CONEM 2022
CONGRESSO NACIONAL DE ENGENHARIA MECÂNICA
TERESINA - PI

ABCM

XI CONGRESSO NACIONAL DE ENGENHARIA MECÃNICA
DE 07 A 11 DE AGOSTO DE 2022, TERESINA-PI, BRASIL

Spatiotemporal Chaos and Patterns in a Dissipative Duffing-type System

Eduardo V. M. Reis, eduardovillela@mecanica.coppe.ufrj.br[1]
Marcelo A. Savi, savi@mecanica.coppe.ufrj.br[1]
[1]Universidade Federal do Rio de Janeiro, COPPE - Department of Mechanical Engineering, Center for Nonlinear Mechanics

Abstract: *Spatiotemporal dynamics arises in systems governed by partial differential equations or coupled lattice maps. A complex dynamics is expected for nonlinear systems, being characterized by chaos and patterns. This paper investigates different responses showing patterns and spatiotemporal chaos on a dissipative Duffing-type system governed by equations with cubic nonlinearity. Duffing systems are largely employed in mechanical energy harvesting systems, defining a relevant topic of engineering research. Numerical simulations are developed employing the finite difference method for spatial discretization and the fourth-order Runge-Kutta method for time discretization. Dissipation effects and external excitation are of concern. Different spatiotemporal patterns are discussed considering distinct combinations of chaos and periodicity.*

Keywords: *Mechanical vibration, spatiotemporal chaos, pattern formation, Duffing system*

1. INTRODUCTION

Spatiotemporal dynamics has proper characteristics being associated with different scientific and engineering situations such as chemistry (Ng and Dubljevic, 2012; Owolabi and Hammouch, 2019), solid mechanics (Yang and Chen, 2005; Reis *et al.*, 2018) and fluid dynamics (Chimanski *et al.*, 2016; Reis and de B Alves, 2021). These problems can be modeled by either partial differential equations (PDE) or by a coupled lattice maps, representing a challenging topic due to nonlinearities that establish complex dynamical responses. On this basis, different spatiotemporal patterns and chaos are some instances of the richness related to these nonlinear systems.

Pattern formation is the dynamical investigation of behavioral patterns of complex systems, being characterized by collective behavior built from individual or local dynamical behavior. Self-organization is one of the key aspects of this kind of behavior. Different dynamical behaviors are instances of patterns, which means that periodic and quasi-periodic responses are some of the spatiotemporal patterns. Biological systems have several mechanisms that can emerge from this kind of strategy, varying from self-assembly of molecular structures, cell organizations and proliferation, cell divisions and control, among others. Owolabi and Hammouch (2019) exploited different spatiotemporal patterns presented in Belousov-Zhabotinskii chemical reaction systems. Biancalani *et al.* (2010) developed a stochastic version of the Brusselator model and observed Turing patterns. Coelho *et al.* (2021) investigated the Swift-Hohenberg nonlinear PDE subjected to nonuniform forcing and observed that the orientation of the strips are driven mainly by forcing gradients. Kohsokabe and Kaneko (2017) studied the boundary-induced pattern formation in a two-component reaction-diffusion system, observing a relationship between the pattern wavelength and the period oscillation of the components.

Spatiotemporal chaos is a complex behavior associated with PDE systems, being associated with an irregular response in both space and time. Predator-prey models are an example of systems that present spatiotemporal chaos. Morozov *et al.* (2004) investigated the routes to chaos and the Allee effect. Ghorai and Poria (2016) investigated the influence of additional food for predators in a diffusive predator-prey model, showing that the system can present spatiotemporal chaos depending on the quality and quantity of the additional food. Gotoda *et al.* (2015) analyzed the Kuramoto-Sivashinsky equation with an additional spatial derivative term standing for dispersion, showing that an exponential decay behavior of the Power Spectrum Density at high frequencies is usually associated with spatiotemporal chaos. Deissler (1987) investigated the convective chaos in a spatiotemporal open-flow problem governed by the Ginzburg-Landau equation, stating that the external noise initiates the transition to turbulence being responsible for the intermittent turbulent behavior observed in the transition region. Afterward, Aranson and Kramer (2002) presented an overview of the different responses given by the Ginzburg-Landau equation.

Duffing-type systems is a classical example characterized by cubic nonlinearities and describing a large number of physical phenomena (Savi and Pacheco, 2002; Kovacic and Brennan, 2011). Among the physical situations governed by Duffing equations, it should be highlighted a network of Moon beams (Moon and Holmes, 1979; Costa *et al.*, 2021) or acoustic metamaterials (Fang *et al.*, 2017). Duffing systems built as a network of oscillators is a typical spatiotemporal system. In this regard, Umberger *et al.* (1989) presented a pioneer study of a chain of Duffing oscillators considering periodic boundary conditions. The nonlinear oscillators were connected by means of linear springs. Pattern formation were investigated when the system undergoes harmonic excitation showing that, for a low degree of coupling, each oscillator responded more independently, which led to chaos. Musielak et al. Musielak *et al.* (2005) studied routes to chaos in a network of Duffing oscillators showing that the increase of the number of degrees of freedom can lead to crisis, instead of period doubling, as the main route to chaos. Romeo and Rega Romeo and Rega (2006) performed an analysis of wave propagation in a chain of Duffing oscillators. Chatterjee *et al.* (2020) considered a chain of Duffing oscillators subjected to harmonic oscillation, calculating the convective Lyapunov exponents and observing the light-cone boundary during transient period. Two different patterns are highlighted: sustained chaos, transient chaos, and nonchaotic regions.

Reis and Savi (2021) studied a spatiotemporal conservative Duffing-type mechanical system governed by PDE with cubic nonlinearity. A Hamiltonian system was employed neglecting external excitation and dissipation. Mathematical tools were defined, establishing an analysis of perturbed orbits in both space and time: local, convective and mean perturbations. Results showed complex behaviors including spatiotemporal chaos.

This paper investigates the spatiotemporal nonlinear dynamics of a dissipative Duffing-type system subjected to external harmonic excitations. It represents a further invertigation initiated by Reis and Savi (2021). Numerical simulations are carried out considering the finite difference scheme for spatial discretization and the fourth order Runge-Kutta method for time discretization. Complex responses are treated showing different responses: periodic and quasi-periodic patterns or spatiotemporal chaos.

2. MATHEMATICAL MODEL

A spatiotemperal Duffing-type system governed by a partial differential equation with cubic nonlinearity is investigated (Reis and Savi, 2021). External stimulus and a linear dissipation are considered. By assuming a dimensionless displacement u, the spatial coordinate $x \in [0,1]$ and time t, the spatiotemporal dynamics is governed by the following equation:

$$\ddot{u} = \sigma u'' + \sigma' u' - 2\xi \dot{u} + \frac{1}{2}\left(u - u^3\right) + \gamma \sin(\Omega t) \tag{1}$$

where $\dot{()}$ means partial time derivative $\partial()/\partial t$, $()'$ stands for spatial partial derivative $\partial()/\partial x$, $\sigma = \sigma(x)$ is a spatial coupling coefficient function, ξ is the dissipation factor, $\gamma = \gamma(x)$ is the amplitude excitation function and Ω is the excitation frequency. It is worthwhile mentioning that the Hamiltonian system treated by Reis and Savi (2021) is recovered by assuming $\xi = \gamma = 0$. Systems with $\sigma' = 0$ are called reciprocal, being characterized by a spatial symmetrical energy propagation. On the other hand, $\sigma' \neq 0$ yields nonreciprocal systems, which are characterized by an asymmetrical energy propagation (Coulais *et al.*, 2017; Brandenbourger *et al.*, 2019). Moreover, the system is spatially decoupled when $\sigma = 0$, a situation where it does not present spatial dependence, being described only by ordinary differential equations (ODEs) that characterizes the Duffing oscillator (Kovacic and Brennan, 2011). Under this assumption, the decoupled system has three equilibrium points: $u = -1$, $u = 1$, and $u = 0$. The first two points are stable whilst the third is unstable. On the other hand, the limit $\sigma \to \infty$ yields a rigid spatial attachment with only one possible solution: $u(x,t) = 0$. Finally, boundary conditions are assumed to be of fixed-fixed kind: $u(0,t) = u(1,t) = 0$.

The canonical form of the system can be rewritten as a first order system giver by

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= \sigma u'' + \sigma' u' - 2\xi \dot{u} + \frac{1}{2}(u - u^3) + \gamma(x)\sin(\Omega t) \end{aligned} \tag{2}$$

System integration is achieved by considering fourth order finite difference scheme for spatial discretization and time discretization is performed with fourth order Runge-Kutta method.

3. PERTURBATION ANALYSIS

A general form of the dynamical system with spatial dependence can be written as $\dot{\mathbf{u}} = f(\mathbf{x}, t, \mathbf{u}, \mathbf{u}', \mathbf{u}'', \mathbf{u}''', ..., \mathbf{u}^{(m)}, \mathcal{P})$, where $\mathbf{x}$ is the spatial coordinates, t is time, $\mathbf{u} = \mathbf{u}(\mathbf{x},t) \in \mathbb{R}^n$ and $\mathcal{P}$ represents a set of parameters. By assuming $\bar{\mathbf{u}}$ as a reference solution, its perturbation $\mathbf{u}_p$ can be obtained from a linearization of the equations of motion. Therefore, it is possible to write,

$$\dot{\bar{\mathbf{u}}} = f(\mathbf{x}, t, \mathbf{u}, \mathbf{u}', \mathbf{u}'', ..., \mathbf{u}^{(m)}, \mathcal{P}) \tag{3a}$$

$$\dot{\mathbf{u}}_p = \left.\frac{\partial f}{\partial \mathbf{u}}\right|_{\bar{\mathbf{u}}} \mathbf{u}_p + \sum_{i=1}^{n} \left.\frac{\partial f}{\partial \mathbf{u}^{(i)}}\right|_{\bar{\mathbf{u}}} \mathbf{u}_p^{(i)} \tag{3b}$$

Specifically for the Duffing-type governing Eq. 2, the perturbation equation in the canonical form is given by

$$\begin{aligned} \dot{u_p} &= v_p\,, \\ \dot{v_p} &= \sigma u_p'' + \sigma' u_p' - 2\xi \dot{u}_p + \frac{1}{2}(1 - 3u^2) u_p \end{aligned} \tag{4}$$

with boundary conditions $u_p(0,t) = u_p(1,t) = 0$.

The perturbed orbit evolution defines the main characteristic of the system dynamics since chaotic behavior is characterized by divergent perturbations with respect to the reference orbit $\bar{\mathbf{u}}$. On the other hand, the convergence between both orbits characterizes a periodic solution. In order to investigate spatial and temporal perturbation characteristics, a local perturbation quantity, $\phi(\mathbf{x}, t)$, is defined as follows

$$\phi(\mathbf{x}, t) = \sqrt{\langle \mathbf{u}_p(\mathbf{x}, t), \mathbf{u}_p(\mathbf{x}, t) \rangle} \tag{5}$$

where $\langle\,\rangle$ yields inner product. In this regard, spatiotemporal characteristics can be evaluated at a specific spatial position in the space-time map. Therefore, an exponential growth of ϕ means that the response is high sensitive to perturbations at this spatial position. On the other hand, an exponential decay of ϕ means otherwise.

Regarding the Duffing-type system, ϕ is defined as follows

$$\phi(x, t) = \sqrt{u_p(x,t)^2 + v_p(x,t)^2} \tag{6}$$

The perturbation quantity ϕ is directly related to the local Lyapunov exponents at a specific spatial position x_0. Therefore, the estimation of the Lyapunov exponents can be done by considering a linear regression of the time series $(t, \log(\phi(x_0), t))$ using a base function $q(t)$, defined as follows

$$\lambda(x_0) = \lim_{t\to\infty} \dot{q}(t) \tag{7}$$

Due to the boundary conditions, one might expect $\lambda(0) = \lambda(1) = 0$. Additionally, a dynamical response with $\lambda(x) > 0\ \forall x\ \in]0,1[$ characterizes spatiotemporal chaos, while $\lambda(x) < 0\ \forall x\ \in]0,1[$ is considered a spatiotemporal periodic pattern. A quasi-periodic pattern instead is characterized by $\lambda(x) \approx 0\ \forall x \in [0,1]$.

4. RESULTS AND DISCUSSION

This section investigates the dynamical behavior of the Duffing-type system. All simulations are carried out assuming $\sigma = 5{\times}10^{-4}$ and $\xi = 5{\times}10^{-3}$. Moreover, the nonuniform excitation function is considered $\gamma(x) = \frac{A}{2}\tanh(40x - 20) + \frac{1}{2}$, being A a constant yielding the excitation amplitude. Therefore, on half of the domain is subjected to small excitations amplitude while the other half is subjected to a larger excitation. Initial condition simulations are such that $u(x,0) = v(x,0) = 0$. Regarding the perturbation function, initial conditions adopted are given by a centered Gaussian shaped perturbation $u_p(x,0) = 2.825\exp\left[-100(x-1/2)^2\right]$, while $v_p(x,0) = 0$.

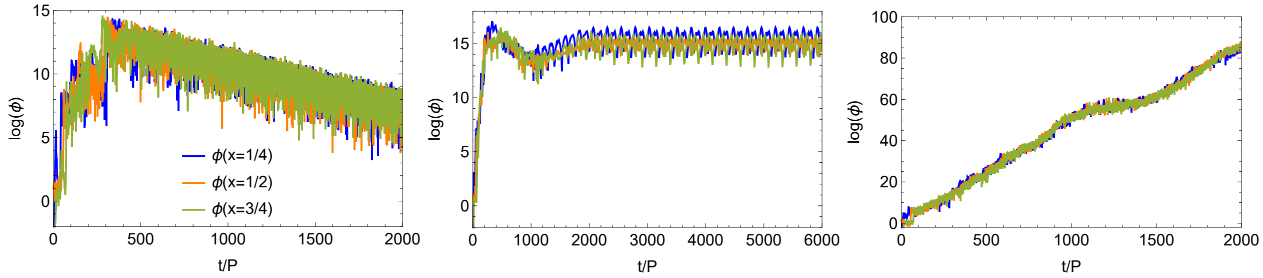
After neglecting a transient period of $t = 600P$, where P stands for the excitation period and $P = 2\pi/\Omega$, Table 1 identifies the different kinds of dynamical response for distinct excitations: spatiotemporal chaos (blue), periodic (red) and quasi-periodic (yellow). The identification is done by monitoring the perturbations and using a linear base function $q(t) = \lambda t + b$, which is related to the local Lyapunov exponent. It is noticeable that the majority of responses is chaotic, but periodic patterns and quasi-periodic responses also take place for higher excitation frequencies.

Table 1: **Different types of dynamical response: spatiotemporal chaos (blue), periodic (red) and quasi-periodic (yellow) patterns.**

$A \diagdown \Omega$	1/2	1	2	4	8
1					
5					
10					
20					

Different responses are selected in order to understand the spatiotemporal dynamical aspects. It is assumed that $\Omega = 8$ and different values of A. On this basis, $A = 5$ stands for a periodic response, $A = 10$ stands for a quasi-periodic one and, finally, $A = 20$ yields a spatiotemporal chaotic response. Fig. 1 shows the perturbation ϕ evaluated for these three responses and three spatial positions. One can observe that each kind of response has distinct types of local perturbation evolution. Periodic response has a signature with local perturbation asymptotic fall throughout time after the transient period. The decrease rate of the fall physically yields the local Lyapunov exponent. Moreover, one should notice that the decay rate of ϕ at the three different spatial positions are practically the same. The quasi-periodic response is characterized by an oscillation of the local perturbation around an average value. Therefore, the local perturbation average tends to be constant. Finally, chaotic response is characterized by an increase of the local perturbation ϕ in all spatial domain.

Fig. 2 presents the local Lyapunov exponent variation through spatial domain. As expected, the quasi-periodic response shows $\lambda \approx 0$ in all spatial domain. Instead, the periodic and chaotic responses show $\lambda(x) < 0$ and $\lambda(x) > \forall x \in]0,1[$, respectively, with exception of boundaries, where a discontinuous step takes place due to boundary conditions and therefore $\lambda(0) = \lambda(1) = 0$. Moreover, the spatial distribution of the local Lyapunov function is almost homogeneous.



(a) Periodic (b) Quasi-Periodic (c) Spatiotemporal Chaos

Figure 1: **Perturbation ϕ evolution in time for different kinds of dynamical response for $\Omega = 8$: (a) Periodic with $A = 5$, (b) Quasi-periodic with $A = 10$ and (c) Spatiotemporal chaotic with $A = 20$.**

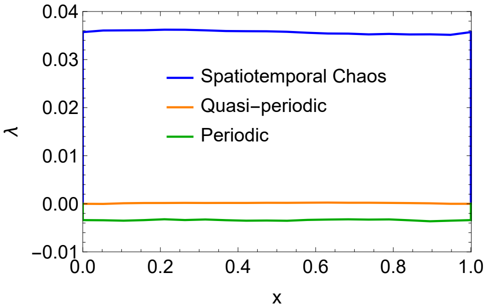
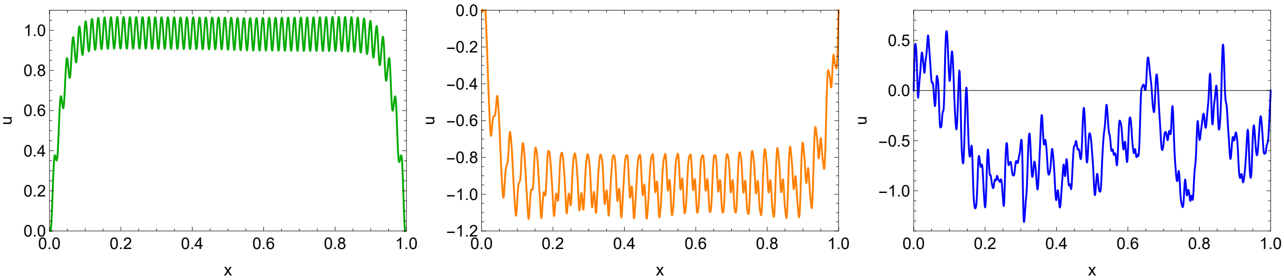


Figure 2: **Local Lyapunov exponents $\lambda(x)$.**

Fig. 3 depicts the displacement time history and its respective frequency spectrum of the middle spatial point $u(x = 1/2, t)$. Frequency response is obtained using the Fast Fourier Transform. Note that it is possible to identify the signatures of each type of dynamical response. Periodic response presents a time history that shows a pattern that repeats itself over time, oscillating around $u \approx 1$, which is a stable equilibrium point for the decoupled system ($\sigma = 0$) (Reis and Savi, 2021). The frequency spectrum is regular, being characterized by the presence of specific frequencies. On the other hand, quasi-periodic response is characterized by the presence of some specific discrete peaks in the frequency spectrum. Nevertheless, these frequencies are incommensurate, i.e., the ratio between these peaks are given by irrational numbers, which makes the the time history different from periodic since it does not repeat each period of time. Finally, the chaotic response is mainly characterized by an irregular pattern in both time and frequency.

The displacement through spatial direction is now of concern. Fig. 4 shows the spatial configurations at $t/P = 2000$. The periodic response presents a symmetric pattern with respect to $x = 1/2$. The quasi-periodic response presents a quasi-symmetric pattern. Finally, the spatiotemporal chaos yields an irregular pattern, where no regularity can be observed at all.

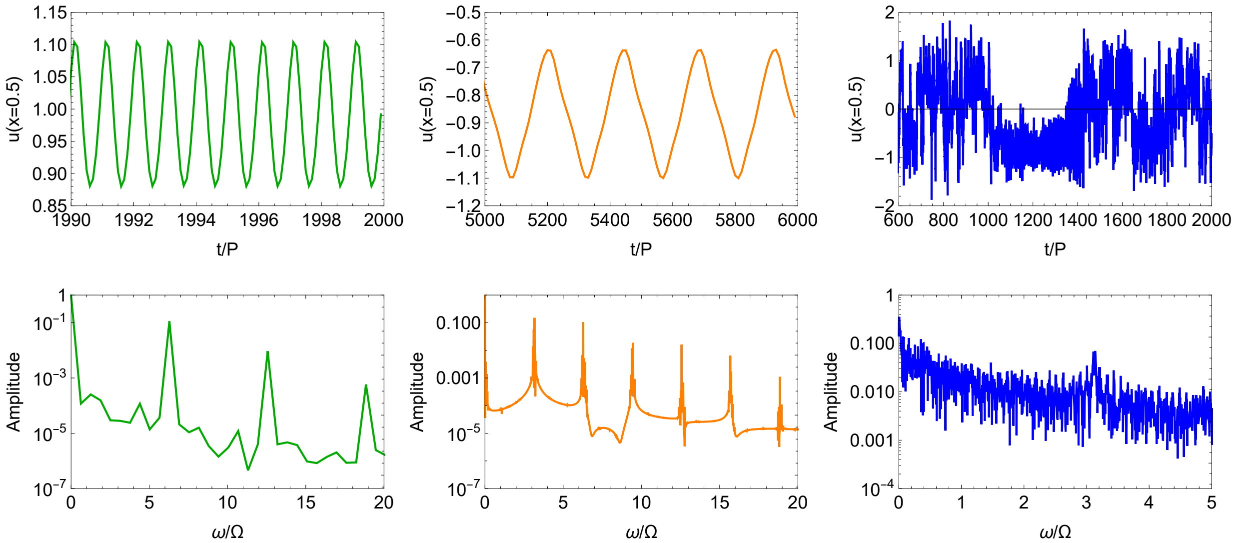


(a) Periodic (b) Quasi-Periodic (c) Spatiotemporal Chaos

Figure 4: **Displacement space distribution for different kinds of dynamical response with $\Omega = 8$: (a) periodic response with $A = 5$; (b) quasi-periodic with $A = 10$; (c) spatiotemporal chaos with $A = 20$.**

5. CONCLUSIONS

This paper deals with the spatiotemporal nonlinear dynamics of a Duffing-type system subjected to nonhomogeneous excitations. Numerical simulations are carried out by employing the fourth order Runge-Kutta method for time discretization and the finite difference method for spatial discretization. A perturbation analysis is developed considering a linearized solution around a reference orbit. The local perturbation allows the definition of the local Lyapunov exponent that is capable to diagnosis different kinds of response. Results show different spatiotemporal dynamical responses can be represented by periodic or quasi-periodic patterns or spatiotemporal chaos. Finally, the local Lyapunov exponent shows



(a) Periodic (b) Quasi-Periodic (c) Spatiotemporal Chaos

Figure 3: **Displacement along time at $x = 1/2$ and its respective Fourier Transformation for different kinds of dynamical response with $\Omega = 8$: (a) Periodic with $A = 5$, (b) Quasi-periodic with $A = 10$ and (c) Spatiotemporal chaotic with $A = 20$.**

to be a good mathematical tool to characterize spatiotemporal dynamical responses.

6. ACKNOWLEDGEMENTS

The authors would like to acknowledge the support of the Brazilian Research funding Agencies FAPERJ, CAPES and CNPq.

7. REFERENCES

Aranson, I.S. and Kramer, L., 2002. "The world of the complex Ginzburg-Landau equation". *Reviews of Modern Physics*, Vol. 74, No. 1, pp. 99–143. ISSN 00346861. doi:10.1103/RevModPhys.74.99.

Biancalani, T., Fanelli, D. and Di Patti, F., 2010. "Stochastic turing patterns in the brusselator model". *Physical Review E*, Vol. 81, No. 4, p. 046215. doi:10.1103/PhysRevE.81.046215.

Brandenbourger, M., Locsin, X., Lerner, E. and Coulais, C., 2019. "Non-reciprocal robotic metamaterials". *Nature Communications*, Vol. 10, No. 1, pp. 1–8. ISSN 20411723. doi:10.1038/s41467-019-12599-3.

Chatterjee, A.K., Kundu, A. and Kulkarni, M., 2020. "Spatiotemporal spread of perturbations in a driven dissipative duffing chain: An out-of-time-ordered correlator approach". *Physical Review E*, Vol. 102, No. 5, p. 052103. doi: 10.1103/PhysRevE.102.052103.

Chimanski, E.V., Rempel, E.L. and Chertovskih, R., 2016. "On–off intermittency and spatiotemporal chaos in three-dimensional rayleigh–bénard convection". *Advances in Space Research*, Vol. 57, No. 6, pp. 1440–1447.

Coelho, D.L., Vitral, E., Pontes, J. and Mangiavacchi, N., 2021. "Stripe patterns orientation resulting from nonuniform forcings and other competitive effects in the swift–hohenberg dynamics". *Physica D: Nonlinear Phenomena*, Vol. 427, p. 133000.

Costa, L.G., Monteiro, L.L.d.S., Pacheco, P.M.C.L. and Savi, M.A., 2021. "A parametric analysis of the nonlinear dynamics of bistable vibration-based piezoelectric energy harvesters". *Journal of Intelligent Material Systems and Structures*, Vol. 32, No. 7, pp. 699–723. doi:10.1177/1045389X20963188.

Coulais, C., Sounas, D. and Alù, A., 2017. "Static non-reciprocity in mechanical metamaterials". *Nature*, Vol. 542, No. 7642, pp. 461–464. ISSN 14764687. doi:10.1038/nature21044.

Deissler, R.J., 1987. "Spatially growing waves, intermittency, and convective chaos in an open-flow-system". *Pysica 25D*, pp. 233–260. doi:10.1016/0167-2789(87)90103-5.

Fang, X., Wen, J., Bonello, B., Yin, J. and Yu, D., 2017. "Wave propagation in one-dimensional nonlinear acoustic metamaterials". *New Journal of Physics*, Vol. 19, No. 5, pp. 21–23. ISSN 13672630. doi:10.1088/1367-2630/aa6d49.

Ghorai, S. and Poria, S., 2016. "Pattern formation and control of spatiotemporal chaos in a reaction diffusion prey–predator system supplying additional food". *Chaos, Solitons & Fractals*, Vol. 85, pp. 57–67. doi: 10.1016/j.chaos.2016.01.013.

Gotoda, H., Pradas, M. and Kalliadasis, S., 2015. "Nonlinear forecasting of the generalized kuramoto-sivashinsky equation". *International Journal of Bifurcation and Chaos*, Vol. 25, No. 5, pp. 1–19. ISSN 02181274. doi: 10.1142/S0218127415300153.

Kohsokabe, T. and Kaneko, K., 2017. "Boundary-induced pattern formation from temporal oscillation: Spatial map analysis". *EPL (Europhysics Letters)*, Vol. 116, No. 4, p. 48005.

Kovacic, I. and Brennan, M.J., 2011. *The Duffing equation: nonlinear oscillators and their behaviour*. John Wiley & Sons.

Moon, F. and Holmes, P.J., 1979. "A magnetoelastic strange attractor". *Journal of Sound and Vibration*, Vol. 65, No. 2, pp. 275–296. doi:10.1016/0022-460X(79)90520-0.

Morozov, A., Petrovskii, S. and Li, B.L., 2004. "Bifurcations and chaos in a predator-prey system with the allee effect". *Proceedings of the Royal Society of London. Series B: Biological Sciences*, Vol. 271, No. 1546, pp. 1407–1414. doi: 10.1098/rspb.2004.2733.

Musielak, D., Musielak, Z. and Benner, J., 2005. "Chaos and routes to chaos in coupled duffing oscillators with multiple degrees of freedom". *Chaos, Solitons & Fractals*, Vol. 24, No. 4, pp. 907–922. doi:10.1016/j.chaos.2004.09.119.

Ng, J. and Dubljevic, S., 2012. "Optimal boundary control of a diffusion-convection-reaction PDE model with time-dependent spatial domain: Czochralski crystal growth process". *Chemical Engineering Science*, Vol. 67, No. 1, pp. 111–119. ISSN 00092509. doi:10.1016/j.ces.2011.06.050.

Owolabi, K.M. and Hammouch, Z., 2019. "Spatiotemporal patterns in the belousov–zhabotinskii reaction systems with atangana–baleanu fractional order derivative". *Physica A: Statistical Mechanics and its Applications*, Vol. 523, pp. 1072–1090. doi:10.1016/j.physa.2019.04.017.

Reis, E.V.M. and de B Alves, L.S., 2021. "Convective and absolute instabilities induced by viscous dissipation in the thermocapillary convection with through-flow". *Journal of Heat Transfer*, Vol. 143, No. 5, p. 052601. doi: 10.1115/1.4050312.

Reis, E.V.M. and Savi, M.A., 2021. "Cob-2021-0492 chaos in a spatiotemporal duffing-type system".

Reis, E., Sphaier, L., Nunes, L. and Alves, L.d.B., 2018. "Dynamic response of free span pipelines via linear and nonlinear stability analyses". *Ocean Engineering*, Vol. 163, pp. 533–543. doi:10.1016/j.oceaneng.2018.06.002.

Romeo, F. and Rega, G., 2006. "Wave propagation properties in oscillatory chains with cubic nonlinearities via nonlinear map approach". *Chaos, Solitons and Fractals*, Vol. 27, No. 3, pp. 606–617. ISSN 09600779. doi: 10.1016/j.chaos.2005.04.087.

Savi, M. and Pacheco, P., 2002. "Chaos in a two-degree of freedom duffing oscillator". *Journal of the Brazilian Society of Mechanical Sciences*, Vol. 24, No. 2, pp. 115–121. doi:10.1590/S0100-73862002000200006.

Umberger, D.K., Grebogi, C., Ott, E. and Afeyan, B., 1989. "Spatiotemporal dynamics in a dispersively coupled chain of nonlinear oscillators". *Physical Review A*, Vol. 39, No. 9, pp. 4835–4842. doi:10.1103/PhysRevA.39.4835.

Yang, X.D. and Chen, L.Q., 2005. "Bifurcation and chaos of an axially accelerating viscoelastic beam". *Chaos, Solitons & Fractals*, Vol. 23, No. 1, pp. 249–258.

8. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.