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# Evaluation of Lateral Soil Resistance of Buried Pipes by Limit Analysis Considering Friction at Pipe-Soil Interface

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*Pipelines are vital means of transportation of liquids and gases over large geographical areas. Buried pipes are submitted to thermal and mechanical loads due to their supports, pipe-soil friction and the surrounding soil mass. Under certain circumstances, these loads may lead to loss of stability and buckling may occur. Then, the evaluation of soil lateral resistance that will cause a imminent breakout is important. This work aims the analysis of the soil lateral resistance by frictional limit analysis formulation, considering the soil mass as a deformable body and the pipe as a rigid one. The influence of the pipe buried depth is also analyzed. First, let compare the results with those ones found in literature and discuss the observed failure modes. Later, friction at pipe-soil interface is considered. From the solution of contact problem, the effective pipe-soil contact region can be determined in contrast with fully bonded pipe-soil solutions found in literature and the possibility of overestimated lateral forces is discussed.*
***Keywords: Buried pipes, Limit analysis, Finite elements.***

## INTRODUCTION

Pipelines are slender structures used for transportation of liquids and gases over large distances. Regarding oil industries applications, in-service hydrocarbons must be transported at hight temperatures and pressure in order to avoid wax formation. Due to expansion resistance, large compressive loads are developed due to thermal stresses, internal pressure and longitudinal friction. If this compressive load overcomes a critical load, sudden large deformation may occur due to loss of stability. The occurrence of buckling depends on soil resistance and friction at pipe-soil interface and the determination of lateral resistance is a key point in order to prevent a sudden failure that can result in economical, environmental and life losses.

Many authors have conducted experimental, analytical and numerical solutions regarding the determination of the lateral resistance on pipes in shallow embedment as found in Marifield *et al.* (2008); Aubeny *et al.* (2005); Randolph and White (2008); Murff *et al.* (1989), among others. However, few researches have dealt with finding the lateral resistance of buried pipes and papers in literature are quite limited. In Chakraborty (2018), lateral resistance of buried pipes are evaluated by finite elements and limit analysis lower bound. Results considering the soil as a von Mises and Mohr-Coulomb material are presented. In Kouretzis *et al.* (2014), a limit analysis approach is applied in the analysis of buried pipeline submitted to a downward load and Liu *et al.* (2015) shows a experimental procedure and a finite element model for sand and clay soils.

In this work, the lateral resistance is evaluated by frictional limit analysis methodology proposed in Figueiredo and Borges (2017, 2020). In this approach, permanent contact between a rigid and a deformable body is considered and Coulomb sliding law. In this case, due to a large stiffness ratio, the pipe is considered as a rigid body while the soil mass is treated as a deformable one. The soil mass is discretized into finite elements and the domain set up in order to accommodate the shear zones and the velocity field. Assuming permanent and known contact region, the pipe-soil interaction is modeled according to frictional unilateral contact conditions. Imposing a lateral velocity field at contact region in order to model the pipe lateral movement, the collapse power, the stress and velocity fields are determined as well as the contact stresses and tangential velocities and the failure modes represented by the slip lines along the soil mass.

In this preliminary study, the soil mass is considered as a von Mises material. This criterion is not suitable to describe the soil behavior under plasticity. However, it can be used to obtain preliminary results and compare to a lower bound solution found in Chakraborty (2018). In this work, the fully bonded condition is assumed and the soil lateral resistance is evaluated considering a growing buried depth-pipe diameter ratio. Different failure modes are investigated. Later, by solving the frictional contact problem, the fully contact condition is discussed.

## LIMIT ANALYSIS THEORY

Limit analysis aims the determination of a limit state that will entail the incipient plastic collapse for elastic perfectly plastic bodies. The classical limit analysis principles are found in Lubliner (1990); Chen and Liu (1990). In order to simulate the lateral resistance problem, the limit analysis approach considering frictional interface proposed in Figueiredo and Borges (2017, 2020) is applied. This methodology considers elements of contact mechanics and instead forces, a velocity field is imposed in order to represent the action of a rigid body onto a deformable one. Permanent contact between the rigid and the deformable body is admitted so as to keep the process ongoing and Coulomb friction law is considered as a sliding law. Concepts of contact mechanics found in Wriggers (2002); Weißenfels and Wriggers (2015) are considered in this proposed limit analysis formulation.

The solution technique relies on the discretization of the continuum media is into finite elements and the solution of the discretized mixed limit analysis formulation. Triangular elements are used, with quadratic interpolation for the velocity field and linear interpolation for the stress field Zouain *et al.* (2014). The solver code is implemented in Fortran, based on an iterative procedure based on Newton algorithm and contraction techniques in order to keep the plastic admissibility on the stress field Borges *et al.* (1996). The set of optimum equations derived from the min-max problem are solved. Then, after a sequence of substitutions and application of a condensation technique, a Linear Complementarity Problem at contact is derived and solved by Lemke algorithm Belegundu and Chandrupatla (2014); Bazaraa *et al.* (2006). As results, the plastic collapse power, the stress and velocity fields, the shear zones, normal and tangential forces at contact are determined.

### Kinematics

First, let define a deformable continuum body $\mathcal{B}$ with an imminent contact with a rigid body $\mathcal{B}_r$. Also, let $\Gamma_D$ and $\Gamma_c$ be a part of the boundary $\Gamma$ in which homogeneous and non-homogeneous velocities are imposed, respectively:

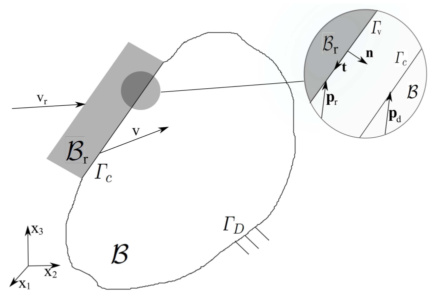

**Figure 1 – Continuum deformable body $\mathcal{B}$ in imminent contact with the rigid body $\mathcal{B}_r$. In detail, $\mathbf{p}_r$ and $\mathbf{p}_d$ are the position vectors of points on the rigid and deformable bodies at imminent contact.**

Now, let define the space $V^0$ of all kinematically admissible velocity field $\mathbf{v}$ that satisfied the homogeneous boundary conditions on boundary $\Gamma_D$:

$$V^0 = \{\mathbf{v} \in \mathcal{U} \mid \mathbf{v} = 0 \quad \text{in}\, \Gamma_D\} \tag{1}$$

where $\mathcal{U}$ is the space of sufficient regular velocities.

Also, one defines $V$ as an affine space of $V^0$ Anton and Busby (2003). It constitutes the function space of all kinematically admissible velocity fields $\mathbf{v}$ satisfying the homogeneous boundary conditions on $\Gamma_D$ part of $\Gamma$ and non-homogeneous boundary conditions at $\Gamma_c$:

$$V = \mathbf{v}_p + V^0 \tag{2}$$

where $\mathbf{v}_p$ is a prescribed velocity field.

The strain rate $\mathbf{d}$ that belongs to the set $W$ is related to the velocity $\mathbf{v} \in V$ through the deformation operator $\mathcal{D}$ as follows:

$$\mathbf{d} = \mathcal{D}\mathbf{v}, \qquad \mathbf{v} \in V \tag{3}$$

The kinematics of contact mechanics consists of unilateral conditions at normal direction and a sliding law for the tangential direction. In limit analysis, once incipient plastic collapse is attained, permanent contact hypothesis is assumed. This condition is achieved by setting null the normal gap $g_n = \mathbf{g}.\mathbf{n}$ Wriggers (1999); Chabrand *et al.* (1998). Additionally, this conditions is ensured if the normal relative velocity $w_n$ between the deformable and rigid bodies is null. Since separation is not admitted, the Signorini conditions Panagiotopoulos (1975) are particularized as follows:

$$w_n = 0,\ r_n < 0,\ r_n w_n = 0 \tag{4}$$

At tangential direction, the relative velocity $w_t$ is defined as the difference $w_t = v_t - v_{rt}$. There is no restriction at this direction and sliding or sticking may occur. Then, a sliding law is required. Denoting the Coulomb friction function as $\mathcal{F}(\mathbf{r}_t, r_n)$ Raous (1999); Michalowski and Mroz (1978); Saxcé and Bousshine (1998), it follows that:

$$\mathcal{F}(\mathbf{r}_t, r_n) = ||\mathbf{r}_t|| + \mu r_n \leq 0 \tag{5}$$

The Coulomb friction law states the occurrence of sliding or sticking regimes:

$$\text{If } \mathcal{F}(\mathbf{r}_t, r_n) = 0, \text{ then } \mathbf{w}_t = -\xi \frac{\partial \mathcal{F}(\mathbf{r}_t, r_n)}{\partial r_t} \tag{6}$$

$$\text{else },\mathbf{w}_t = \mathbf{0} \tag{7}$$

and the complementarity relation: (8)

$$\mathcal{F}(\mathbf{r}_t, r_n) w_t = 0 \tag{9}$$

where: $\mathbf{r}_t = r_t\mathbf{t}$, $r_n < 0$, $\xi \geq 0$ is a proportionally constant and $\mu$ is the friction coefficient.

## Equilibrium

The equilibrium between external and internal efforts is posed in terms of the principle of virtual power (PVP), according to Taroco *et al.* (2017). Let $W^{'}$ and $W$ denoting the spaces of stress and strain rate fields, respectively. Considering kinematic in (3), the internal power for any pair $\sigma \in W'$ and $\mathbf{d} \in W$ is defined by the duality product:

$$\langle \sigma, \mathcal{D}\mathbf{v} \rangle = \int_{\mathcal{B}} \sigma \cdot \mathcal{D}\mathbf{v}\, d\mathcal{B}, \quad \mathbf{v} \in V \tag{10}$$

Dissimilar to problems with conventional supports, the PVP considering reaction forces at contact boundary $\Gamma_c$ is applied as follows Taroco *et al.* (2017):

$$\langle \sigma, \mathcal{D}\mathbf{v}^0 \rangle \ = \langle \mathbf{a}, \mathbf{v}^0 \rangle_{\Gamma_t} + \langle \mathbf{r}, \mathbf{v}_c^0 \rangle_{\Gamma_c} \tag{11}$$

where: $\mathbf{v}^0 \in V^0$, $\mathbf{v}_c^0 \in V^0$ are velocities at contact boundary $\Gamma_c$. Active forces are not imposed in this formulation: $\mathbf{a} = \mathbf{0}$.

Using (2) and considering the hypothesis of permanent contact ($\mathbf{w}_n = \mathbf{0}$), the equilibrium equation is established, according to Figueiredo and Borges (2020):

$$\langle \sigma\mathbf{n}, \mathbf{v}_r \rangle_{\Gamma_c} = \langle \sigma, \mathcal{D}\mathbf{v} \rangle \ - \langle \mathbf{r}_t, \mathbf{w}_t \rangle_{\Gamma_c} \tag{12}$$

where: $\Pi_e = \langle \sigma\mathbf{n}, \mathbf{v}_r \rangle_{\Gamma_c}$ is the external power, $\mathbf{v}_r$ is the velocity of the rigid body and the term related to normal direction was suppressed since $r_n w_n = 0$ according to the complementarity of the unilateral conditions from (4).

### Constitutive

First, let define $P$ as the set of admissible stress field:

$$P = \{\sigma \in P \mid f(\sigma) \leq 0 \text{ in } \mathcal{B}\} \tag{13}$$

where: $f(\sigma)$ is the yield function.

In order to relate strain rates and stress fields, the normality law derived from Hill's maximum dissipation principle and based on subdifferential of plastic dissipation function $\chi(\mathbf{d}^p)$ is defined as follows (Lubliner (1990); Maugin (1992)):

$$\chi(\mathbf{d}^p) = \sup_{\sigma^* \in P} \langle \sigma^*, \mathbf{d}^p \rangle \tag{14}$$

The normality law is a consequence of this principle. Admitting the existence of a convex set $P$ it follows:

$$\mathbf{d}^p = \dot{\lambda} \nabla f(\sigma), \tag{15}$$

$$f(\sigma)\dot{\lambda} = 0 \quad \dot{\lambda} \geq 0, \quad f(\sigma) \leq 0 \tag{16}$$

### Limit analysis formulation

The aim of limit analysis consists in finding the external collapse power $\Pi_e$ that will entail the incipient plastic collapse under permanent regime configuration, the stress field $\sigma \in P$ that satisfied the equilibrium in (12), the kinematically admissible velocity $\mathbf{v} \in V$, the strain rate field $\mathbf{d}^p$ related to the velocity field $\mathbf{v}$ through equation (3).

The solution scheme is based on discrete form of the limit analysis formulation stated in Figueiredo and Borges (2017, 2020). The velocity and stress fields are quadratic and linearly interpolated, respectively. Considering the discrete form the strain operator as $\mathbf{B}$ (Borges *et al.* (1996); Zouain *et al.* (2014)), the discrete form of the mixed limit analysis formulation is derived as follows:

$$\Pi_e = \min_{\mathbf{v}, \mathbf{w}_t} \max_{\sigma, \mathbf{r}_t} (\sigma \cdot \mathbf{B}\mathbf{v} - \mathbf{r}_t \cdot \mathbf{w}_t) \tag{17}$$

$$\text{such that :}$$

$$\mathbf{f}(\sigma) \leq \mathbf{0} \text{ in } \mathcal{B} \tag{18}$$

$$\mathcal{F}(\mathbf{r}_t, r_n) \leq \mathbf{0} \text{ at } \Gamma_c \tag{19}$$

$$\mathbf{w}_n = \mathbf{0} \text{ at } \Gamma_c \tag{20}$$

The solution of the above optimization problem with constraints is based on a Newton algorithm as described in Borges *et al.* (1996); Zouain *et al.* (2014) and Figueiredo and Borges (2020). From solution, the collapse power (associated with cutting forces), the velocity and stress fields, reaction forces and the plastic multiplier are determined.

## ANALYSIS OF BURIED PIPES

The long pipe of diameter $D$ is buried at a depth $H$ in a soil mass. The soil mass is a von Mises material in this preliminary study, although Drucker-Prager or Mohr-Coloumb criteria are more suitable to represent this material behavior under plasticity. Von Mises crterion was applied due to simplicity and to fulfill the objective of better understanding the problem to be analyzed.

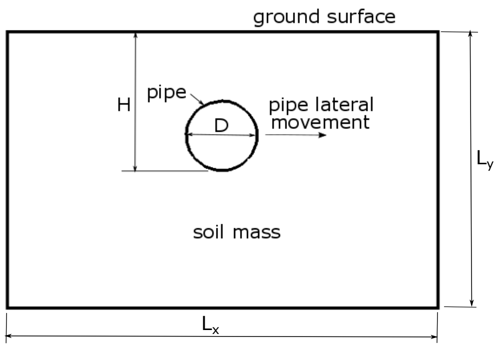


**Figure 2 – Problem definition for soil lateral resistance evalutation.**

In Fig. 2, the dimensions $L_x$ and $L_y$ are defined so that the resulting slip lines and velocity fields are fully contained in the analysis domain. First, in order to compare to the lower-bound solution found in Chakraborty (2018), the fully bonded contact between the pipe and the surrounding soil mass is considered. Later, friction is considered at pipe-soil interface and the hypothesis of contact over entire pipe-soil interface is discussed.

## Comparison with a lower-bound solution

The analysis carried out by Chakraborty (2018) was based on the lower bound theorem of limit analysis and finite elements discretization. In this work, were performed a parametric study varying the embedment radio $H/D$ and the magnitude of the internal friction angle $\phi$, from $0^o$ up to $40^o$. Mohr-Coulomb yield criterion was used by the author. In this approach, the nondimensional lateral capacity factors $F_c$ and $F_\gamma$ due to soil cohesion and soil weight were determined, respectively. The fully bonded condition was adopted.

This work presents preliminary results considering von Mises criterion ($\phi = 0^o$). The soil self-weight is not computed. Then, the ultimate load $P_u$ and the lateral resistance are related as follows:

$$P_u = cDF_c \tag{21}$$

Relating these quantities, the lateral resistance $F_c$ for embedment ratio from 2 up to 10 are compared:
where: $c$ is the material cohesion and $D$ is the pipe diameter.

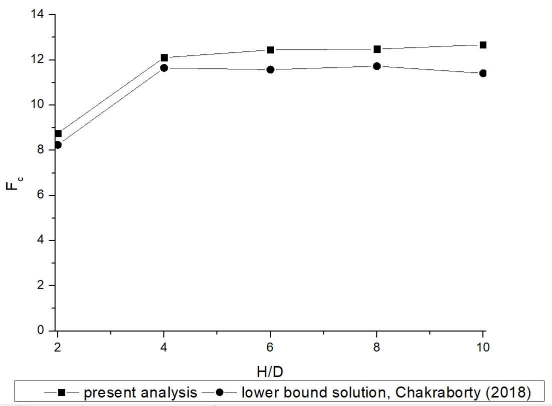


**Figure 3 – Soil lateral resistance.**

On should observe that the present limit analysis mixed solution is compared to a lower bound one. Due to this, these results do not need to match necessarily. Later, failure modes regarding the embedment ratios are present as well the velocity field. The following figures are related to $H/D = 2$ ($H/D = 4$ presents a similar result) and $H/D = 8$ ($H/D = 6$ and $H/D = 10$ present a similar behaviour).

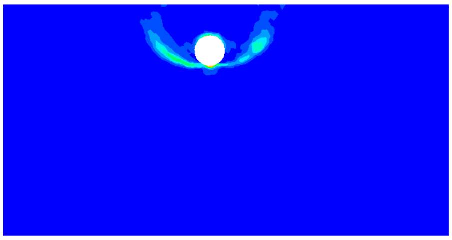

**Figure 4 – Failure mode for a shallow embedment - H/D=2.**

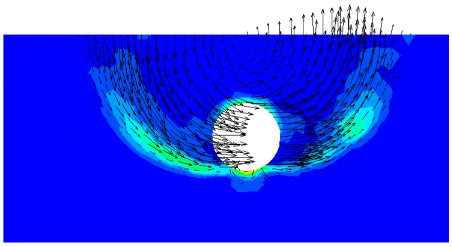

**Figure 5 – Velocity field for H/D=2.**

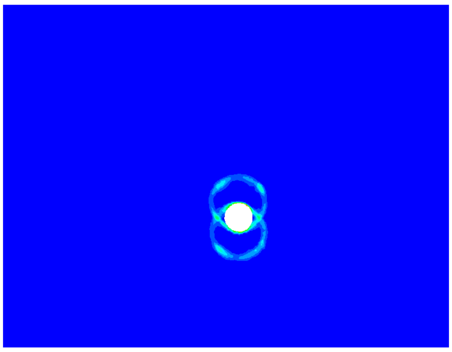

**Figure 6 – Failure mode for a shallow emdedment - H/D=8.**

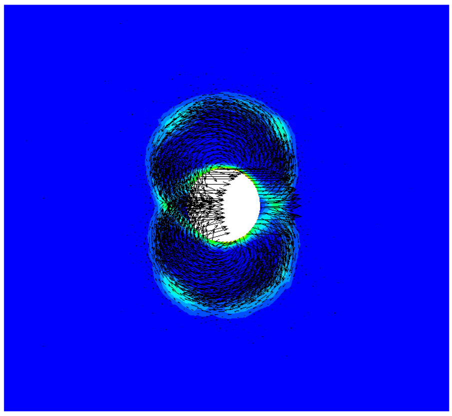

**Figure 7 – Velocity field for H/D=8.**

Regarding Figures 4 and 6, there are dissimilar failure modes according to embedment ratio. For the shallow embedment $H/D = 2$, the slip lines meet the ground surface and tend to form a depression on the left region and a pile on the right, as observed in Figure 5. However, for embedment ratio $H/D = 8$ there is a deep failure mode as presented in Figure 6. According to Figure 7, there is the tendency of material circulation around the pipe at plastic collapse. A similar behavior for the depth failure mode behavior was also observed in Kouretzis *et al.* (2014).

## Pipe-soil frictional analysis

In the fully bonded case, permanent contact and sticking were imposed around the circumference at pipe-soil interface. Now, a frictional contact at pipe-soil interface is considered. First, let admit permanent contact all around the pipe-soil interface, as admitted in the previous analysis. Then, the feasibility of this hypothesis can be discussed by solving the contact problem and plotting the normal stress distribution all over contact circumference.

In this formulation, the condition of permanent contact is obtained by imposing a kinematic condition. The contact stresses are unknown variables, being the stresses restricted to $\sigma_n < 0$ if there is contact. Positive values imply loss of contact.

Following these principles, cases for embedment ratios $H/D = 2$ and $H/D = 6$ and friction coefficients at pipe-soil interface from 0.10 to 0.30 are analyzed. The lateral resistance was evaluated as well as the determination of stresses at pipe-soil interface. In the following figures the distribution of normal stresses are plotted along the circumference length. The origin of the local coordinate $s$ is located at the circumference lower point, oriented at counter-clockwise direction.

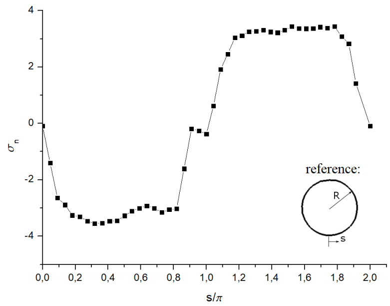


**Figure 8 – Distribution of contact normal stresses for H/D=2, $\mu$=0.10.**

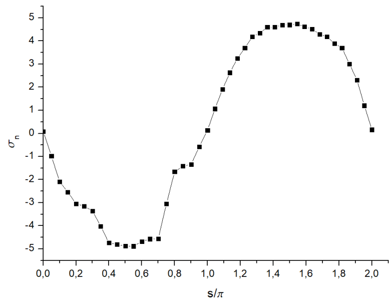


**Figure 9 – Distribution of contact normal stresses for H/D=6, $\mu$=0.30.**

Observing the distribution of normal stresses at contact in Figures 8 and 9, at imminent plastic collapse there is loss of contact for $s/\pi > 1$. From this results, some analysis were carried out considering contact only at half circumference, for $0 \leq s/\pi \leq 1$.

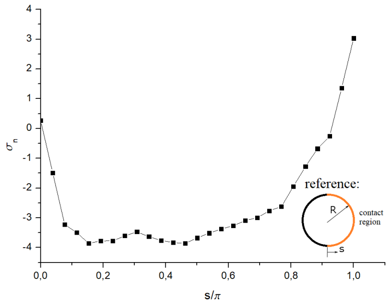


**Figure 10 – Distribution of contact normal stresses for H/D=2, $\mu$=0.10.**

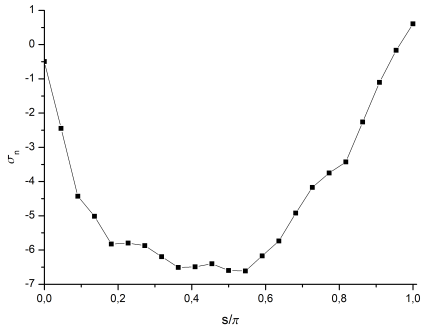


**Figure 11 – Distribution of contact normal stresses for H/D=8, $\mu$=0.30.**

The distributions of normal stresses at contact interface in Figures 10 and 11 are predominantly negative, i.e., imposing the permanent contact by kinematic condition, the resulting normal stresses at contact are according to the necessary condition for permanent contact $\sigma_n < 0$. Comparing the lateral resistances concerning both contact cases (full contact x effective contact zones):

**Table 1 – Comparison between full contact x effective contact results for lateral resistance.**

| $\mu$ | H/D | Fc (full contact) | Fc (effective contact) |
|---|---|---|---|
| 0.1 | 2 | 7.4315 | 3.9204 |
| 0.3 | 6 | 10.2190 | 6.1042 |
| 0.3 | 8 | 10.4707 | 7.6673 |

Comparing the results in table 1, overestimated lateral forces are observed if full contact hypothesis is adopted.

Considering the full contact at pipe-soil interface, the obtained failure modes are similar to those ones in Figures 4 and 6. However, there are dissimilar failure modes if effective contact is considered:

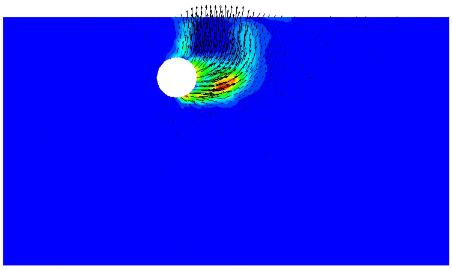

**Figure 12 – Failure mode for H/D=2, $\mu$=0.10.**

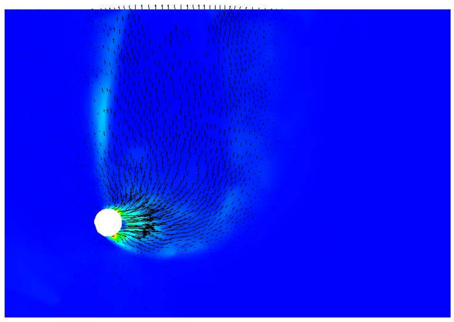

**Figure 13 – Failure mode for H/D=8, $\mu$=0.30.**

At plastic collapse, for the embedment ratio $H/D = 8$, Figure 13 presents a velocity field moving toward the upper surface, dissimilar to the depth failure mode observed in Figure 6. Considering the fully contact hypothesis, the failure mode is similar to Figure 6, presenting a recirculating velocity field around the pipeline circumference.

## CONCLUSIONS

Pipelines are slender structures submitted to compressive loads. If they reach a critical value, buckling may occur. The occurrence of buckling depends on the surrounding soil resistance and friction at pipe-soil interface. In this paper, the soil lateral resistance is evaluated by limit analysis theory considering frictional interfaces. The pipe is considered as a rigid body and the soil is treated as a deformable one. In order to evaluate the soil lateral resistance, a horizontal velocity field is imposed at pipe-soil interface.

In these preliminary results, the soil was considered as a von Mises material. First, the magnitude of lateral resistance and failure modes were compared to those ones found in literature, considering a fully bonded contact. In this case, dissimilar failure modes were observed and they depends on the embedment ratio $H/D$, entailing a shallow or a depth failure mode. Also, the lateral resistance forces were carried out by the present limit analysis methodology and compared to a lower bound solution found in literature. The results were coherent since results form a mixed formulation were compared to lower bound results.

Later, friction was considered at pipe-soil interface. Regarding the distribution of normal stresses at this interface, permanent contact of entire circumferential interface is not guaranteed at plastic collapse. Imposing a permanent contact by a kinematic condition, it was observed a non-negative violation for the normal stresses restriction. It implies contact loss and the fully contact hypothesis is not feasible. Then, the resistance lateral forces may be overestimated if fully contact is assumed.

As future work, the Drucker-Prager criterion should be implemented, in order to better represent the soil behavior under plasticity. Also, in geological problems, the self weight plays an important role in the evaluation of soil capacity. Then, a formulation considering it as a dead load should be developed and implemented.

## ACKNOWLEDGMENTS

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