

# Presentation of a New Family of Strain Measures Based on the Hyperbolic Tangent Function

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*Abstract: The main purpose of this article is to present an objective, symmetrical and versatile family of strain measures based on the hyperbolic tangent function, both in its uniaxial and in its tensor form (3D): the Generalized Hyperbolic Tangent (GHT) strain measure. It is a goal of this work to examine the properties of the strain measure created and to discuss in which practical applications its use could be advantageous. The investigation of the new family of strain measures behavior will be done through the analytical study of pure deformation modes and through the analysis of a spatial truss using the Positional Finite Element Method (PFEM). The specific formulae for implementation of the Hyperbolic Tangent strain measure into the PFEM is presented. The mathematical definition of the GHT strain in its tensor form is made by transitioning from the uniaxial definition. To establish its conjugate stress tensor, the spectral decomposition of tensors will be used. With the variation of a single constant ( $n$ ), versatility is obtained and calibration may be used to adequate the modeling of different materials. The results obtained from the analysis of trusses using the PFEM and analytical investigation of pure deformation modes of solids indicate that the GHT measures are suitable for engineering applications at small and moderate strains. In the present work, only the energy-conservative elastic, isotropic and homogeneous behaviors of materials are considered.*

**Keywords:** Moderate Strains, Hyperelasticity, Strain Measures, Tangent Hyperbolic

## INTRODUCTION

In the regime of moderate and large strains, the choice of the strain measure to be used is very important, this is because such a choice must consider the physical suitability, the mathematical complexity of the derived expressions and the computational efficiency of the entire formulation to be implemented. In large strain hyperelasticity, for example, the Green strain tensor is convenient because it doesn't require polar decomposition of the Deformation Gradient, so the computational cost is lower, which makes the Green strain tensor a popular choice. Another example is the common use of the Logarithmic strain in plasticity because of the need to use the multiplicative decomposition of the Deformation Gradient in large strain regime, this because the logarithmic function has the property that the logarithm of a product is equal to the sum of the logarithms of the terms, which leads to an additive decomposition of the total strain into elastic and plastic parts.

Furthermore, the choice of the strain measure also defines the strain rate and the stress, since the strain measure and the strain rate must conjugate with the stress, in work and power, respectively. Bazant et. al (2012) and Ji et. al (2010) analyze errors caused by the use of non-energetic-conjugate quantities in Finite Element Method commercial softwares. For small strain values, all strain measures converge, but the same cannot be said for stress, stress rate and strain rate, because they diverge even for small stretches. This aspect can be very relevant in the study of viscosity (material behavior is time-dependent) and in hypoelastic based plasticity formulations, which present constitutive equations in rate form.

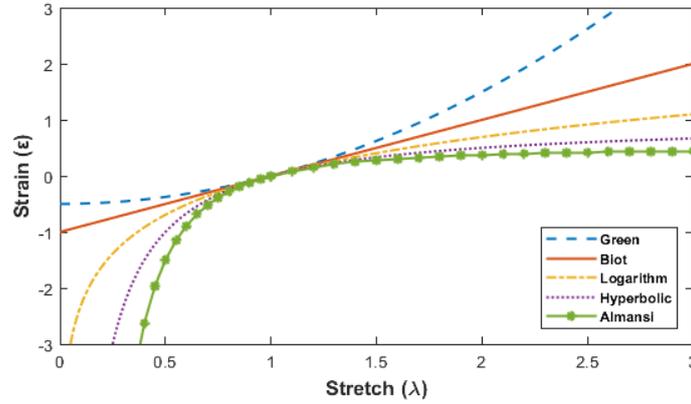
A considerable number of authors have created new strain measures and new family of strain measures, one can cite: Doyle e Ericksen (1956); Seth (1962); Hill (1968); Curnier and Rakotomanana, (1991); Curnier and Zysset (2006); Itskov (2004); Darijani et. al (2010); Darijani e Naghdabadi (2010); Darijani e Naghdabadi, (2013); Korobeynikov (2019); Beex (2019). In this context, the present work presents a new objective, symmetrical and versatile family of strain measures based on the hyperbolic tangent function.

## PROBLEM STATEMENT

In the 1960s, the so-called classical strain measures were grouped into a single family by Seth (1962) and then by Hill (1968), thus becoming the Seth-Hill family. Equation (1) presents this family in the uniaxial form (1D): Green  $\rightarrow m = 2$ , Biot  $\rightarrow m = 1$ , Logarithmic  $\rightarrow m = 0$ , Hyperbolic  $\rightarrow m = -1$  e Almansi  $\rightarrow m = -2$ , with  $\lambda$  being the stretching ratio.

$$\begin{cases} (1/m) \cdot (\lambda^m - 1) & m \neq 0 \\ \ln(\lambda) & m = 0 \end{cases} \quad (1)$$

Figure 1 graphically represents Eq. (1).



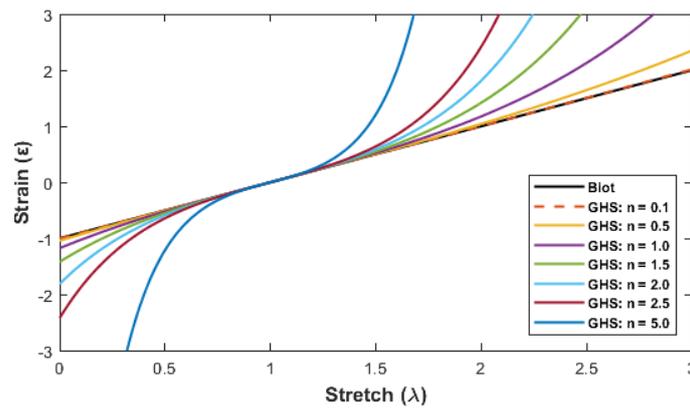
**Figure 1 – Classical strain measures.**

Beex (2019) explains that, among the measures of the Seth-Hill family, the ones with  $m > 0$  are problematic in compression because they do not tend to  $-\infty$  when the stretch tends to 0, whereas the ones with  $m < 0$  are problematic in tension, because they do not present a monotonic stress increase. The Logarithmic strain measure ( $m = 0$ ) tends to  $-\infty$  in compression and presents a monotonic increase in tension, however it has very asymmetric behavior, with stiffness that tends to infinity in compression and decreasing stiffness in traction.

Observing the graph of the Seth-Hill family (Figure 1), it is noted that only the Biot measure has symmetry between compression and tension, the others, on the contrary, have strong asymmetries. This becomes a problem when one wants to model a material with approximately symmetrical behavior.

Greco and Peixoto (2021) presented a family of objective and symmetrical strain measures based on the Sine-Hyperbolic function: the Generalized Sine-Hyperbolic (GHS) family – Eq. (2) and Figure 2. This family is suitable for the regime of large deformations, mainly in the modeling of hyperelastic materials that have increasing stiffness, both in compression and tension.

$$\varepsilon_{SHG} = \frac{1}{n} \sinh[n(\lambda - 1)] \quad (2)$$



**Figure 2 – Generalized Hyperbolic Sine strain measures. Adapted from: Greco and Peixoto (2021).**

In the present article, a new family of strain measures is presented: the Generalized Hyperbolic Tangent (GHT) family – Eq. (4) and Figure 3. The idea is now to consider materials that have a decreasing stiffness as a characteristic, both in traction and compression, in order to contemplate the materials that cannot be modeled by the GHS family. The GHT family is believed to be suitable for the moderate strain regime, since it does not tend to  $-\infty$  when stretch tends to 0 and in tension it does not present a monotonic increase in stress.

$$\varepsilon_{TH} = \tanh(\lambda - 1) \quad (3)$$

$$\varepsilon_{THG} = \frac{\tanh[n \cdot (\lambda - 1)]}{n} \quad (4)$$

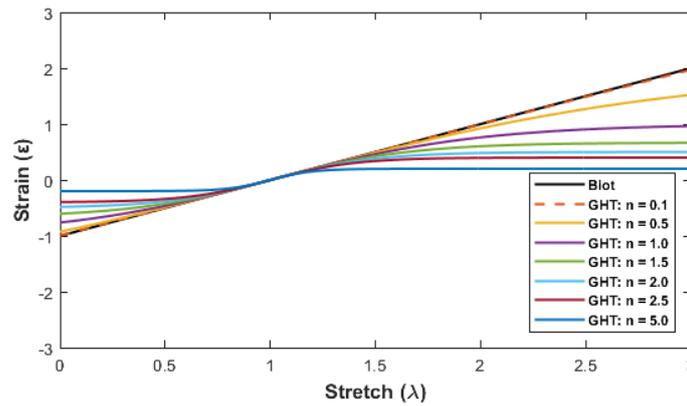


Figure 3 – Generalized Hyperbolic Tangent strain measures.

A possible area of application of the GHT family is in the representation of elastoplastic behavior and damage in materials that work under moderate strains. In this type of modeling, the materials tend to have approximately symmetrical behavior and loss of stiffness along the stress-strain curve. It is believed that the use of strain measures with a behavior closer (and more coherent) to the behavior of the material to be modeled is conceptually important, allowing an adequate conciliation between micromechanical models and the mathematical formulation. As an example, we can cite the Polyvinyl Chloride (PVC), whose material curve is shown in Figure 4.

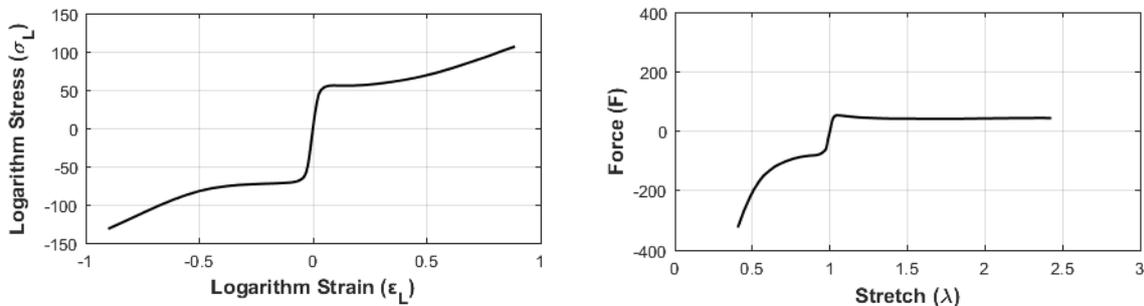


Figure 4 – Left Graph: PVC Logarithmic (“true”) stress vs. Logarithmic (“true”) strain curve; Right Graph: Force vs. Stretch curve. Data Source: Alves and Martins (2009).

In the analysis of trusses (bar elements) for moderate and large deformations, it is common to use the Logarithmic measure due to the fact that this measure allows the multiplicative decomposition of the Deformation Gradient,  $[F]$ , which in the 1D case is the stretch  $\lambda$  ( $\lambda = \lambda^e \lambda^p$ ), while an additive decomposition of the total strain can be used ( $\epsilon_L = \epsilon_L^e + \epsilon_L^p$ ). Driemeier *et al.* (2005), Greco and Ferreira (2009), Felipe (2019) and Felipe *et al.* (2019) are a few examples of studies that use the Logarithmic strain in the analysis of trusses. However, this measure has a strong asymmetry, which makes the “correction” due to plastification and degeneration of the material (which are clearly energetically dissipative phenomena) difficult to reconcile with considerations about the micromechanics of these phenomena.

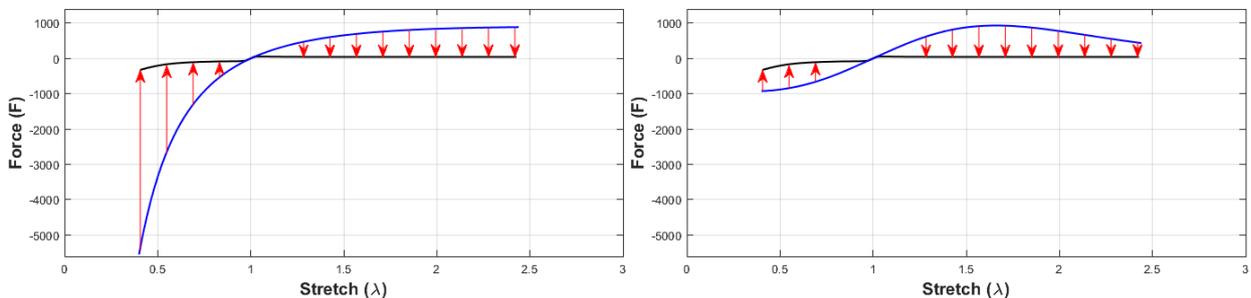


Figure 5 – Correction imposed on the elastic prediction for PVC. Left graph: Logarithmic strain measure; Right graph: Hyperbolic Tangent strain measure.

Observing Figure 5 it becomes clear that the “correction” using the Logarithmic strain is very different for compression and tension, while the “correction” when the Hyperbolic Tangent strain is used is more reasonable. One even wonders if smaller values of “corrections” may lead to smaller computational cost. Though further investigation is needed to confirm if the GHT family is really more coherent for elastoplastic and damage mechanics formulations, the discussion about how the use of different strain measures affects the parameters of those formulations needs to be made, because such parameters usually have physical meanings assigned to them, especially when a micromechanical model is used in the formulation conception.

Despite citing elastoplastic and damage mechanics models as an example of application of the THG family, this work will only focus on hyperelastic models. This because the characterization of the behavior of the THG family from the analysis of its hyperelastic Hookean-type model is a prerequisite for its use in elastoplastic and damage models, since, as a rule, the stress is considered to originate exclusively from the elastic portion of the deformation and, to be calculated, hyperelastic models are used.

Finally, it is noteworthy that, knowing the definition of the hyperbolic sine and hyperbolic tangent functions, the Generalized Hyperbolic family, presented in Greco and Peixoto (2021), and the THG family presented here can be unified in a single family, the Generalized Hyperbolic (HG) strain:

$$\varepsilon_{HG} = \left\{ \frac{e^{[n.(\lambda-1)]} - e^{-[n.(\lambda-1)]}}{2.A + (e^{[n.(\lambda-1)]} + e^{-[n.(\lambda-1)]}).B} \right\} \cdot n^{-1} \rightarrow A \in \{1, 0\}; B \in \{1, 0\}; A + B = 1; n \in \mathbb{R} \quad (5)$$

The graph presented by Figure 6 shows components of the HG family, which is highly versatile with the use of constants with clear physical meaning and is composed of symmetrical and objective measures.

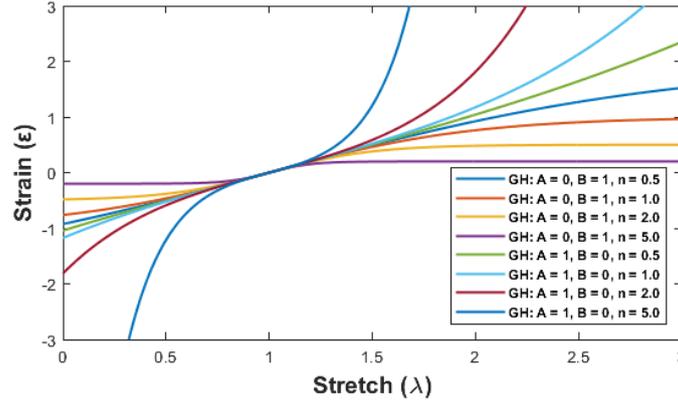


Figure 6 – Generalized Hyperbolic strain measures.

## HYPERBOLIC TANGENT STRAIN TENSOR

Equation (6) defines the 3D version of the Hyperbolic Tangent strain measure, that is, the Hyperbolic Tangent strain tensor,  $[\varepsilon^{HT}]$ , with  $[U]$  being the Right Stretch tensor and  $[I]$  the identity tensor. The Biot Strain tensor,  $[\varepsilon^B]$ , is defined by Eq. (7).

$$[\varepsilon^{HT}] = \tanh([U] - [I]) \quad (6)$$

$$[\varepsilon^B] = [U] - [I] \quad (7)$$

Knowing that the stretch  $\lambda$  for the uniaxial case is a 1D version of the tensor  $[U]$ , it is easy to associate the Eq. (3) and (6), which define the Hyperbolic Tangent measure for the 1D and 3D cases, and Eq. (1) and (7), which does the same for the Biot measure.

The HT tensor is rewritten as:

$$[\varepsilon^{HT}] = \tanh([\varepsilon^B]) \quad (8)$$

The hyperbolic functions of second order tensors obey the following identities:

$$\sinh([A]) = \frac{e^{[A]} - e^{-[A]}}{2} \quad (9)$$

$$\cosh([A]) = \frac{e^{[A]} + e^{-[A]}}{2} \quad (10)$$

$$\tanh([A]) = \frac{e^{[A]} - e^{-[A]}}{e^{[A]} + e^{-[A]}} \quad (11)$$

Given a generic diagonal second order tensor  $[X]$  shown by Eq. (12), its exponential is calculated as presented by Eq. (13).

$$[X] = \begin{bmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & X_{33} \end{bmatrix} \quad (12)$$

$$e^{[X]} = \begin{bmatrix} e^{X_{11}} & 0 & 0 \\ 0 & e^{X_{22}} & 0 \\ 0 & 0 & e^{X_{33}} \end{bmatrix} \quad (13)$$

Given a generic diagonalizable matrix  $[Y]$ , its exponential is determined in the following manner:

$$[Y] = [Z][Y'][Z^{-1}] \quad (14)$$

On Eq. (14),  $[Y']$  is the tensor whose diagonal is formed by the eigenvalues of  $[Y]$ , and  $[Z]$  is the tensor whose columns are the eigenvectors of  $[Y]$ .

For strain and stress tensors,  $[\varepsilon]$  and  $[\sigma]$ , the eigenvalues are the principal strains and stress, respectively, and the eigenvectors are the principal directions. The eigenvectors of the Biot strain tensor and the eigenvectors of the right stretch tensor coalesce, thus the Transformation Tensor  $[Q]$  (formed by eigenvectors) is equal for these two tensors  $[Q_{\varepsilon^B}] = [Q_U]$ . With all the information and definitions showed so far, one can rewritten the HT tensor:

$$[\varepsilon^{HT}] = \tanh([\varepsilon^B]) = \frac{e^{[\varepsilon^B]} - e^{-[\varepsilon^B]}}{e^{[\varepsilon^B]} + e^{-[\varepsilon^B]}} \quad (15)$$

With  $[\Lambda^B]$  being the diagonal tensor formed by the principal strains of the Biot strain tensor, one has

$$e^{[\varepsilon^B]} = [Q_U]e^{[\Lambda^B]}[Q_U^{-1}] \quad (16)$$

The eigenvalues of the right stretch tensor,  $\lambda_i$ , represent the stretches in the principal directions. The tensor whose diagonal is formed by these values,  $[\Lambda]$ , has the following relationship with  $[\Lambda^B]$ :

$$[\Lambda^B] = [\Lambda] - [I] \quad (17)$$

Eq. (16) can be rewritten as:

$$e^{[\varepsilon^B]} = [Q_U] \begin{bmatrix} e^{\lambda_1-1} & 0 & 0 \\ 0 & e^{\lambda_2-1} & 0 \\ 0 & 0 & e^{\lambda_3-1} \end{bmatrix} [Q_U^{-1}] \quad (18)$$

Substituting Eq. (18) in Eq. (15) one finds:

$$\tanh([\Lambda]) = [Q_U] \frac{e^{[\Lambda^B]} - e^{-[\Lambda^B]}}{e^{[\Lambda^B]} + e^{-[\Lambda^B]}} [Q_U^{-1}] \quad (19)$$

When dealing with square matrices (and second order tensors), the division operation is equivalent to multiplying by the inverse of the divisor matrix. This approach fails to have a unique answer if the divisor matrix is not invertible or if the dividend matrix and the divisor matrix do not commute. In the case of the Eq. (19), since both terms of the division are diagonal matrices, both criteria are met and the result is unique.

## HYPERBOLIC TANGENT STRESS TENSOR

Hill (1968) generalized the strain measures through Eq. (20), where  $\lambda_i$  and  $\vec{N}_i$  are the eigenvalues and the corresponding eigenvectors of  $[U]$ , respectively. According to this author, the generic function  $f$  must return null value and have its derivative equal to 1 when the stretch is equal to 1, in addition to always having a positive derivative.

$$f([U]) = \sum_i f(\lambda_i) \vec{N}_i \otimes \vec{N}_i \quad (20)$$

Using the notation presented by Eq. (20), the Tangent-Hyperbolic strain tensor can be rewritten as follows:

$$\varepsilon^{HT}([U]) = \sum_i \tanh(\lambda_i - 1) \vec{N}_i \otimes \vec{N}_i \quad (21)$$

This new way of presenting the HT tensor is important for the definition of its conjugate tensor: the Hyperbolic Tangent stress tensor. Farahani and Naghdabadi (2003) presented a general formula (Eq. (22)) which allows defining the conjugate stress of any strain measure.

$$\begin{cases} \bar{\sigma}_{ii}^f = \frac{g'(\lambda_i)}{f'(\lambda_i)} \bar{\sigma}_{ii}^g & i = 1, 2, 3 \\ \bar{\sigma}_{ij}^f = \frac{g(\lambda_i) - g(\lambda_j)}{f(\lambda_i) - f(\lambda_j)} \bar{\sigma}_{ij}^g & i \neq j \end{cases} \quad (22)$$

In this formula the conjugate stresses,  $[\bar{\sigma}^f]$  e  $[\bar{\sigma}^g]$ , of two generic strain measures,  $f([U])$  e  $g([U])$ , are related to each other. Through Eq. (22) it is possible to define the Hyperbolic Tangent stress tensor using common stress tensors used in continuum mechanics theory and numerical applications, such as Piola-Kirchhoff stress tensor  $[\sigma^G]$  and Jaumann stress tensor  $[\sigma^B]$ , that are conjugate to Green strain tensor  $[\varepsilon^G]$  and Biot strain tensor  $[\varepsilon^B]$ , respectively:

$$\varepsilon^G([U]) = \frac{1}{2} \sum_i (\lambda_i^2 - 1) \bar{N}_i \otimes \bar{N}_i \quad (23)$$

$$\varepsilon^B([U]) = \sum_i (\lambda_i - 1) \bar{N}_i \otimes \bar{N}_i \quad (24)$$

$$\begin{cases} \bar{\sigma}_{ii}^{HT} = \cosh^2(\lambda_i - 1) \bar{\sigma}_{ii}^B & i = 1, 2, 3 \\ \bar{\sigma}_{ij}^{HT} = \frac{\lambda_i - \lambda_j}{\tanh(\lambda_i - 1) - \tanh(\lambda_j - 1)} \bar{\sigma}_{ij}^B & i \neq j \end{cases} \quad (25)$$

$$\begin{cases} \bar{\sigma}_{ii}^{HT} = \lambda_i \cosh^2(\lambda_i - 1) \bar{\sigma}_{ii}^G & i = 1, 2, 3 \\ \bar{\sigma}_{ij}^{HT} = \frac{\lambda_i^2 - \lambda_j^2}{2[\tanh(\lambda_i - 1) - \tanh(\lambda_j - 1)]} \bar{\sigma}_{ij}^G & i \neq j \end{cases} \quad (26)$$

## GENERALIZED HYPERBOLIC TANGENT STRAIN TENSOR AND ITS STRESS CONJUGATE

The tensor form of the Generalized Hyperbolic Tangent strain is defined by Eq. (27).

$$[\varepsilon^{GHS}] = \frac{\tanh(n[\varepsilon^B])}{n} = \left(\frac{1}{n}\right) \left(\frac{e^{n[\varepsilon^B]} - e^{-n[\varepsilon^B]}}{e^{n[\varepsilon^B]} + e^{-n[\varepsilon^B]}}\right) = \left(\frac{1}{n}\right) [Q_U] \left(\frac{e^{n[\Lambda^B]} - e^{-n[\Lambda^B]}}{e^{n[\Lambda^B]} + e^{-n[\Lambda^B]}}\right) [Q_U^{-1}] \quad (27)$$

The tensor form of the Generalized Hyperbolic Tangent stress can be defined by Eq. (28) and (29).

$$\begin{cases} \bar{\sigma}_{ii}^{HT} = \cosh^2[n(\lambda_i - 1)] \bar{\sigma}_{ii}^B & i = 1, 2, 3 \\ \bar{\sigma}_{ij}^{HT} = \frac{n(\lambda_i - \lambda_j)}{\tanh[n(\lambda_i - 1)] - \tanh[n(\lambda_j - 1)]} \bar{\sigma}_{ij}^B & i \neq j \end{cases} \quad (28)$$

$$\begin{cases} \bar{\sigma}_{ii}^{HT} = \lambda_i \cosh^2[n(\lambda_i - 1)] \bar{\sigma}_{ii}^G & i = 1, 2, 3 \\ \bar{\sigma}_{ij}^{HT} = \frac{\lambda_i^2 - \lambda_j^2}{2\{\tanh[n(\lambda_i - 1)] - \tanh[n(\lambda_j - 1)]\}} \bar{\sigma}_{ij}^G & i \neq j \end{cases} \quad (29)$$

## POSITIONAL FINITE ELEMENT METHOD FORMULAE FOR TRUSSES

Greco and Peixoto (2021) presented the Generalized Hyperbolic Sine strain measure and its formulae for implementation into a PFEM algorithm capable of analyzing truss structures. The authors then proceeded to analyze the result of truss benchmarks and compare the behavior of different strain measures. Following the same notation and the same algorithm, the implementation of the Hyperbolic Tangent relies on Eq. (30) (definition of the strain), (31) (internal force) and (32) (Hessian Matrix).

$$\varepsilon_{TH} = \tanh \left[ \frac{\sqrt{(x_1^2 - x_1^1)^2 + (x_2^2 - x_2^1)^2 + (x_3^2 - x_3^1)^2}}{l_0} - 1 \right] \quad (30)$$

$$(F_k^{\beta(int)}) = \frac{\partial U_e^j}{x_k^\beta} = A_0^{(j)} \cdot \sigma_{TH} \cdot \operatorname{sech}^2 \left( \frac{l - l_0}{l_0} \right) \cdot (-1)^\beta \cdot \frac{(x_k^2 - x_k^1)}{l} \quad (31)$$

$$\begin{aligned} (H_{ik}^{\alpha\beta})^j = & A_0^{(j)} \cdot (-1)^\alpha \cdot (-1)^\beta \cdot \left\{ \frac{(x_i^2 - x_i^1)}{l} \cdot \frac{(x_k^2 - x_k^1)}{l} \cdot \left[ \frac{E_t(\varepsilon_{TH})}{l_0} \cdot \operatorname{sech}^4(\lambda - 1) \right. \right. \\ & \left. \left. - \sigma_{TH} \cdot \operatorname{sech}^2(\lambda - 1) \cdot \left( 2 \cdot \tanh(\lambda - 1) + \frac{1}{l} \right) \right] - \frac{\sigma_{TH}}{l} \cdot \delta_{ik} \cdot \operatorname{sech}^2(\lambda - 1) \right\} \end{aligned} \quad (32)$$

## APPLICATIONS

In this section the new family of strain measures will be used in the analysis of a truss structure and its properties will be investigated through the study of its behavior in pure deformation modes of solids.

## Truss analysis using PFEM and the GHT strain measure

Greco and Peixoto (2021) used the Generalized Hyperbolic Sine strain in the analysis of four trusses, comparing the results with the ones obtained with the classical strain measures. In such article, the goal was to demonstrate that important divergences arise from the use of different strain measures in moderate and large strain regime, highlight the importance of symmetry and test the Positional Finite Element Method (PFEM) formulations developed.

Here, the goals are to address the issue of symmetry of strain measures once more and proof the validity of the PFEM formulae developed for the GHT strains – Eq. (30), Eq. (31) and Eq. (32). To achieve those goals, the analysis of just one benchmark, the Circular Dome Truss, it is believed to be enough.

### Circular Dome Truss

The Circular Dome Truss showed in Figure 7 was exhibited for the first time by Papadrakakis (1981), since then it has become a benchmark for post-buckling analysis of trusses. The structure is composed of 168 bars and 73 nodes and has an axial stiffness of 10 000 [N] for each element. The analysis was performed using the PFEM through the use of the algorithm presented in Greco and Peixoto (2021), but now using Eq. (30), Eq. (31) and Eq. (32). Displacement control was used, with vertical position increments being imposed on the central node. Both the force  $P$  and the displacement are considered positive downwards.

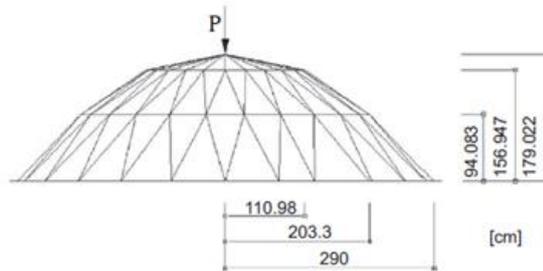


Figure 7 – Circular Dome Truss with its most important dimensions. Source: Greco e Ferreira (2009).

Figure 8 displays the results. On the left graph one has the classical strain measures, the Hyperbolic Sine (HS) strain measure (GHS with  $n = 1$ ) and Hyperbolic Tangent (HT) strain measure (GHT with  $n = 1$ ). One notes that when large strains are reached, the classical measures diverge significantly from each other, especially when the vertical displacement is greater than 200. However, the Biot, HS and HT measures remain very close to each other throughout the entire equilibrium trajectory, being an indication that symmetry is an important factor in itself. The graph on the right shows the curves referring to the measures of the GHT family with the use of different values of  $n$ , which generates very different measures, although, obviously, they all remain symmetrical. The result is that the symmetrical measures curves (right graph) are clearly closer to each other than the asymmetrical measures curves (classical measures in the left graph), which is another indication of the relevance of symmetry.

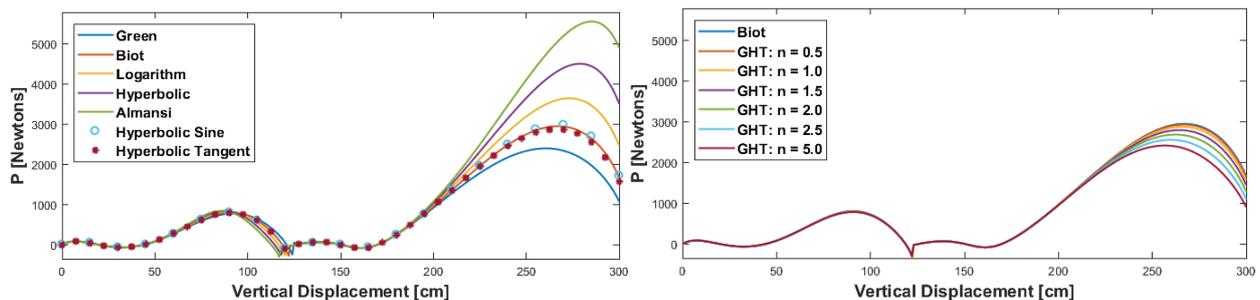


Figure 8 – Force vs. Vertical Displacement. Left: Classical, Hyperbolic Sine and Hyperbolic Tangent strain measures; Right: GHT strain measures.

The formulae implemented to represent the GHT strain measure in the PFEM showed a very good stability, given that the curves displayed in Figure 8 represent a complex equilibrium trajectory, with many snap-throughs. For the use in truss structures (unidimensional), the GHT family had results that attest its adequacy.

### Pure Modes of Solids

In articles that present new strain measures or in the ones in which the classical measures are investigated, it is very common to carry out an analytical study of the behavior of the hyperelastic model formed by using the proposed strain measure and its conjugate stress in the context of Hooke's law, the so-called Hookean-type models (Batra (1998); Batra (2001); Darijani and Naghdabadi (2010); Korobeynikov (2019); Beex (2019)). In these studies, pure deformation modes

are used, establishing intervals of physical coherence and indicating the types of applications suitable for the strain measures.

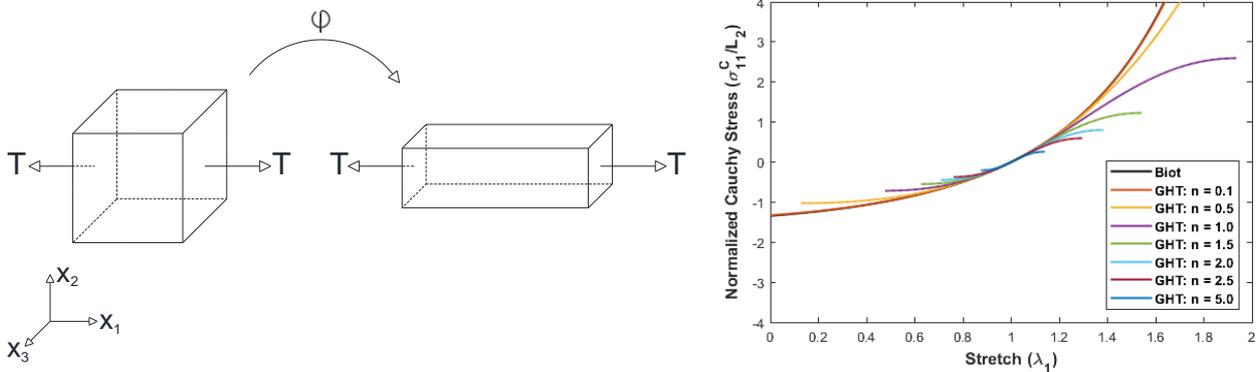
Here the Simple Axial Extension (SAE), Equi-Biaxial Loading (E-BL) and the Simple Shear (SS) modes will be studied. The material is considered hyperelastic, isotropic, incompressible and homogeneous.

**Simple Axial Extension**

The SAE is the first pure mode to be examined and the development of Batra (1998) is used here, considering the deformation motion represented by the mathematical expressions below (Eq. (33)) and assuming that the deformation is produced by surface tractions parallel to the  $x_1$ -axis, applied only on the end faces of the prismatic body, which has its mantle considered traction free. In Eq. (33),  $x_i$  are the current coordinates,  $X_i$  are the reference coordinates and  $\lambda_i$  are the stretches in the principal directions, with  $\lambda_1 \neq \lambda_2 = \lambda_3$ .

$$x_1 = \lambda_1 X_1; \quad x_2 = \lambda_2 X_2; \quad x_3 = \lambda_3 X_3; \tag{33}$$

Figure 9 shows on the left side a representation of the SAE mode and on the right the results obtained. The graph presents only the portion of the curves that has physical coherence, meaning that the GHT with  $n = 1$  loses physical meaning when  $\lambda_1 \approx 1.93$  in tension, for example. From these results, it can be seen that smaller values of  $n$  allow a wider range of application of the strain measures of the GHT family, with  $n$  values between 0 and 1 being the most appropriate for the regime of moderate and large strains. For  $n \geq 1$  the GHT strain measures have limits both in tension and compression, for  $n = 0.5$  the measure loses coherence in compression, and for  $n = 0.1$  the measure is really close to the Biot strain measure and has no limits of physical meaning.

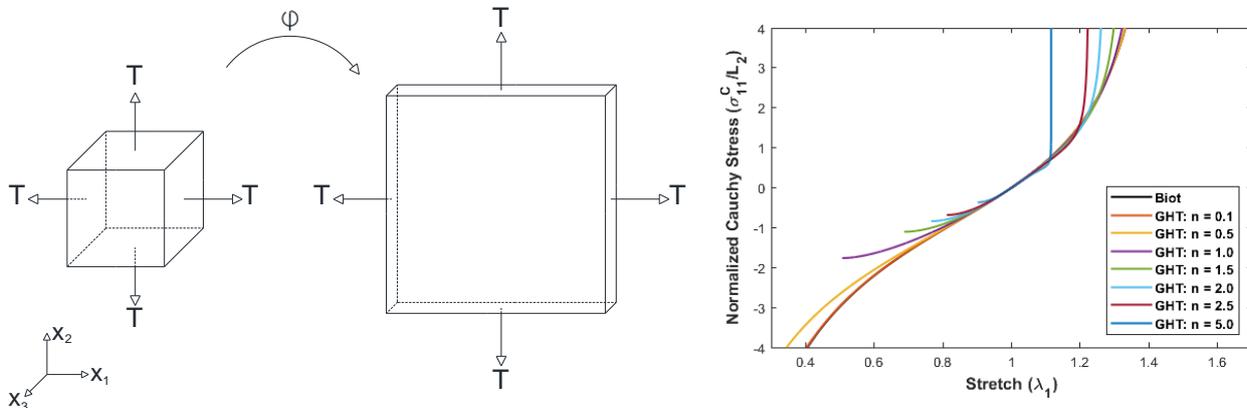


**Figure 9 – SAE mode. Normalized Cauchy Stress vs. Stretch in the  $x_1$ -axis direction.**

**Equi-Biaxial Loading**

The same script used in the SAE mode is used here for the E-BL pure mode. The difference is that now the deformation is produced by equal surface tractions parallel to both  $x_1$ -axis and  $x_2$ -axis (Figure 10 left side). Equation (34) represents the E-BL pure mode, with  $\lambda_1 = \lambda_2 \neq \lambda_3$ .

$$x_1 = \lambda_1 X_1; \quad x_2 = \lambda_2 X_2; \quad x_3 = \lambda_3 X_3; \tag{34}$$



**Figure 10 – E-BL mode. Normalized Cauchy Stress vs. Stretch in the  $x_1$ -axis direction.**

The graph in Figure 10 (right side) shows that, in tension, the higher  $n$  is the faster the stress tends to infinity as  $\lambda_1$  increases. Although there is no loss of physical coherence, lower values of  $n$  generate more reasonable results in tension. In compression, there is a loss of physical meaning for values of  $n \geq 1$ , and the greater  $n$ , the smaller the interval. Only values of  $n \leq 1$  are suitable to represent moderate and large deformations.

### Simple Shear

The Simple Shear pure mode is mathematically expressed by Eq. (35), with  $\gamma$  being the Shear Parameter. The analytical study of the SS pure mode is done following almost entirely the procedure created by authors Hashiguchi e Yamakawa (2012).

$$x_1 = X_1 + \gamma \cdot X_2 \quad x_2 = X_2 \quad x_3 = X_3 \quad (35)$$

Figure 11 shows on the left side the representation of the SS mode and on the right side the graph with the shear component of the Nominal stress tensor ( $P_{12}$ ) vs. the Shear Parameter. With exception of the GHT measure with  $n = 0.1$ , all other strain measures loose physical coherence at some point of the curve, with the interval of physical coherence increasing as the value of  $n$  decreases. Only values of  $n \leq 1$  are suitable to represent moderate and large deformations.

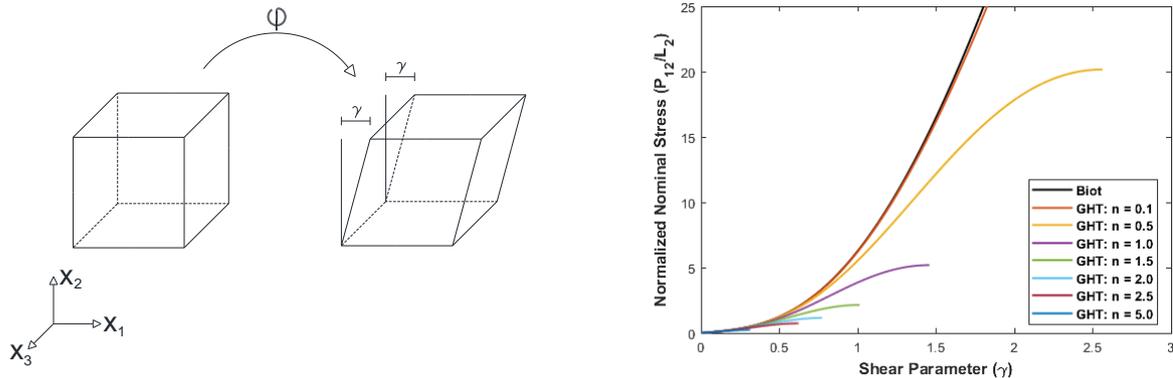


Figure 11 – SS mode. Normalized Nominal Stress vs. Shear Parameter.

## CONCLUSIONS

The Generalized Hyperbolic Tangent strain tensor, an objective, symmetrical and versatile family of strain measures based on the hyperbolic tangent function, was presented both in its uniaxial and in its tensor form (3D). The mathematical definition for the multiaxial (tensor) form was developed starting from the uniaxial definition. The formulae for implementation in the PFEM for trusses was also determined.

The results obtained from the analysis of a truss benchmark, the Circular Dome Truss, attest the validity of the PFEM formulae developed for the GHT strains and the adequacy of the new family of strains for the uniaxial case in moderate and large strain regime. The question about the importance of symmetry was addressed once more and again, in accordance with the results of Greco and Peixoto (2021), there were indications that symmetry is in itself a relevant factor to have closer equilibrium trajectories when using different strain measures.

An investigation about the behavior of the GHT family for solids under pure mode deformations was carried out and the results revealed that smaller values of  $n$ , specially between 0 and 1, lead to larger intervals of validity and more reasonable results in the moderate and large strain regime. For  $n > 1$ , the GHT family is adequate only for small strains.

Further investigation about the GHT family need to be made to determine where its application will be more useful. Especial interest should be given for its application in elastoplastic and damage formulations, especially in truss structures, because the behavior of the GHT family seems to lead to more reasoned energy dissipation corrections.

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