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Application of the Hybrid-Mixed Finite Element Method with Stabilized Nodal Enrichment on 2D Problems

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Abstract: This work addresses the development of non-conventional variants of the Finite Element Method (FEM) for plane elasticity based on the combination of the Stable Generalized Finite Element Method (SGFEM) with Hybrid-Mixed Stress Formulation (HMSF). For the HMSF, three approximation fields are involved: stresses and displacement in the domain and displacement on the static boundary. In the combined HMSF-SGFEM approach, the stress domain field enrichment is provided by the product of the Partition of Unity (PoU) and polynomials enrichment functions. It is noteworthy that the nodal enrichment structure in SGFEM is different from that applied in the Generalized Finite Element Method (GFEM). In addition, the resource of the so-called nodal enrichment available by SGFEM and GFEM conceptually enlarges the approximation bases of the stress field of the HMSF, without the need to introduce new nodal points in the domain. The performance of this new approach (HMSF-SGFEM) is illustrated and compared with results from classical FEM. The results, in terms of strain energy and stress field, obtained from the application of this new methodology, HMSF-SGFEM, point to the quality of this unconventional methodology presented, once the stress field enrichment did not destroy strain energy results and improved the stress field. Lastly, the application of nodal enrichment in problems with coarse meshes allowed obtaining results comparable to those obtained through the same problems, but with more refined mesh and without enrichment.

Keywords: Hybrid-Mixed Stress Formulation, Stabilized Generalized Finite Element Method, Nodal Enrichment, non-conventional Finite Element Method

INTRODUCTION

In continuum mechanics, physical phenomena are described mathematically through the interactions of the three fields: displacements (u), stresses (σ) and strains (ε). Thus, a reliable mathematical model must contain restrictions on the displacements, stresses, and strains that allow the equilibrium equations, compatibility, constitutive laws and boundary conditions of each phenomenon analyzed. These models can be called Boundary Value Problem (BVP), and among the numerical methods, the conventional "displacement" form of Finite Element Method (FEM) is one of the most widespread and applied tools in the approximate solution to a BVP.

The main reasons for the acceptance of the FEM, in the technical and scientific environment, are its simplicity and conceptual elegance, combined with the robustness of the mathematical formulation and the easy implementation via computer programming for analysis of structural components (Castro, 1996). Because of the growing number of professionals working with FEM commercial software, there are new needs and opportunities to develop new methods that improve their applicability in order to obtain reliable results in specific cases. Therefore, the importance of the development of new mathematical formulations and numerical models that are able to realistically simulate the mechanical phenomena involved.

However, the implementation of the classic FEM can yield unsatisfactory results, for example, when low order polynomial functions are used. This is the case of problems involving regions with singularities (characterized by high-stress gradients), (Szabó and Babuŝka, 1991). There are also other situations that tend to limit classical FEM application; problems involving almost incompressible elasticity are a good example, because even if discretization using classical finite elements is reasonable, rough solutions can be obtained, (Bathe,1996).

In order to improve the application and obtain reliable results in specific cases where the classic FEM has low accuracy or efficiency, several alternative formulations have been researched and applied recently. For this reason, it will be proposed in this work the application of the Stable Generalized Finite Element Method (SGFEM) nodal enrichment technique in the Hybrid-Mixed Stress Formulation (HMSF), giving rise to the HMSF-SGFEM.

The HMSF simultaneously approximates the stress and displacement fields in the problem domain. Its hybridization is given by the approximation of the displacement also on the static boundary. As in the HMSF no fundamental equation needs to be satisfied a priori, the HMSF allows the use of a wide range of functions to define the approximation bases (Vicente da Silva; Castro; Pereira, 2015).

Both GFEM and SGFEM can be summarized as the expansion of the approximate (polynomial) bases of the FEM without the need to introduce new nodal points in the domain. This resource is called nodal enrichment. It is worth noting an important feature of these methods is that their implementations preserve the conventional structure of the FEM. Since the SGFEM is a variation of the GFEM, when its classical application - FEM plus nodal enrichment, which has as main objective to improve the conditioning of the stiffness matrix generated by the enriched approximation functions (Duarte, Babuška and Oden (2000), Babuška and Banerjee (2012), Gupta et al. (2013) and Lins (2015)).

Thus, as a contribution of this work, we propose the implementation of a computational algorithm developed in Python for HMSF, in the sense of integration with the objected-oriented programming toolkit named SCIEnCE (São Carlos Integrity Environment for Computational Engineering), whose detailed description can be found in Piedade Neto et al. (2013). Consequently, our main objective would be to enable other HMSF applications, such as the HMSF-GFEM and HMSF-SGFEM, applied to the plane isotropic problems, highlighting the necessary conditions for the convergence of the problems and the analysis of the results.

HYBRID-MIXED FINITE ELEMENT METHOD WITH STABILIZED NODAL ENRICHMENT ON 2D PROBLEM

This section describes the theoretical aspects related to HMSF as well as how this formulation can be extended by introducing custom enrichment functions on nodes using the nodal enrichment technique proposed in the SGFEM.

The HMSF is a non-conventional numerical formulation called mixed because approximates two unrelated fields, stress σ and displacement u on the domain Ω, see Fig. 1. The independence of the domain and boundary Γ displacement field defines the hybrid scheme. The main advantage over a classical FEM is the direct approximation of the stress field, making the post-processing to define the stress field unnecessary. Another important and desired feature in hybrid formulations is their affinity with p-refinement, as they can exploit hierarchical function sets (Businaro and Bussamra, 2020).

The SGFEM can be seen as a variation of GFEM, and both are numerical methods of nodal enrichment, which, in short, can be defined as the expansion of the approximate bases involved, without the need to introduce new nodes in the finite element domain of the discretization of a given problem.

Hybrid-Mixed Stress Formulation (HMSF)

For the problems of this work, an elastic body subject to certain loads is considered, where $\mathbf{b}$ is a vector that contains the volume forces and $\bar{\mathbf{t}}$ is a vector that contains the surface forces. This elastic body can be represented by a domain Ω confined by a regular boundary Γ. This boundary is composed by the union of the Dirichelt/Essential Boundary Conditions Γ_u and the Neumann/Natural Boundary conditions Γ_t ($\Gamma = \Gamma_u + \Gamma_t$), see Fig. 1.

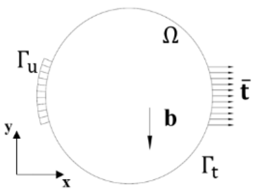

Figure 1 – Elastic Body subjected to body forces and traction.

Mathematically, the weight functions that describe the HMSF are P_Ω associated with the equilibrium equation, R_Γ related to the Neumann boundary conditions and M_Ω associated with the compatibility equation.

$$\int_\Omega \mathbf{P}_\Omega^\mathbf{T}(\mathbf{L}^\mathbf{T}\boldsymbol{\sigma} + \mathbf{b})\, d\Omega = 0 \tag{1}$$

$$\int_{\Gamma_t} \mathbf{R}_{\Gamma_t}^\mathbf{T}(\mathbf{N}\boldsymbol{\sigma} - \bar{\mathbf{t}})\, d\Gamma = 0 \tag{2}$$

$$\int_\Omega \mathbf{M}_\Omega^\mathbf{T}(\mathbf{f}\boldsymbol{\sigma} - \mathbf{L}^\mathbf{T}\mathbf{u})\, d\Omega = 0 \tag{3}$$

where the matrix $\mathbf{L}$ represents the divergent differential operator, $\boldsymbol{\sigma}$ and $\mathbf{u}$ are, respectively, the stress tensor and displacement vector in the Ω domain. $\mathbf{N}$ represents the matrix that contains the components of the direction normal to the boundary. And $\mathbf{f}$ represents the flexibility matrix.

Altering Eq. (3) through integration by parts in order to prescribe Dirichlet's conditions Γ_u, $\mathbf{u} = \bar{\mathbf{u}}$, the following relation can be written.

$$\int_{\Omega} \mathbf{M}_{\Omega}^{\mathrm{T}} \mathbf{f} \boldsymbol{\sigma} \, d\Omega + \int_{\Omega} (\mathbf{L}\mathbf{M}_{\Omega})^{\mathrm{T}} \mathbf{u} \, d\Omega - \int_{\Gamma_t} (\mathbf{N}\mathbf{M}_{\Omega})^{\mathrm{T}} \mathbf{u}_{\Gamma} \, d\Gamma - \int_{\Gamma_u} (\mathbf{N}\mathbf{M}_{\Omega})^{\mathrm{T}} \bar{\mathbf{u}} \, d\Gamma = 0 \tag{4}$$

In Eq. (4) the fields considered independent are highlighted, justifying the HMSF proposal: displacements $\mathbf{u}$ and stresses σ in the domain Ω (mixed) and independent displacements $\mathbf{u}_\Gamma$ in the part Γ_t of the boundary (hybrid). It is admitted, therefore, that: the stresses and displacements are incompatible in the domain; the displacement approximations in the domain and boundary are different and are enforced to be compatible in a weak form, through the previous relation (Eq. 4.). In other words, the boundary values can be approximated independently of the approximations adopted on domain. And since $\bar{\mathbf{u}}$ and $\bar{\mathbf{t}}$ are known in the boundaries (Γ_u and Γ_t) of the Ω domain, t and u can be approximated in these same boundaries (Góis, 2009).

Thus, the numerical approximations are defined in the domain Ω and in the boundary Γ_t, and the three approximate and independent fields can be indicated using the interpolation of nodal values, in which $\mathbf{S}_\Omega$, $\mathbf{U}_\Omega$, and $\mathbf{U}_{\Gamma t}$ represent the matrices that hold the approximation functions, of the stress field and displacement in the domain and displacement in the boundary, respectively. And $\mathbf{s}_\Omega$, $\mathbf{u}_\Omega$, and $\mathbf{u}_{\Gamma t}$ can be interpreted as vectors that hold the weights of approximations.

$$\boldsymbol{\sigma} = \mathbf{S}_{\Omega} \mathbf{s}_{\Omega}, \text{ in } \Omega \tag{5}$$

$$\mathbf{u} = \mathbf{U}_{\Omega} \mathbf{q}_{\Omega}, \text{ in } \Omega. \tag{6}$$

$$\mathbf{u}_{\Gamma} = \mathbf{U}_{\Gamma_t} \mathbf{q}_{\Gamma_t}, \text{ in } \Gamma_t. \tag{7}$$

Imposing the same procedure adopted in FEM, in which the Galerkin method is applied in the construction of the weight functions $\mathbf{M}_\Omega$, $\mathbf{P}_\Omega$ and $\mathbf{R}_\Gamma$, results in the following restrictions:

$$\mathbf{M}_{\Omega} = \mathbf{S}_{\Omega} \qquad \mathbf{P}_{\Omega} = \mathbf{U}_{\Omega} \qquad \mathbf{R}_{\Gamma_t} = \mathbf{U}_{\Gamma_t} \tag{8}$$

Finally, the linear system of the HMSF is given by:

$$\begin{bmatrix} \mathbf{F}_{\Omega} & \mathbf{A}_{\Omega} & -\mathbf{A}_{\Gamma_t} \\ \mathbf{A}_{\Omega}^{\mathrm{T}} & 0 & 0 \\ -\mathbf{A}_{\Gamma_t}^{\mathrm{T}} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{s}_{\Omega} \\ \mathbf{q}_{\Omega} \\ \mathbf{q}_{\Gamma} \end{Bmatrix} = \begin{Bmatrix} \mathbf{e}_{\Gamma_u} \\ -\mathbf{Q}_{\Omega} \\ -\mathbf{Q}_{\Gamma_t} \end{Bmatrix}. \tag{9}$$

In which, the matrices that define the HMSF system are:

$$\mathbf{F}_{\Omega} = \int_{\Omega} \mathbf{S}_{\Omega}^{\mathrm{T}} \mathbf{f} \mathbf{S}_{\Omega} \, d\Omega \tag{10}$$

$$\mathbf{A}_{\Omega} = \int_{\Omega} (\mathbf{L}\mathbf{S}_{\Omega}^{\mathrm{T}}) \mathbf{U}_{\Omega} \, d\Omega \tag{11}$$

$$\mathbf{A}_{\Gamma_t} = \int_{\Gamma_t} (\mathbf{N}\mathbf{S}_{\Omega})^{\mathrm{T}} \mathbf{U}_{\Gamma_t} \, d\Gamma \tag{12}$$

$$\mathbf{e}_{\Gamma_u} = \int_{\Gamma_u} (\mathbf{N}\mathbf{S}_{\Omega})^{\mathrm{T}} \bar{\mathbf{u}} \, d\Gamma \tag{13}$$

$$\mathbf{Q}_{\Omega} = \int_{\Omega} \mathbf{U}_{\Omega}^{\mathrm{T}} \mathbf{b} \, d\Omega \tag{14}$$

$$\mathbf{Q}_{\Gamma_t} = \int_{\Gamma_t} \mathbf{U}_{\Gamma_t}^{\mathrm{T}} \bar{\mathbf{t}} \, d\Gamma \tag{15}$$

At this point, the approximation functions are constructed by the FEM technique. In this work, bilinear quadrilateral elements will be used in the domain, whereas linear elements will be used in the boundary.

Concerning to the solvability of the HMSF system, it should be stated that it cannot be directly solved, since the resulting system is linearly dependent. In this work, the Babuška procedure firstly presented in (Stroboulis, Babuška, Copps, 2000) and used in (Góis, 2009) is applied to solve the problem.

A brief review of GFEM and SGFEM

The GFEM, originally introduced by Melenk and Babuska (1996) and Duarte and Oden (1996), is based on the concept Partition of Unit (PoU) and this can be considered an advantage of the method. Because of this feature is necessary to change only a part of the FEM code to incorporate special shape functions with the compact support (Babuška and Banerjee, 2012). Despite this, the GFEM may present numerical difficulties in some cases depending on the enrichment functions chosen, for example, the ill-conditioning of the stiffness matrix and the presence of blending elements. This kind of element appears when the enrichment is limited to a restricted region of the domain or number of patches, as shown in Fig. 2. (Sato, 2017).

The SGFEM is a variation of the GFEM, whose objective is to maintain the good convergence of the approximation, but with a conditioning, in the same order of magnitude as that presented in the classical FEM.

For the application of the GFEM and SGFEM in plane analysis (2D), it is necessary to define supports or clouds formed by a set of finite elements that share a common node *j* (see Fig 2). The main difference between the enrichment methods, SGFEM and GFEM, when compared to FEM is in the definition of shape functions, in these methods the shape functions (ϕ) are defined within each cloud by the product between the enrichment functions (L_{jk}) (polynomial or not) and the functions of the unit partition (φ_j) belonging to the element.

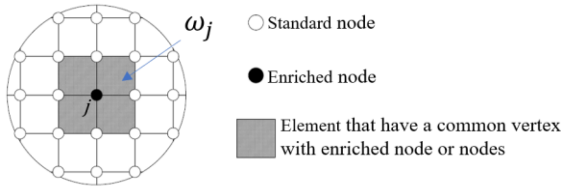

Figure 2 – Cloud example for a two-dimensional discretization (quadrilateral elements).

Therefore:

$$\phi_{jk} = \varphi_j L_{jk} \tag{16}$$

where *j* identifies the nodal cloud, *k* (=1,..., *nl*) identifies the enrichment function, and *nl* is the total number of enrichment functions adopted for each cloud.

Considering that the unity is taken as the first component of the set of enrichment functions, the global approximation space for the GFEM can be split into a conventional FEM part and an enrichment part (Lins, Proença, Duarte, 2019). Therefore, restricting only to plane problems, the GFEM approximation for each component of the displacement field is given by:

$$u = \sum_{i=1}^{n} \varphi_j \left\{ U_j + \sum_{k=1}^{nl} L_{jk} b_{jk} \right\} \tag{17}$$

$$v = \sum_{i=1}^{n} \varphi_j \left\{ V_j + \sum_{k=1}^{nl} L_{jk} c_{jk} \right\} \tag{18}$$

where n is the total number of nodes in the discretized domain, φ_j are the original shape functions (PoU) for the displacement field, U_j and V_j are the degrees of freedom in displacements associated with the originals shape functions and e_{jk} are the new nodal parameters corresponding to each enrichment.

The basic difference between SGFEM and GFEM consists in modifying the enrichment functions by means of an interpolant. Thus, the enrichment function of the SGFEM is obtained in the cloud via the difference between the enrichment function L_{jk} and its interpolant I_{wj}, therefore:

$$L_{jk}^{SGFEM} = L_{jk} - I_{wj}\left(L_{jk}\right) \tag{19}$$

The interpolant I_{wi} is defined as:

$$I_{wj(x,y)} = \sum_{i=1}^{m} \varphi_j(x,y)\, L_{jk}(x_i, y_i) \tag{20}$$

where (*xj*, *yj*) are the coordinates of node *j* of the element in question and *m* is the number of the element nodes. After building the enrichment function, the procedure for enriching the shape functions is similar to that applied in GFEM:

$$\emptyset_{JK}^{SGFEM} = \varphi_j L_{jk}^{SGFEM} \tag{21}$$

Hybrid-Mixed Finite Element Method with Stabilized Nodal Enrichment on 2d Problem

In HSMF-SGFEM approach, a PoU functions create a confirming approximation which can be improved by a nodal enrichment strategy. In this work, quadratic polynomials are adopted as enrichment, i.e.:

$$L_{jk}^{SGFEM}(x) = \left(x - x_j\right)^2 - \sum_{i=1}^{4} \varphi_j \left(x_i - x_j\right)^2 \tag{22}$$

$$L_{jk}^{SGFEM}(y) = \left(y - y_j\right)^2 - \sum_{i=1}^{4} \varphi_j \left(y_i - y_j\right)^2 \tag{23}$$

It is worth mentioning that such the enrichments will be applied only in the stress field. Due to the HMSF formulation, this enrichment is able also to change the approximations in displacement fields. This improvement in the displacement field, due to the matrix that holds the stress approximation functions, $\mathbf{S}_{\Omega}$, is present not only in the definition of stress matrix $\mathbf{F}_{\Omega}$, Eq. (10), but also in the domain and in the boundary displacement matrices, respectively $\mathbf{A}_{\Omega}$ and $\mathbf{A}_{\Gamma t}$, as shown in the Eq. (11) and Eq. (12).

In this work, numerical evaluations of the application of the Hybrid-Mixed Finite Element Method with Stabilized Nodal Enrichment on two 2D problems will be presented. The adopted approach is based on selective enrichment, that is, the enrichment is applied to a few chosen nodes of the problem domain. The results obtained are compared in terms of convergence rate in strain and displacement energy. In addition, graphical results of a specific mesh will be presented, in order to compare the stress and displacement fields with the conventional method (FEM) and also with the responses adopted as references. The results obtained in each presented problem are briefly analyzed during its presentation. Finally, the main conclusions are summarized.

NUMERICAL RESULTS

This item presents the numerical results of two problems: the plate under traction (Fig. 3), and the Cook panel (Fig. 10). Using the developed program within SCIEnCE tool previously mentioned, it was possible to solve these problems considering four different numerical methods: the classic FEM; HMSF; HMSF-GFEM; HMSF-SGFEM. To HMSF-GFEM and HMSF-SGFEM simulations were performed for each mesh configuration with the enrichment (Eq. (22) + Eq. (23)) applied on some nodes selected near to loading.

Plate under traction

The first benchmark problem used to analyze the potentialities of the proposed formulation is a plate under traction problem described in Fig. 3a. For simplicity, measurement units were not adopted. The Young Modulus E = 1000, Poisson's ratio $\upsilon = 0.3$, traction $\mathbf{p} = 10$ (to this problem, $\mathbf{b}$=0) and plane stress conditions are assumed. Aiming to simplify the simulation was considered only the symmetric part depicted in Fig. 3b. The proposed formulation is evaluated in terms of strain energy and displacement, where the reference values are, respectively, $U_{ref} = 0.369$, originally obtained in ANSYS, for strain energy and the $ux_{ref} = 0.032$ for horizontal displacement at the point A.

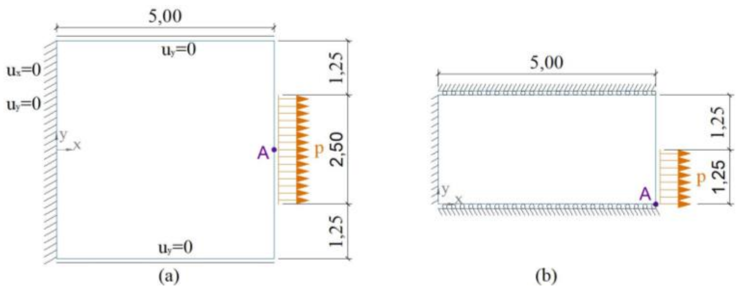

Figure 3 - Plate under traction: (a) geometry and boundary conditions and (b) analyzed symmetric part

Using plane bilinear quadrilateral elements, the discretizations for this problem were made with the following grid sizes: 4x2, 8x4, 16x8 and 32x16 elements as shown in Fig. 4.

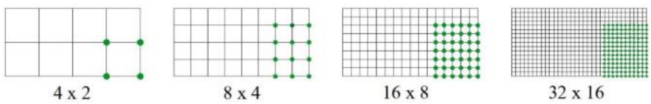

Figure 4 – Adopted meshes for plate under traction simulation. The enriched nodes are highlighted in green.

Figure 5 shows the comparison between the methods in terms of the strain energy as a function of degrees of freedom. While Fig. 6 compares the horizontal displacement at the point A indicated in Fig. 3.

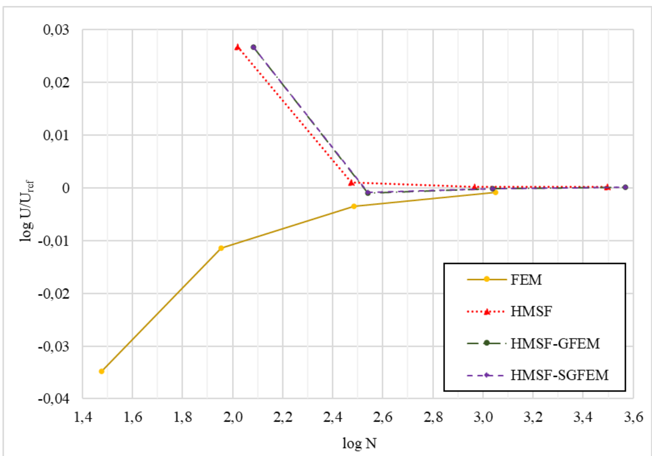

Figure 5 - The comparative of $\log(U/U_{ref})$ x log(N) in the FEM, HMSF, HMSF-GFEM and HMSF-SGFEM, where N holds the number of degrees of freedom of the discretization.

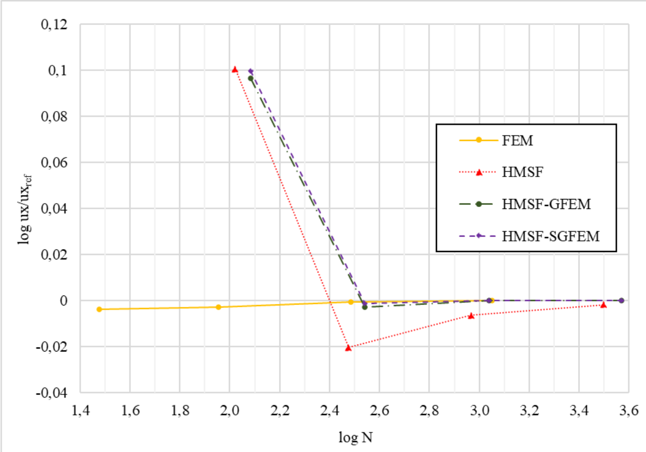

Figure 6 - The comparative of displacement in x at the point A, log(ux/uxref) x log(N) in the FEM, HMSF, HMSF-GFEM and HMSF-SGFEM, where N holds the number of degrees of freedom of the discretization.

In Figure 5, it is possible to observe that non-conventional Finite Element Method HMSF, HMSF-GFEM and HMSF-SGFEM present a higher rate of convergence in terms of strain energy than that obtained by classical FEM. The good convergence rate recovered by the HMSF has been improved with the application of the enrichment GFEM and SGFEM, not only in the convergence rate in terms of energy but also in the graphical representation of the stress field, see Fig. 8.

Analyzing the convergence rates in horizontal displacement, Fig. 6, it is concluded that the classical FEM can recover considerably better solutions than the HMSF, mainly in less refined meshes such as the 4x2 and 8x4 shown in Fig. 4. However, the application of enrichment, GFEM and SGFEM, in HMSF proved to be efficient in improving the solutions obtained in terms of displacements. This can be seen in both the convergence curves in Fig. 6 and in the graphical representation shown in Fig. 8.

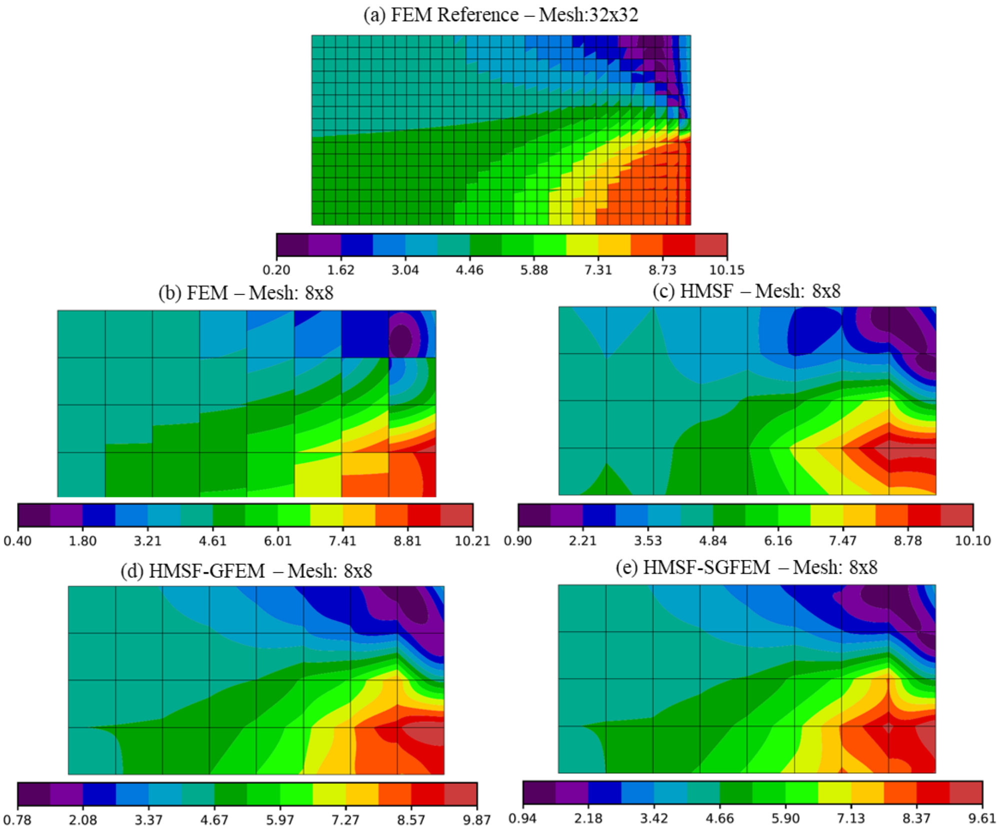

Figure 7 – The von Mises stress field for the Plate under traction problem, with the discretization of 8x8 bilinear quadrilateral elements. Using the symmetry shown in Fig. 3 b.

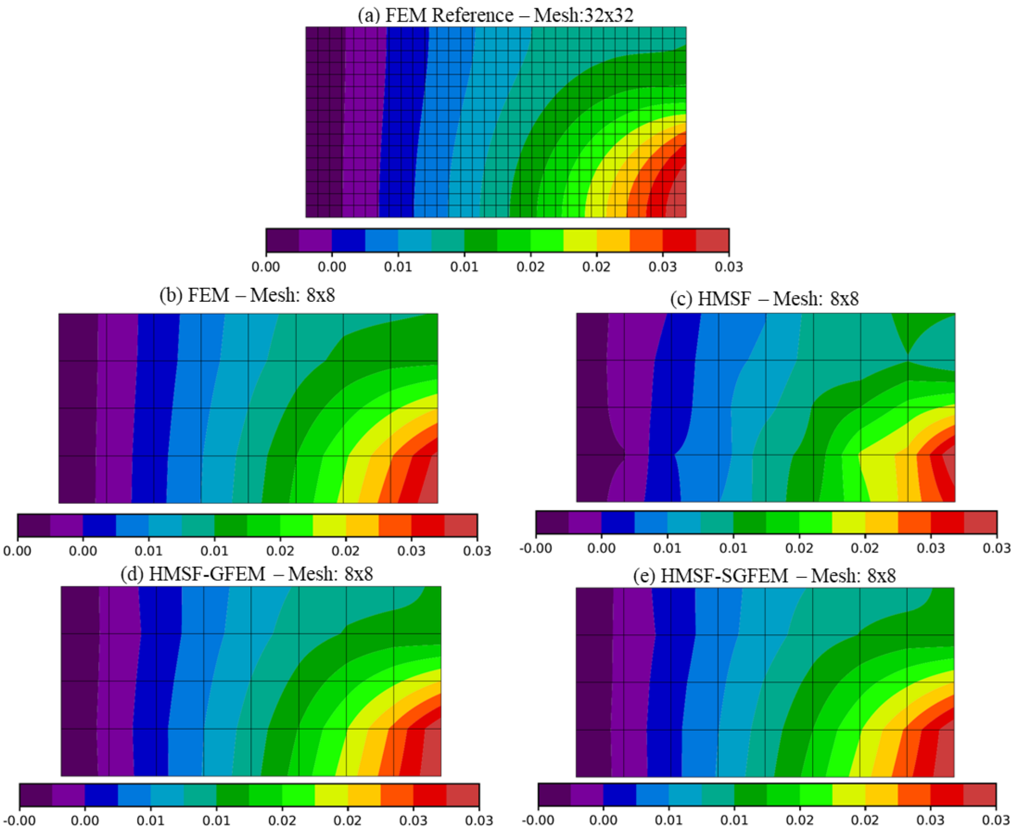

Figure 8 – Displacement field in x, for the Plate under traction problem with the discretization of 8x8 bilinear quadrilateral elements. Using the symmetry shown in Fig. 3 b.

Cook Panel

Proposed by Cook (1987), this problem consists of a sheet with unit thickness, where one end is clamped and the opposite end is subjected to a uniformly distributed load q=0.00625 units of force/units of length. Once again for simplicity, measurement units were not adopted. Young's modulus E = 1000, Poisson's ratio υ = 1/3 and plane stress conditions are assumed. The geometry of this problem is shown in Fig. 9. The proposed formulation is evaluated in terms of strain energy and displacement, where the reference values are, respectively, U_{ref} = 0.012, originally obtained in ANSYS, for strain energy and the uy_{ref} = 0.239 for vertical displacement at the point A.

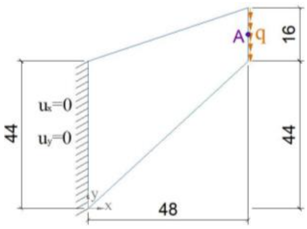

Figure 9 – Cook Panel problem

Using plane bilinear quadrilateral elements, the discretizations for this problem were made with the following grid sizes: 2x2, 4x4, 8x8 and 16x16 elements as shown in Fig. 10. Another objective of the analysis of this problem is to verify the sensitivity to distortion of bilinear quadrilateral elements in the convergence of the solutions obtained via the proposed methods.

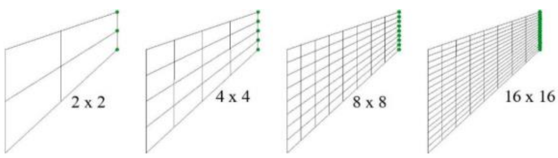

Figure 10 – Adopted meshes for Cook panel simulation. The enriched nodes are highlighted in green.

Figure 11 shows the comparison between the strain energy as a function of degrees of freedom while Fig. 12 compares the vertical displacement at the point A indicated in Fig. 9.

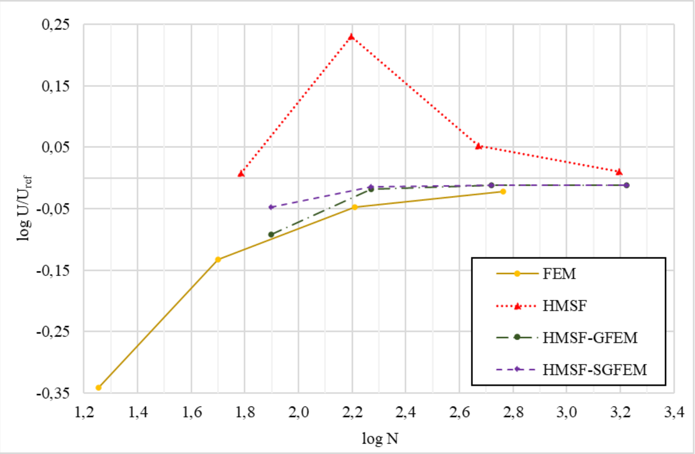

Figure 11 - The comparative of log(U/Uref) x log(N) in the FEM, HMSF, HMSF-GFEM and HMSF-SGFEM, where N holds the number of degrees of freedom of the discretization.

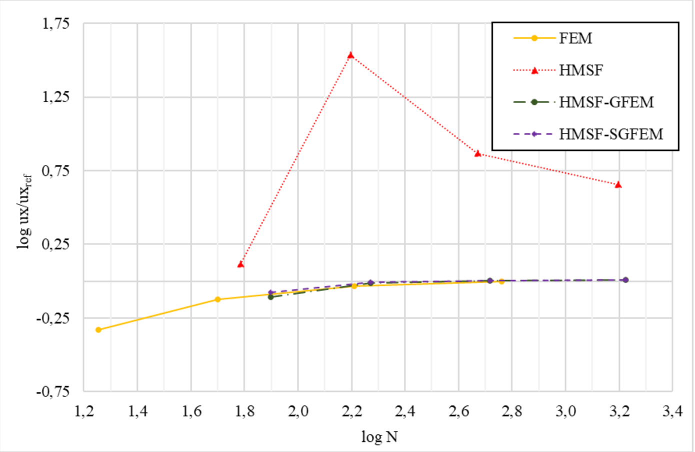

Figure 12 - The comparative of displacement in x at the point A, log(ux/uxref) x log(N) in the FEM, HMSF, HMSF-GFEM and HMSF-SGFEM, where N holds the number of degrees of freedom of the discretization.

It is observed that the HMSF presents poor conversion rates in this problem, both in terms of energy (Fig. 11) and in the representation of displacements (Fig. 12) when compared to the FEM. For example, the representation of displacement fields of HMSF, Fig. 14(c), does not represent the expected behavior. These poor solutions verified in the HMSF may have occurred due to the presence of spurious modes and a large number of degrees of freedom when compared with the conventional displacement elements, which are known disadvantages of mixed-hybrid elements (Pereira and Freitas, 1996).

However, the application of enrichment in a few nodes close to the Neumann boundary conditions proved to be highly efficient in the recovery of solutions when applying HMSF-GFEM and HMSF-SGFEM, see Fig 11 and Fig. 12. It is noteworthy that the application of enrichment on the nodes shown in Fig. 10, in the 8x8 mesh was sufficient to obtain better graphic results than those obtained by the conventional FEM method and similar to the reference solutions, see Fig. 13 and Fig. 14.

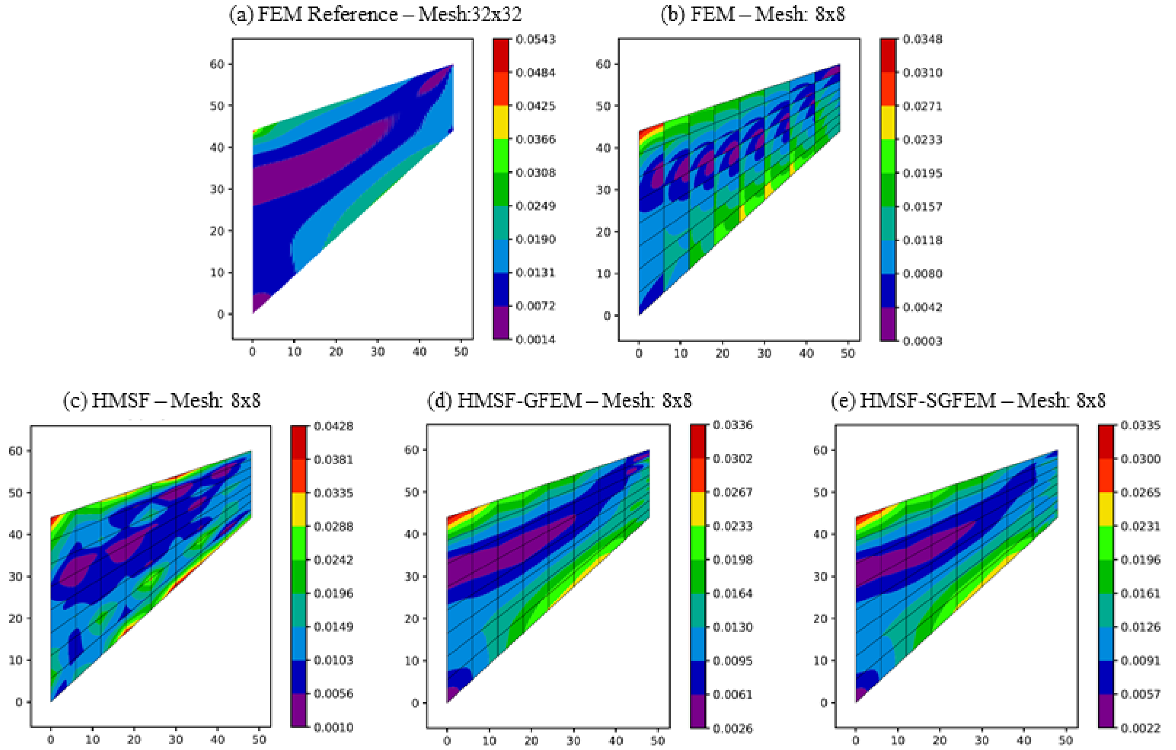

Figure 13 – The von Mises stress field for the tension Cook Panel problem, with the discretization of 8x8 quadrilateral elements.

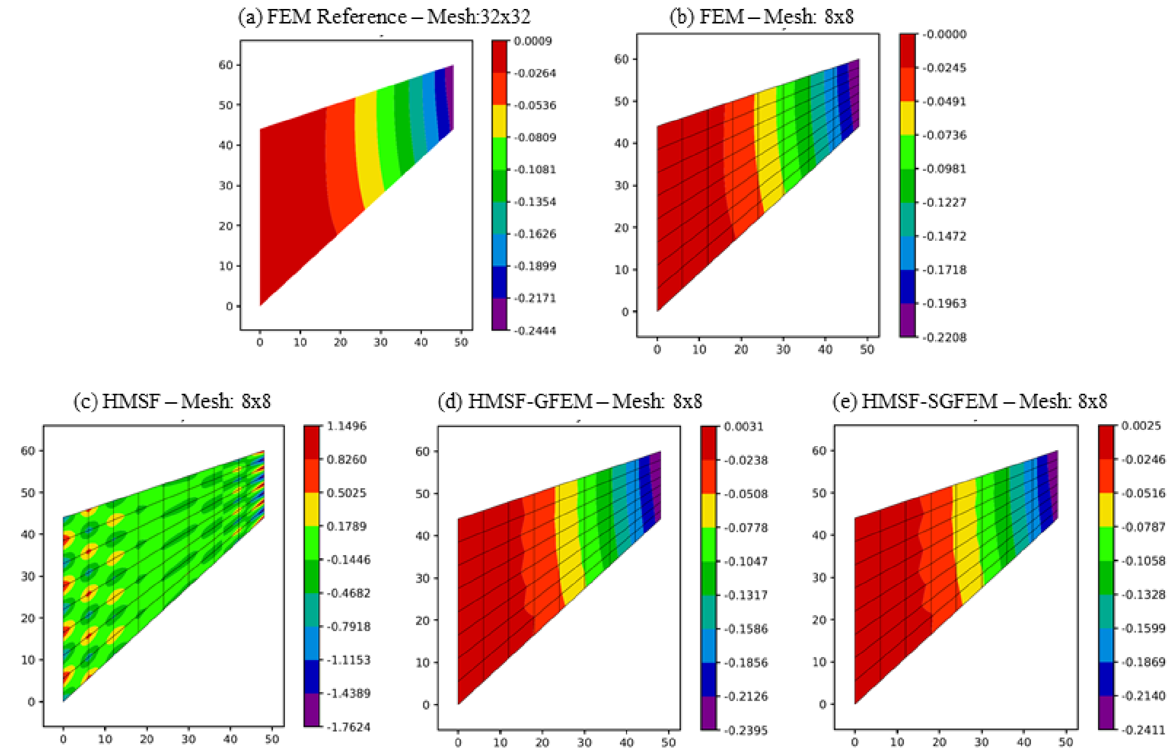

Figure 14 – Displacement field in y for the tension Cook Panel problem, with the discretization of 8x8 quadrilateral elements.

CONCLUSIONS

In the problems analyzed and presented in this work, the enrichment applied to the stress field increased the convergence rates of the solutions, both in stresses and in displacements. This is because, by definition of the HMSF method, the submatrices of the approximations in displacements (in the domain and in the contour) contain in their formulations the approximations of the stress field, so that the enrichment in this field is sufficient to also modify the approximations in displacements. It can also be observed that selective enrichment, that is, the enrichment applied to specific nodes of a problem, proved to be sufficient to increase the quality of the solutions. Therefore, in most cases, there is no need to apply enrichment to the entire problem domain, as is the case with the application of GFEM and SGFEM enrichments in conventional displacement elements (FEM).

It is worth mentioning that normally the best performance of the HMSF-GFEM and HMSF-SGFEM methods occur mainly in the application of enrichment of the nodes close to the regions with higher stress concentration, this can be observed, mainly, in the problem of the plate under traction (Fig. 3). However, in some cases the enrichment of selected nodes close to the regions of natural boundary conditions, even not being regions of stress concentration, can also be efficient to improve the responses obtained, as shown in the Cook Panel problem (Fig. 9).

The formulation proposed here, HMSF-SGFEM, was developed in such a way that it was possible to create an algorithm that would allow the study of its application and performance analysis in different plane problems of linear elasticity. As the bases of the initial approximations of the HSMF constitute the partition of the unit, both the GFEM and the SGFEM can be applied directly to this formulation.

Finally, it is concluded that the initial objectives were achieved, since the HMSF structure in SCIEnCE was initially developed here. As future proposals for the continuity of this research, the following stand out: the study of problems with singularities; adding new enriching functions, such as singular and discontinuous and the improvement in the efficiency of the HMSF-GFEM and HMSF-SGFEM solution using the characteristic of the high sparsity level of the HMSF coefficient matrix.

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