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Characterization of Oscillatory Motion in Multi Degree of Freedom Systems

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Abstract. In many engineering applications, it is important to determine the effect of damping on the overall response because damped systems can exhibit oscillatory or non-oscillatory behaviors in free vibration. Characterization of the free motion of a single degree of freedom damped system is well understood: the nature of damped free motion can be determined by inspection of the viscous damping ratio, which is is a non-negative number defined by the coefficients of the system, and oscillatory behavior can be observed in underdamped systems. While a similar criteria for determining oscillatory behavior of free motion of multi degree of freedom damped systems is desired, the situation in this case is less clear because of coordinate coupling through the system's coefficient matrices. In this article, it is shown how coordinate decoupling can be used to construct a damping ratio for multi degree of freedom systems. The system damping ratio presented herein is a direct extension of the single degree of freedom case, providing a non-negative number that depends on the system's coefficient matrices that allows for the determination of weather oscillatory behavior is present or not. Its applicability is illustrated with an example from structural dynamics.
Keywords: free vibration, damping, damping ratio, decoupling, passive systems

INTRODUCTION

In many engineering applications, it is important to determine the effect of damping on the overall response because damped systems can exhibit oscillatory or non-oscillatory behaviors in free vibration. Besides theoretical interest, this task is important in engineering designs where one needs to know how oscillations can be suppressed by varying certain system parameters. For instance, the design of sensitive instruments must consider relatively high amounts of damping to avoid resonant amplitude distortion and ensure stability (Papargyri-Beskou and Beskos, 2002).

Characterization of the free motion of a single-degree-of-freedom system is well understood: the nature of damped motion can be determined by inspection of the viscous damping ratio. While a similar criterion for characterizing free vibration of damped multi-degree-of-freedom systems is desired, the situation here is less clear. A well known approach to determine the nature of free motion utilizes spectral analysis. For these systems, the spectrum is composed of negative real eigenvalues, associated with exponentially decaying modes, and complex conjugate eigenvalues associated with oscillatory modes, whose negative real part is a rate of decay and whose imaginary part is a damped frequency of vibration (Tisseur and Meerbergen, 2001). This leads directly to free response characterization of multi-degree-of-freedom systems in terms of its modal degrees of freedom. For example, an overdamped system is defined as the one with all modes overdamped, while an underdamped one possesses only oscillatory modes. By the same token, a critically damped system must have real and repeated eigenvalues. Mixed damping arises when, in free vibration, at least one mode oscillates and one does not. Even though spectral analysis provides a complete characterization of free response, it is not convenient for design purposes because it requires full knowledge of the eigenvalues for every combination of system parameters. This task is impractical for large dynamical systems and, for this reason, researchers have often sought other criteria that involve relations between the mass, damping and stiffness matrices of the system.

The first known attempt is due to Duffin and Duffin (1955), who established a necessary condition for a system to be overdamped through an inequality between quadratic forms of damping and stiffness matrices. Nicholson (1978) demonstrated that a system is underdamped if an inequality in terms of the eigenvalues of the mass, damping and stiffness matrices is satisfied. These criteria belong to the first family of approaches found in the literature involving analysis of inequalities. Satisfaction of Duffin's condition would guarantee all eigenvalues to be negative and distinct, but it requires inspection of the inequality in the entire vector space. Similarly, Nicholson's inequality is not trivial to verify for large structural systems.

Another family of criteria involve inspection of definiteness of some matrices. Müller (1979) obtained a necessary condition in terms of the definiteness of a combination of the coefficient matrices to determine if a system is underdamped. Inman and Andry (1980) proposed necessary conditions for underdamping through the positive definiteness of a matrix involving the mass, damping and stiffness coefficients. In addition, they also derived conditions for overdamped behavior, which has been demonstrated to be false (Barkwell and Lancaster, 1992). Afterwards, Nicholson (1983) constructed necessary and sufficient conditions for overdamping based on the calculation of two positive definite and Hermitian matrices. However, there is no systematic manner for determining these matrices. Barkwell and Lancaster (1992) suggested necessary and sufficient conditions for non-oscillatory free response in terms of the coefficient matrices and a positive scalar parameter. While this provides an alternative to Duffin's approach (Duffin and Duffin, 1955) that depends on only one scalar parameter, the obvious drawback is that its determination is by trial and error. Bhaskar (1997) derived sufficient conditions for overdamping, critical damping and underdamping in terms of the definiteness of combinations between the spectral e modal damping matrices. The conditions for critical damping and underdamping are essentially the same as those of Inman and Andry (1980). Bulatović (2001) derived a simple sufficient condition involving only the positive semi-definiteness of a particular combination of the coefficient matrices and a minimum eigenvalue related to the stiffness matrix. Later, Bulatović (2004) presented a necessary and sufficient condition formulated in terms of the definiteness of a Hankel's matrix whose elements depend on the coefficients of the characteristic polynomial that stems from the quadratic eigenvalue problem associated with the damped system. It should be noted that verification of definiteness of matrices is not a simple task and can be challenging for large systems.

Three other approaches differ from the previous cases. Beskos and Boley (1980) introduced the concept of "critical damping surfaces". These surfaces represent the loci of combinations of damping values corresponding to critically damped motions, and thus separate regions of partial or complete underdamping from those of overdamping. Their method is limited to systems with few degrees of freedom because of the need to differentiate the characteristic determinant. Later, Papagyri-Beskou and Beskos (2002) derived an approximate computational method for determining critical damping surfaces. Another attempt has also been made by Inman and Jiang (1987) and Gray et al. (1982) to construct a damping ratio matrix to determine criticality for the system. However, Bhaskar (1997) observed that this approach derives erroneous conclusions and obtained corrected criteria in modal coordinates. The most recent work is due to Morzfeld, Kawano and Ma (2013), who introduced a viscous damping function to characterize free vibration. The effect of viscous damping on the free motion is then determined by minimization and maximization of this viscous damping function. However, optimization of such function may be problematic because the iterations can get trapped around local extrema in applications.

It is evident that, while all previous methods can theoretically indicate the nature of free motion in multi-degree-of-freedom systems, none of them have found ways into engineering design and analysis because of their complexity of implementation. A criterion that is simple to use and depends depends only on the given system parameters, such as the coefficient matrices, is desired. In this light, the purpose of this work is to report on a damping ratio for multi-degree-of-freedom systems, constructed as a direct extension of the damping ratio for single degree of freedom case. The geometric and arithmetic mean of the system's eigenvalues are used to demonstrate that the system damping ratio indicates oscillatory behavior when less than unit. Contrary to previous methods, this system damping ratio is convenient for implementation because it is a positive number calculated from the coefficient matrices. An example is supplied to illustrate its application.

FREE MOTION CHARACTERIZATION OF MULTI-DEGREE-OF-FREEDOM SYSTEMS

Extension of Damping Ratio to Multi-Degree-of-Freedom Systems

The equation of motion of an n-degree-of-freedom linear mechanical system in free vibration can be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \tag{1}$$

where $\mathbf{q}$ and $\mathbf{0}$ are n-dimensional column vectors of generalize coordinates and zeros, respectively, and over dots represent differentiation with respect to time t. This equation possesses three real and square coefficient matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ of order $n \times n$ and their form allow the system concerned to be classified. For instance, an undamped gyroscopic system possesses a skew-symmetric coefficient of velocity. If a system is elastic and non-circulatory, then the coefficient of displacement is symmetric. And so one may go on. Of interest in this work is the class of non-gyroscopic, non-circulatory, passive systems characterized by three constant, symmetric and positive definite matrices, i.e., the mass $\mathbf{M}$, damping $\mathbf{C}$, and stiffness $\mathbf{K}$ matrices. For brevity, this class of systems is referred to as passive or as damped linear systems. While these properties ensure stability, they are not arbitrary. In fact, they are based on physical considerations from the theory of Lagrangian dynamics (O'Reilly, 2020). For example, symmetry of $\mathbf{M}$ is based upon the fact that the quadratic form of kinetic energy can always be defined in terms of a symmetric matrix. In addition, any linear system is passive if the rigid-body modes are eliminated, which is not an essential restriction (Chopra, 2017). There should be no denying that the bulk of existing literature on linear vibration and structural dynamics deals implicitly or explicitly with passive systems.

It should be noted that a damped single-degree-of-freedom system represents a realization of Eq. (1) when $n = 1$. In this case, free motion is easily identified through inspection of the damping ratio. Recall that the equation of motion of a damped system with one degree of freedom is

$$m\ddot{q} + c\dot{q} + kq = 0, \tag{2}$$

where $m > 0$ is the mass, $c > 0$ is the viscous damping coefficient and $k > 0$ is the stiffness. The damping ratio is a non-negative number defined by the coefficients of the system:

$$\zeta = \frac{c}{2m\omega_n} = \frac{m^{-1}c}{2}\frac{1}{\sqrt{m^{-1}k}}, \tag{3}$$

where $\omega_n = \sqrt{k/m} > 0$ is the natural frequency. Free motion behavior is then identified by the value of ζ. For instance, oscillatory behavior is present whenever $\zeta < 1$. Is it possible to extend such a criterion to multi-degree-of-freedom systems? Define a system damping ratio by

$$\zeta_n = \frac{\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{C}\right)}{2n}\frac{1}{\sqrt[2n]{\det\left(\mathbf{M}^{-1}\mathbf{K}\right)}}, \tag{4}$$

where $\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{C}\right)$ is the trace of $\mathbf{M}^{-1}\mathbf{C}$ and $\det\left(\mathbf{M}^{-1}\mathbf{K}\right)$ is the determinant of $\mathbf{M}^{-1}\mathbf{K}$. Since $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are positive definite, the system damping ratio $\zeta_n > 0$ (Horn and Johnson, 2012). More importantly, when $n = 1$, each matrix is just the corresponding positive coefficient in Eq. (2) and ζ_n in (4) reduces to the well-known damping ratio ζ for single-degree-of-freedom systems in Eq. (3).

Nature of Free Motion

How can the system damping ratio (4) be used to indicate the nature of free motion? To answer this question, it is necessary to look at the roots of the characteristic polynomial

$$\det(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}) = 0. \tag{5}$$

Solution of Eq. (5) yields $2n$ eigenvalues λ_j $(j = 1, 2, ..., 2n)$, of which $2n_c$ form complex conjugate pairs with negative real part and the remaining $2n_r = 2n - 2n_c$ are real and negative (Tisseur and Meerbergen, 2001). Interestingly, the $2n$ system eigenvalues λ_j are the same as those of the $2n \times 2n$ state companion matrix given by

$$\mathbf{A} = [\mathbf{0}, \mathbf{I}; -\mathbf{M}^{-1}\mathbf{K}, -\mathbf{M}^{-1}\mathbf{C}.] \tag{6}$$

It is known that the trace and determinant of $\mathbf{A}$ can both be related to its eigenvalues (Horn and Johnson, 2012). As a consequence, it is possible to take into account its block nature to obtain (Silvester, 2000)

$$\prod_{j=1}^{2n}\lambda_j = \lambda_1\lambda_2\cdots\lambda_{2n} = \det(\mathbf{A}) = \det\left(\mathbf{M}^{-1}\mathbf{K}\right) = \frac{\det(\mathbf{K})}{\det(\mathbf{M})} \tag{7}$$

and

$$\sum_{j=1}^{2n}\lambda_j = \operatorname{tr}(\mathbf{A}) = -\operatorname{tr}\left(\mathbf{M}^{-1}\mathbf{C}\right). \tag{8}$$

It follows that the system damping ratio (4) can be expressed in terms of the $2n$ eigenvalues

$$\zeta_n = -\frac{\sum_{j=1}^{2n}\lambda_j}{2n}\frac{1}{\left(\prod_{j=1}^{2n}\lambda_j\right)^{1/2n}} = -\frac{\text{arithmetic mean of eigenvalues}}{\text{geometric mean of eigenvalues}}. \tag{9}$$

Here, the terms arithmetic and geometric mean are used loosely since each λ_j $(j = 1, 2, \cdots, 2n)$ is either a real negative number or a complex number with negative real part. The arithmetic mean of all $-\lambda_j$ can be written in terms of n pairs as

$$-\frac{1}{2n}\sum_{j=1}^{2n}\lambda_j = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{-\lambda_j - \lambda_{n+j}}{2}\right). \tag{10}$$

Each pair $\{\lambda_j, \lambda_{n+j}\}$ $(j = 1, 2, \cdots, n)$ can be arbitrarily chosen, but the pairing of eigenvalues that follow from system decoupling (modal analysis, for example) is more convenient, as evidenced later on. In such case, a complex conjugate pair yields

$$-\lambda_j - \lambda_{n+j} = 2|\mathrm{Re}(\lambda_j)|, \tag{11}$$

i.e., always twice the absolute value of the real part of a complex eigenvalue λ_j. The remaining pairs are formed by the real negative eigenvalues, so each pairing in (10) results in a positive real number. Therefore, the pairing results in the arithmetic mean of n positive numbers

$$\frac{-\lambda_j - \lambda_{n+j}}{2}, \tag{12}$$

and the notorious inequality governing arithmetic and geometric means is applicable (Gradshteyn and Ryzhik, 2007):

$$\frac{1}{n}\sum_{j=1}^{n}\left(\frac{-\lambda_j - \lambda_{n+j}}{2}\right) \geq \left[\prod_{j=1}^{n}\left(\frac{-\lambda_j - \lambda_{n+j}}{2}\right)\right]^{1/n}. \tag{13}$$

It follows that

$$\zeta_n \geq \left[\prod_{j=1}^{n}\left(\frac{-\lambda_j - \lambda_{n+j}}{2}\right)\right]^{1/n} \frac{1}{\left(\prod_{j=1}^{2n}\lambda_j\right)^{1/2n}} = \left[\prod_{j=1}^{n}\left(\frac{-\lambda_j - \lambda_{n+j}}{2\sqrt{\lambda_j\lambda_{n+j}}}\right)\right]^{1/n}. \tag{14}$$

To clarify the meaning of Eq. (14), coordinate decoupling is necessary. If system (1) is decoupled by modal analysis, each modal coordinate p_j has coefficients given by the system eigenvalues from Eq. (5):

$$\ddot{p}_j - (\lambda_j + \lambda_{n+j})\dot{p}_j + (\lambda_j\lambda_{n+j})p_j = 0, \tag{15}$$

where each pair λ_{n+j} and λ_j are conjugate pairs if λ_j is complex. Note the importance of using the same pairing in Eq. (10) that stems from coordinate decoupling. Since each modal equation represents a single-degree-of-freedom system, the damping ratio of the jth scalar equation of the decoupled system can be given by Eq. (3):

$$\zeta_{pj} = -\frac{\lambda_j + \lambda_{n+j}}{2}\frac{1}{\sqrt{\lambda_j\lambda_{n+j}}} = -\frac{\text{arithmetic mean of roots}}{\text{geometric mean of roots}}. \tag{16}$$

Again, the terms arithmetic and geometric means are used liberally because the roots may be complex or negative. Consequently, the jth modal degree of freedom is underdamped ($\zeta_{pj} < 1$) when λ_j and λ_{n+j} are complex conjugates. If it is overdamped, $\zeta_{pj} > 1$ and this means λ_j and λ_{n+j} are distinct negative real numbers. In the case it is critically damped, $\zeta_{pj} = 1$, in which case $\lambda_j = \lambda_{n+j} < 0$. Therefore, Eq. (14) can be expressed as

$$\zeta_n \geq (\zeta_{p1}\zeta_{p2}\cdots\zeta_{pn})^{1/n}, \tag{17}$$

where ζ_{pj} $(j = 1, \ldots, n)$ is given by (16). Clearly, if

$$\zeta_n < 1, \tag{18}$$

at least one $\zeta_{pj} < 1$, so at least one modal degree of freedom is underdamped. Because of coordinate coupling, the system response is oscillatory if at least one scalar equation in the decoupled system is underdamped. It is possible to conclude that free response of a damped system is oscillatory if the system damping ratio $0 < \zeta_n < 1$. This provides a simple and numerically efficient mean to characterize oscillatory behavior in multi-degree-of-freedom systems.

ILLUSTRATIVE EXAMPLE

The linearized equation of motion (1) is widely used in earthquake engineering to model the dynamic behavior of, for example, multistory buildings, nuclear power plants, or base-isolated structures (Chopra, 2017). Consider the idealized 10-story building in Fig. 1 subject to free vibration. Each coordinate q_j $(j = 1,\ 2,\ \cdots,\ 10)$ is measured with respect to the ground, characterizing a system with $n = 10$ degrees of freedom. For sake of simplicity, it is assumed all floor levels have the same mass $m = 1$ and the stiffness of each floor beam is $k_1 = 16$, $k_2 = 10$, $k_3 = 10$, $k_4 = 8$, $k_5 = 9$, $k_6 = 7$, $k_7 = 5$,

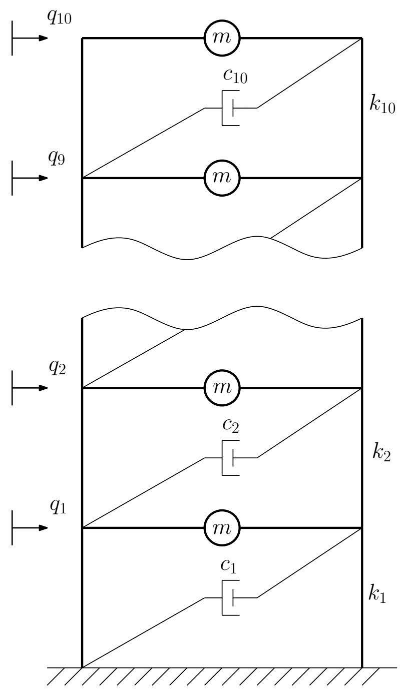

Figure 1 – A fictitious 10-story building.

$k_8 = 10$, $k_9 = 7$, $k_{10} = 14$. If energy dissipation is associated with the deformation of each story, the viscous dampers may be visualized as shown in the figure (Chopra, 2017). Moreover, it is assumed damping is proportional with $\mathbf{C} = \mathbf{K}/3$.

The nature of free motion can be fully characterized by the spectrum of this system. For instance, calculation of the eigenvalues from Eq. (5) reveals that this system possess 9 pairs of complex conjugate eigenvalues, corresponding to underdamped modal degrees of freedom, and 2 distinct real eigenvalues, that are related to an overdamped modal degree of freedom. Because of coordinate coupling, this system will experience oscillatory behavior in free vibration for arbitrary initial conditions. Tab. 1 summarizes this observation, where eigenvalues were computed numerically from the state companion matrix (6) with the function *eig* from MATLAB.

Table 1 – Eigenvalue pairs that form each modal degree of freedom and corresponding damping ratio.

Eigenvalue pairs	ζ_{pj}
$\lambda_1 = -0.0342 + 0.4515i = \bar{\lambda}_2$	0.0755
$\lambda_3 = -5.7625 + 1.1700i = \bar{\lambda}_4$	0.9800
$\lambda_5 = -0.2824 + 1.2708i = \bar{\lambda}_6$	0.2170
$\lambda_7 = -0.8279 + 2.0693i = \bar{\lambda}_8$	0.3715
$\lambda_9 = -4.8716 + 2.3446i = \bar{\lambda}_{10}$	0.9011
$\lambda_{11} = -1.3751 + 2.5219i = \bar{\lambda}_{12}$	0.4787
$\lambda_{13} = -4.2154 + 2.7428i = \bar{\lambda}_{14}$	0.8382
$\lambda_{15} = -2.3210 + 2.9222i = \bar{\lambda}_{16}$	0.6220
$\lambda_{17} = -3.5573 + 2.9478i = \bar{\lambda}_{18}$	0.7700
$\lambda_{19} = -6.8084$, $\lambda_{20} = -5.3632$	1.0071

Alternatively, free vibration behavior can be characterized with inspection of the system damping ratio in Eq. (4). It reveals that

$$\zeta_n = 0.9717. \tag{19}$$

Since $\zeta_n < 1$, oscillatory behavior is expected. Indeed, Fig. 2 shows all floors can experience oscillatory behavior with general initial conditions. This response was obtained numerically with *ode45*, from MATLAB, The initial conditions for the simulation are $\mathbf{q}(0) = \mathbf{0}$ and $\dot{\mathbf{q}}(0) = [9, 5, 11, 20, 0, 2, 16, 8, 2, 12]^T$, the former randomly generated with the function *randi*.

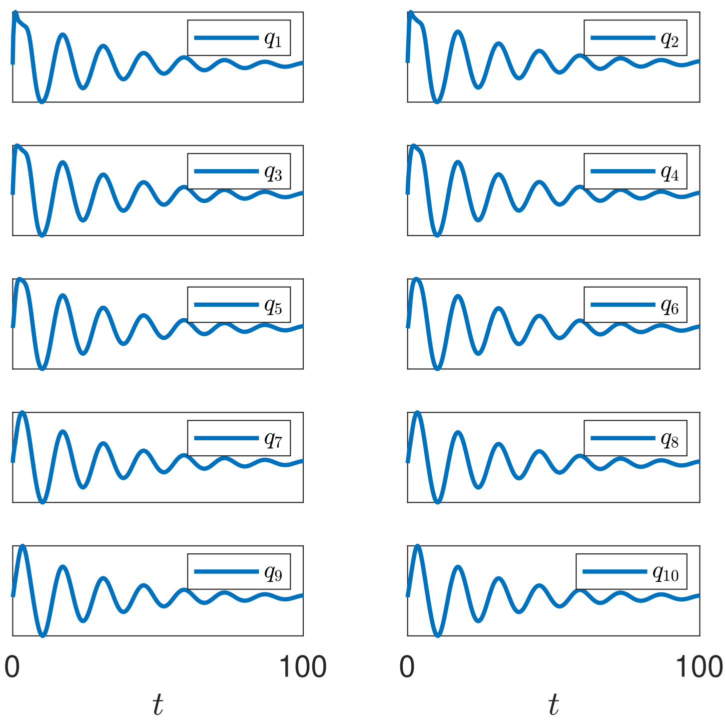

Figure 2 – Free Response of all degrees of freedom of the fictitious 10-story building.

A comparison between the computational cost between evaluation of the system damping ratio and the eigenvalue approach can be performed. Since full determination of the system eigenvalues is necessary, it is expected such approach to be costly for large system. It so happens that this occurs even for this system with 10 degrees of freedom, which is not considered large. For example, the computational cost, in terms of floating-point operations, can be estimated by the run time of each approach using MATLAB's internal stopwatch timer, as shown in Fig. 3. Each approach was realized 10^6 times in sequence, and the time for each computation was recorded. The average time using the eigenvalue approach was 1.0802×10^{-4}s, while evaluation of system damping ration only took 2.1477×10^{-6}s in average. A complete picture is provided in Fig. 3, which shows the 10-iteration average evolution, i.e., the average of time required for every sequence of 10 computations. It is clear the evaluation of system damping ratio is less costly, meaning it can be efficiently implemented in applications.

It should be noted that all other methods expounded earlier are of no use for this system, because they can only predict when all degrees of freedom are underdamped, when they are all overdamped or critically damped, but not when mixed damping is present. In addition, Beskos and Boley's method (Beskos and Boley, 1980) is only applicable to fewer degrees of freedom.

CONCLUSIONS

A system damping ratio that predicts oscillatory behavior in multi-degree-of-freedom damped systems has been reported. Contrary to previous methods, the system damping ratio is a real number that is a direct extension from the damping ratio for single-degree-of-freedom systems, indicating oscillatory behavior when less than unit. This result was established from the arithmetic and geometric means of the system eigenvalues. Moreover, the system damping ration can be used in systems with mixed damping, contrary to other approaches in the literature. Finally, an application from structural dynamics was used to illustrate its use, showing it can be more computationally efficient than traditional methods.

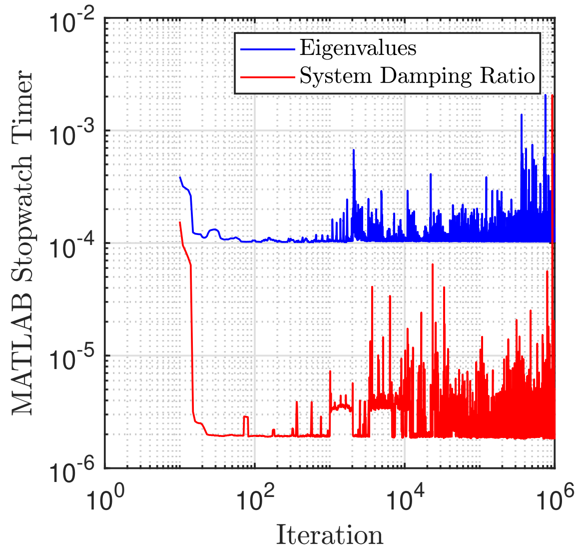

Figure 3 – Time required for each approach. Graph in log scale.

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