es^+} e^{k(x-l)}$	$\frac{1-sc^+ e^{-kl}}{1-es^+} e^{-kx}$	$-\frac{1+ec^-}{1-es^+}\sin(kx)$	$\cos(kx)$	$k_4l=14.1371655$ $k_5l=17.2787596$ $k_nl\approx(2n+1)\frac{\pi}{2}, n>5$
clamped-pinned 	$\frac{sc^- + e^{-kl}}{1-es^+} e^{k(x-l)}$	$-\frac{1-sc^+ e^{-kl}}{1-es^+} e^{-kx}$	$-\frac{1+ec^-}{1-es^+}\sin(kx)$	$\cos(kx)$	$k_1l=3.9266023$ $k_2l=7.0685827$ $k_3l=10.2101761$
free-pinned 	$\frac{sc^- - e^{-kl}}{1-es^-} e^{k(x-l)}$	$\frac{1+sc^+ e^{-kl}}{1-es^-} e^{-kx}$	$-\frac{1+ec^+}{1-es^-}\sin(kx)$	$\cos(kx)$	$k_4l=13.3517688$ $k_5l=16.4933614$ $k_nl\approx(4n+1)\frac{\pi}{4}, n>5$
clamped-sliding 	$\frac{sc^+ - e^{-kl}}{1+ec^-} e^{k(x-l)}$	$-\frac{1+sc^- e^{-kl}}{1+ec^-} e^{-kx}$	$-\frac{1-es^-}{1+ec^-}\sin(kx)$	$\cos(kx)$	$k_1l=2.3650204$ $k_2l=5.497804$ $k_3l=8.6393798$
free-sliding 	$\frac{sc^+ + e^{-kl}}{1+ec^+} e^{k(x-l)}$	$\frac{1-sc^- e^{-kl}}{1+ec^+} e^{-kx}$	$-\frac{1-es^+}{1+ec^+}\sin(kx)$	$\cos(kx)$	$k_4l=11.7809724$ $k_5l=14.9225651$ $k_nl\approx(4n-1)\frac{\pi}{4}, n>5$
clamped-free 	$-\frac{sc^- - e^{-kl}}{1-es^-} e^{k(x-l)}$	$-\frac{1+sc^+ e^{-kl}}{1-es^-} e^{-kx}$	$-\frac{1+ec^+}{1-es^-}\sin(kx)$	$\cos(kx)$	$k_1l=1.8751041$ $k_2l=4.6940911$ $k_3l=7.8547574$ $k_4l=10.9955407$ $k_5l=14.1371684$ $k_nl\approx(2n-1)\frac{\pi}{2}, n>5$

$sc^+ = \sin(kl)+\cos(kl)$, $es^+ = \left(e^{-kl}+2\sin(kl)\right)e^{-kl}$, $ec^+ = \left(e^{-kl}+2\cos(kl)\right)e^{-kl}$,
$sc^- = \sin(kl)-\cos(kl)$, $es^- = \left(e^{-kl}-2\sin(kl)\right)e^{-kl}$, $ec^- = \left(e^{-kl}-2\cos(kl)\right)e^{-kl}$.

5 DISCUSSION

The proposed approach was based on the Lie symmetry method and provided a suitable way to express mode shape functions of Euler-Bernoulli beams subject to classical and non-classical BCs. The resulting mode shapes can be numerically assessed accurately at low and high frequencies. The procedure for obtaining suitable expressions of the mode shapes involved considering the invariance of the mode shape equation of an Euler-Bernoulli beam and those of its BCs. To avoid exponential terms of large magnitude at very high frequencies, which can reach computer limit representation values (10^{308} for double-precision floating-point arithmetic), the mode shape expressions have been regularized by considering a

Table 4 – Parameters for non-classical BCs

	general non-classical boundaries 
$\alpha_1\, \mathrm{e}^{kx}$	$\Big\{\Big(k+\frac{\mathcal{M}_2}{EI}\Big)\Big(EIk^4+\frac{\mathcal{M}_1}{EI}\mathcal{V}_1\Big)\mathrm{e}^{-kl}+\Big[-EIk^5+(2\,\mathcal{M}_1+\mathcal{M}_2)\,k^4+\frac{\mathcal{M}_1+2\,\mathcal{M}_2}{EI}\mathcal{V}_1 k-\frac{\mathcal{M}_1\mathcal{M}_2}{(EI)^2}\mathcal{V}_1\Big]\cos(kl)$ $+\Big[EIk^5+\big(\mathcal{M}_2 k^2+2\,\mathcal{V}_1\big)k^2-\frac{\mathcal{M}_1}{EI}\big(2\,\mathcal{M}_2 k^2+\mathcal{V}_1\big)k-\frac{\mathcal{M}_1\mathcal{M}_2}{(EI)^2}\mathcal{V}_1\Big]\sin(kl)\Big\}\frac{\mathrm{e}^{k(x-l)}}{\beta}$
$\alpha_2\, \mathrm{e}^{-kx}$	$\Big\{-\Big(k-\frac{\mathcal{M}_2}{EI}\Big)\Big(EIk^4+\frac{\mathcal{M}_1}{EI}\mathcal{V}_1\Big)+\Big[EIk^5+(2\,\mathcal{M}_1+\mathcal{M}_2)\,k^4-\frac{\mathcal{M}_1+2\,\mathcal{M}_2}{EI}\mathcal{V}_1 k-\frac{\mathcal{M}_1\mathcal{M}_2}{(EI)^2}\mathcal{V}_1\Big]\mathrm{e}^{-kl}\cos(kl)$ $+\Big[EIk^5-\big(\mathcal{M}_2 k^2+2\,\mathcal{V}_1\big)k^2-\frac{\mathcal{M}_1}{EI}\big(2\,\mathcal{M}_2 k^2+\mathcal{V}_1\big)k+\frac{\mathcal{M}_1\mathcal{M}_2}{(EI)^2}\mathcal{V}_1\Big]\mathrm{e}^{-kl}\sin(kl)\Big\}\frac{\mathrm{e}^{-kx}}{\beta}$
$\alpha_3 \sin(kx)$	$\Big[\Big(k-\frac{\mathcal{M}_2}{EI}\Big)\Big(EIk^4+2\,\mathcal{V}_1 k-\frac{\mathcal{M}_1}{EI}\mathcal{V}_1\Big)+\Big(k+\frac{\mathcal{M}_2}{EI}\Big)\Big(EIk^4-2\,\mathcal{V}_1 k-\frac{\mathcal{M}_1}{EI}\mathcal{V}_1\Big)\mathrm{e}^{-2kl}$ $-2\Big(k\cos(kl)-\frac{\mathcal{M}_2}{EI}\sin(kl)\Big)\Big(EIk^4+\frac{\mathcal{M}_1}{EI}\mathcal{V}_1\Big)\mathrm{e}^{-kl}\Big]\frac{\sin(kx)}{\beta}$
$\alpha_4 \cos(kx)$	$\cos(kx)$
β	$\Big[-\Big(k-\frac{\mathcal{M}_2}{EI}\Big)\Big(EIk^4-2\,\mathcal{M}_1 k^3-\frac{\mathcal{M}_1}{EI}\mathcal{V}_1\Big)+\Big(k+\frac{\mathcal{M}_2}{EI}\Big)\Big(EIk^4+2\,\mathcal{M}_1 k^3-\frac{\mathcal{M}_1}{EI}\mathcal{V}_1\Big)\mathrm{e}^{-2kl}$ $+2\Big(k\sin(kl)+\frac{\mathcal{M}_2}{EI}\cos(kl)\Big)\Big(EIk^4+\frac{\mathcal{M}_1}{EI}\mathcal{V}_1\Big)\mathrm{e}^{-kl}\Big]$

”preconditioner” e^{-kl} in the numerator and denominator terms of the transformation parameters.

Table 5 shows a qualitative comparison between the proposed approach and two relevant works from the literature which investigate the computation of the mode shapes of Euler-Bernoulli beams at high frequencies. In these works, the round-off errors associated with the computation of the mode shapes are overcome using algebraic manipulations for some numerically unstable terms or the whole expression of the mode shapes. In contrast, the proposed approach works by defining new subspaces that are mathematically relevant for expressing the invariance properties of the appropriate ODE and BCs.

Table 5 – Comparison between the proposed approach and other methods issued from the literature

Approach	Proposed	(Gonçalves; Peplow; Brennan, 2018)	(Khasawneh and Segalman, 2019)
Strategy	invariance conditions	algebraic manipulation	algebraic manipulation
Frequency range	all	all	high frequencies
Round-off errors	—	—	small at low frequencies
Boundary conditions	classical and non-classical	classical	classical

6 CONCLUSION

A Lie symmetry approach has been proposed to compute the mode shape functions of uniform Euler-Bernoulli beams with accurate precision at low and high frequencies. By exploring the invariance properties of the DE of a beam together with those of its BCs, a well-posed BVP has been proposed whose resolution yields the desired mode shape functions. The proposed approach has been used to compute the mode shape functions of beams subject to classical BCs, and it has been generalized to address more complex BCs, including impedance terms with added masses and springs. Thus, Lie symmetry analysis appears to be a powerful tool based on a robust mathematical theory to find exact solutions to BVPs. Future works may include the analysis of more complicated structures like beams with periodic arrays of nonlinear springs.

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APPENDIX

The following characteristic equations have been obtained via the proposed approach and then multiplied by e^{-kl} to obtain more stable equations at high frequencies (Gonçalves; Peplow; Brennan, 2018).

A. Euler-Bernoulli beam of length l subject to some classical BCs.

(a) Clamped-clamped and free-free beams:

$$\cos(kl)(1+e^{-2kl}) - 2e^{-kl} = 0.$$

(b) Clamped-pinned and free-pinned beams:

$$\left(1-e^{-2kl}\right)\cos(kl) - \left(1+e^{-2kl}\right)\sin(kl) = 0.$$

(c) Clamped-sliding and free-sliding beams:

$$\left(1-e^{-2kl}\right)\cos(kl) + \left(1+e^{-2kl}\right)\sin(kl) = 0.$$

(d) Clamped-free beam:

$$\cos(kl)(1+e^{-2kl}) + 2e^{-kl} = 0.$$

B. Euler-Bernoulli beam of length l subject to general non-classical BCs.

$$e^{-kl}\det(\mathbf{M}) = 0,$$

where

$$\mathbf{M} = \begin{bmatrix} k+\frac{\mathcal{M}_1}{EI} & k-\frac{\mathcal{M}_1}{EI} & \frac{\mathcal{M}_1}{EI} & -k \\ k^3-\frac{\mathcal{V}_1}{EI} & -k^3-\frac{\mathcal{V}_1}{EI} & -k^3 & -\frac{\mathcal{V}_1}{EI} \\ \left(k-\frac{\mathcal{M}_2}{EI}\right)e^{kl} & \left(k+\frac{\mathcal{M}_2}{EI}\right)e^{-kl} & -k\sin(kl)-\frac{\mathcal{M}_2}{EI}\cos(kl) & \frac{\mathcal{M}_2}{EI}\sin(kl)-k\cos(kl) \\ \left(k^3+\frac{\mathcal{V}_2}{EI}\right)e^{kl} & -\left(k^3-\frac{\mathcal{V}_2}{EI}\right)e^{-kl} & \frac{\mathcal{V}_2}{EI}\sin(kl)-k^3\cos(kl) & k^3\sin(kl)+\frac{\mathcal{V}_2}{EI}\cos(kl) \end{bmatrix}.$$

The characteristic equation given by $e^{-kl}\det(\mathbf{M}) = 0$ is general and may be applied to retrieve some characteristic equations for beams with classical BCs or mixed classical and non-classical BCs. For instance, for a clamped-clamped beam, one has $\mathcal{M}_i \to \infty$ and $\mathcal{V}_i \to \infty$, $i = 1,2$. In this case, the characteristic equation $e^{-kl}\det(\mathbf{M}) = 0$ yields:

$$e^{-kl}\det\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & -1 \\ -e^{kl} & e^{-kl} & -\cos(kl) & \sin(kl) \\ e^{kl} & e^{-kl} & \sin(kl) & \cos(kl) \end{bmatrix} = 0,$$

which gives the frequency equation for a clamped-clamped beam, see case A.(a).

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