Dynamic Models for Transmission Lines and Hoses

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Abstract: In this paper simplified models for hydraulic transmission lines and hoses are presented, for both time and frequency domain simulation. Hydraulic transmission lines are omnipresent elements in hydraulic systems. In most cases it is justified to regard pipelines as lumped capacitances when hydraulic systems are analysed and simulated. In some cases, however, wave propagation effects in lines may become significant. In particular this is important to consider in order to predict pressure response to fluid transients, and to predict pressure amplitudes in system with periodic excitations, such as from hydraulic pumps. Their impact on hydraulic control system can also be of importance.

Keywords: transmission lines, hydraulic, hose, wave propagation, simulation

NOMENCLATURE

A:Cross-section area q:Flow α:Friction frequency
C:Hydraulic capacitance R:Hydraulic resistance ρ:Fluid density
L:Hydraulic inductance s:Laplace variable β:Fluid bulk modulus
l:Length T:Time delay η:Dynamic viscosity
p:Pressure Z_c:Characteristic impedance

INTRODUCTION

Noise in hydraulic systems is one of the major problems with hydraulic systems. However, this can in many cases be analysed at low cost using the proper design tools. For prediction of pump pulsations in hydraulic systems modelling in the frequency domain is preferable. In this way attenuators can be designed to suppress e.g. frequencies generated by the pump. Here, a transmission line element is modelled with a four-pole equation. One aspect of hydraulic transmission lines that is complicated is the frequency dependent friction. The exact solution involves expressions with Bessel function that are costly to evaluate using standard packages. However, very efficient approximates that are valid for the whole frequency range of interest have been developed, which means that this kind of models can be used for design optimization. Furthermore, the effects of damping from visco-elastic walls in hoses can also be modelled. This can have a great effect, especially for damping at high frequencies.

BASIC EQUATIONS FOR A TRANSMISSION LINE

A general transmission line can be described by the four-pole equation. See e.g. Viersma [10]. Here capitals are used to indicate Laplace transformed variables.
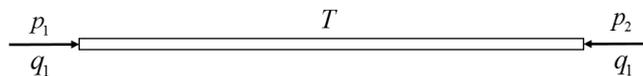

Figure 1 – Transmission line

$$\begin{pmatrix} -Q_2 \\ P_2 \end{pmatrix} = \begin{pmatrix} A_L & B_L \\ C_L & D_L \end{pmatrix} \times \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} \tag{1}$$

There are two important parameters. These are the time delay, T, due to the limited signal propagation speed, and there is the characteristic impedance Z_c. According to Viersma 1980 frequency dependent friction can be handled by introducing the frequency dependent friction factor N. The elements in the matrix can be written as:

$$A_L = D_L = \cosh T s \sqrt{N} \tag{2}$$

$$B_L = -\frac{1}{Z_c \sqrt{N}} \sinh T s \sqrt{N} \tag{3}$$

$$C_L = -Z_c \sqrt{N} \sinh T s \sqrt{N} \tag{4}$$

The time delay T can be calculated from the oil properties β and oil density ρ and the length l.

$$T = l\sqrt{\rho/\beta} \tag{5}$$

and the characteristic impedance can be calculated from the oil properties and the cross section area A.

$$Z_c = \frac{1}{A}\sqrt{\rho\beta} \tag{6}$$

Using (1) and multiplying by 2 the following expressions are obtained.

$$\frac{1}{Z_c\sqrt{N}}(e^{Ts\sqrt{N}} - e^{-Ts\sqrt{N}})P_1 = 2Q_2 + (e^{Ts\sqrt{N}} + e^{-Ts\sqrt{N}})Q_1 \tag{7}$$

$$(e^{Ts\sqrt{N}} + e^{-Ts\sqrt{N}})P_1 == 2P_2 + Z_c\sqrt{N}(e^{Ts\sqrt{N}} - e^{-Ts\sqrt{N}})Q_1$$

Adding (7) and (8) and dividing with 2 yields

$$P_1 e^{Ts\sqrt{N}} = P_2 + Z_c\sqrt{N}(Q_2 + Q_1 e^{Ts\sqrt{N}}) \tag{8}$$

Rearranging yields

$$P_1 e^{Ts\sqrt{N}} - Z_c\sqrt{N}Q_1 e^{Ts\sqrt{N}} = P_2 + Z_c\sqrt{N}Q_2 \tag{9}$$

Introducing the wave variables C_1 and C_2 such that

$$P_1 = C_1 + Z_c Q_1 \tag{10}$$
$$P_2 = C_2 + Z_c Q_2 \tag{11}$$

One often used line model is the one proposed by Trikha (1975). This model uses the method of characteristics and is reasonably accurate. The line is divided in sections with the length ha where h is the time step used in the simulation and a is the speed of sound in the line. Pressures and flows are computed for each part of the line. Obviously, this can be time consuming if the line is long.

The model used here have similarities to a model proposed by Karam and Leonard (1973). In this model only the pressures and flows at the ends of the line are computed which greatly reduces the computational effort. A model based on this approach with improved accuracy was described by Krus et al (1991) where it was shown how this approach could be mated to the method with characteristics to obtain a very robust, accurate and economical model.

In that model, which will be described here, only state variables at the ends of a line are calculated, and not in any internal nodes as in other models. Despite this, the model gives good results both in the time and frequency domain when compared to more elaborate models. If for some reason internal state variables are wanted they can be obtained by representing a line with several line elements. The model was further developed in Johnston 2006, It was also adopted for Modelica by 2011.

The introduction of the transmission line models with wave propagation is also a very effective way to connect different components in time domain simulation of systems. Essentially there are two kinds of components, those who calculates wave variables, and components that calculate flow and pressures. For a more detailed discussion on this subject see Krus et al. (1990).

At each such component, the following system of equations is solved (in the time domain).

$$q = q(p) \tag{12}$$
$$p = c + Z_c q$$

Solving (9) and (11) for C_1 and C_2 yields

$$C_1 = e^{-Ts\sqrt{N}}(P_2 + Z_c\sqrt{N}Q_2) + Z_c(\sqrt{N} - 1)Q_1 \tag{13}$$

$$C_2 = e^{-Ts\sqrt{N}}(P_1 + Z_c\sqrt{N}Q_1) + Z_c(\sqrt{N} - 1)Q_2 \tag{14}$$

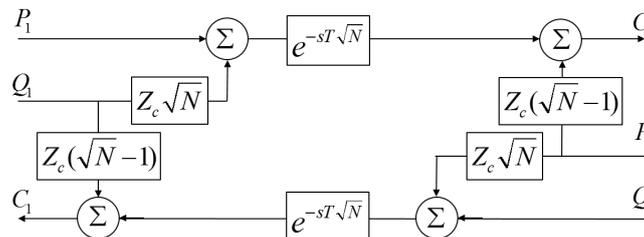In Fig. (2) the corresponding block diagram is shown.

Figure 2 – Block diagram of a transmission line in the frequency domain

Distributed resistance

For the case of a uniformly distributed resistance the expression for N is (see e.g. Viersma (1980)):

$$N(s) = \frac{\alpha}{s} + 1 \tag{15}$$

where

$$\alpha = \frac{R}{Z_c T} \tag{16}$$

Here, R is the total resistance of the line. See e.g. Viersma (1980) For laminar flow.

$$R = \frac{8\pi\eta l}{A^2} \tag{17}$$

The factor N for the distributed friction is hereafter referred to as N_R, in order to distinguish it from the general case with also frequency dependent friction. The frequency response of a transmission line with a blocked outlet can now be evaluated using the expression for N_R for distributed friction. The example in Fig. 3 is shown in dimensionless form. The only quantity that has to be specified is the ratio R/Z_c. Here it is set to 0.3. In this way the effect on the reduction of the resonances can be seen. Except for the first peak, corresponding to the DC-level, all the others are having almost exactly the same level.
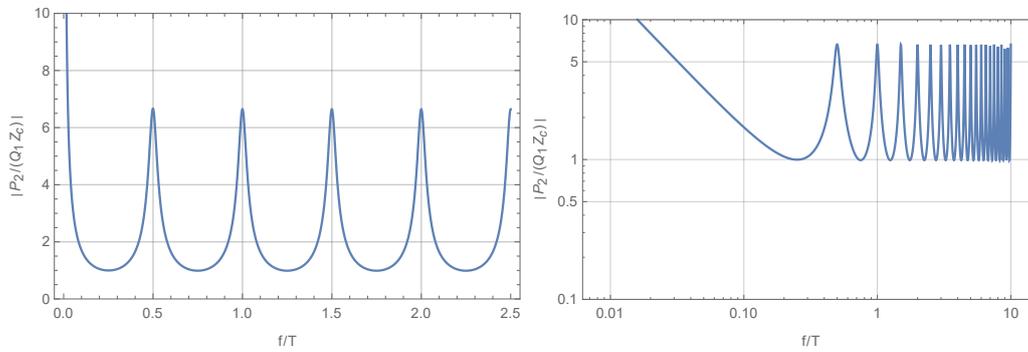

Figure 3 – The frequency spectra of the pressure at the outlet of a transmission line with a blocked outlet in linear and log-log scale.

Frequency dependent resistance

According to Hams (1982) and Viersma (1980) frequency dependent friction can be handled by calculating N as:

$$N(s/\alpha) = -\frac{J_0\left(i\sqrt{8\frac{s}{\alpha}}\right)}{J_2\left(i\sqrt{8\frac{s}{\alpha}}\right)} \tag{18}$$

To plot the function the Laplace variable s is substituted by $i\omega$. The non-dimensional frequency Ω is then introduced which is:

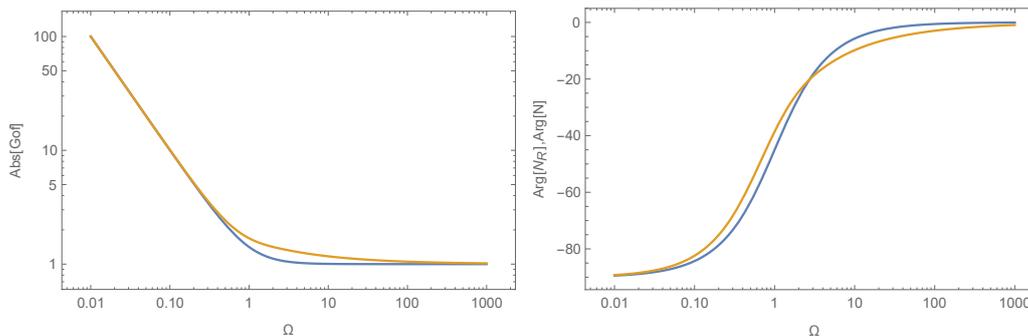$$\Omega = \frac{\omega}{\alpha} \tag{19}$$

Figure 4 – The absolute values and phase of the factor $N_R(i\Omega)$ for distributed friction and frequency dependent friction $N(i\Omega)$. For high frequencies the frequency dependent friction has a much higher damping. They are similar except in the midrange where the frequency dependent friction has a higher value.

In order to be able to deal with the frequency-dependent friction one approach is to introduce the transfer function $G_f(s)$

$$G_f(s) = e^{-Ts(\sqrt{N(s/\alpha)}-1)} \tag{20}$$

If a pressure pulse is propagating in the line the $G_f(s)$ represents a filter that is acting on the pressure signal. The block diagram of the transmission line then becomes like in Fig. 5.
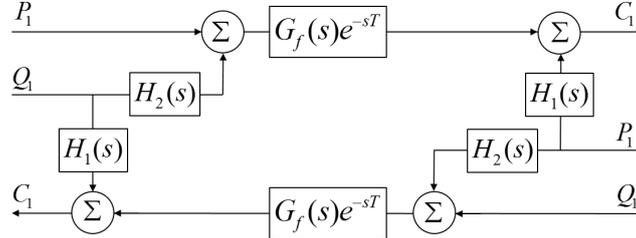

Figure 5 – Block diagram of transmission line with filters introduced for the effects of distributed frequency dependent friction.

Using a frequency domain description of N, the filter $G_f(s)$ can be evaluated. In order to get a general plot of $G_f(i\omega)$. the non-dimensional frequency Ω is used:

Rewriting (20) we obtain if s is substituted for $i\Omega\alpha$

$$G_f(i\Omega\alpha) = e^{-Ti\Omega\alpha(\sqrt{N(i\Omega)}-1)} \tag{21}$$

Rearranging (**??**) we obtain

$$G_{of}(i\Omega) = G_f(i\Omega\alpha)^{\frac{1}{\alpha T}} = e^{-i\Omega(\sqrt{N(i\Omega)}-1)} \tag{22}$$

This is a function that is independent of the properties of the line, since it is only a function of the dimensionless frequency Ω. This function was introduced in Krus et al (1991). The plot of this function for N expressed by (15) and (18) is shown in Fig 6. Clearly there is a dramatic difference, especially compared to the results from just looking at the N functions by themselves as in Fig. 4
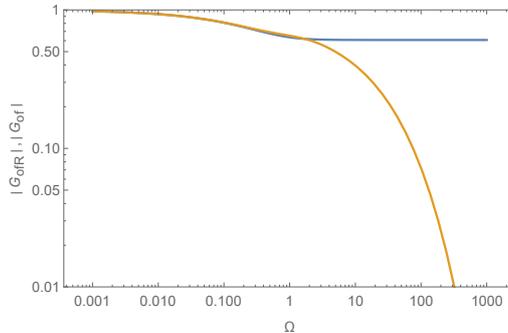

Figure 6 – The transfer function of distributed resistance $G_{ofR}(i\Omega)$ and frequency dependent friction $G_{of}(i\Omega)$. For high frequencies the frequency dependent friction has a much higher damping.

The expression with the Bessel functions can sometimes be inefficient for computations. Even though they are implemented in standard packages with many software, computations, e.g. simulations in the frequency domain can be much more efficient if good approximations can be found. One popular expression have been from Trikha (1975). It is:

$$N_{A1}(i\Omega) = 1 + \frac{1}{i\Omega} + \frac{0.1515}{1+0.3030i\Omega} + \frac{0.1620}{1+0.04i\Omega} + \frac{0.020}{1+0.001i\Omega} \tag{23}$$

Another approximate expression introduced here, is:

$$N_{A2} = 1 + \frac{1}{i\Omega} + \frac{1/2}{1+\sqrt{k_N i\Omega/2}} \tag{24}$$

where $k_N = 0.8$. A comparison of the resulting dimensionless transfer functions is shown in Fig. 7.
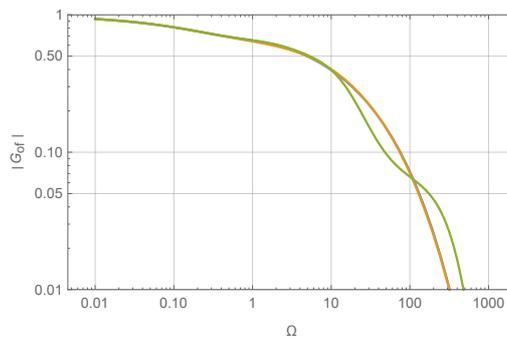
Figure 7 – The transfer function of frequency dependent friction $G_{of}(i\Omega)$ using the exact expression and with the two approximate expressions. The Trikha approximation is the one with the largest deviation.

The frequency response of a transmission line with a blocked outlet can now be evaluated using the expression for N. Also in these examples the quotient R/Z_c is set to 0.3.
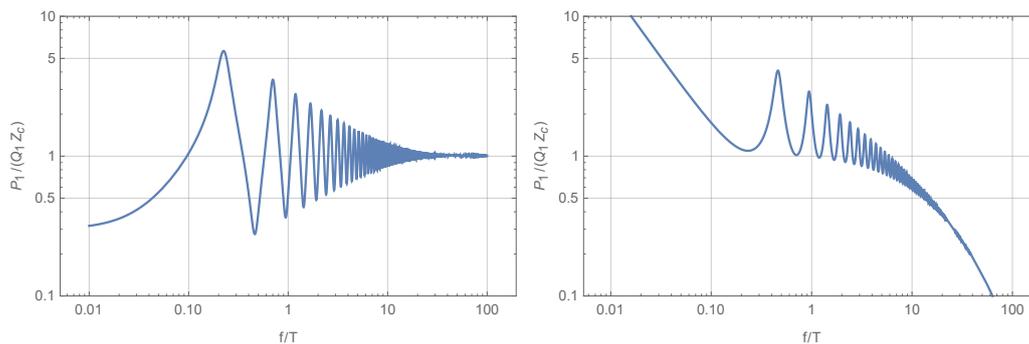

Figure 8 – The frequency spectra of the pressure at the inlet with an open inlet (left) and at the outlet for a transmission line with a blocked outlet (right) in log-log scales.

It can be seen here that the resonance peaks get progressively more damped with the frequency.

The transfer function cannot be inverse transformed exactly into the time domain. It is, however, to calculate it exactly through convolution of a desired input signal with the impulse response that is obtained through inverse Fourier transformation into the time domain. The result is show for a unit input step and an open end corresponding to the same case as 9. In Johsnston (2006) an approximate model for this is presented.
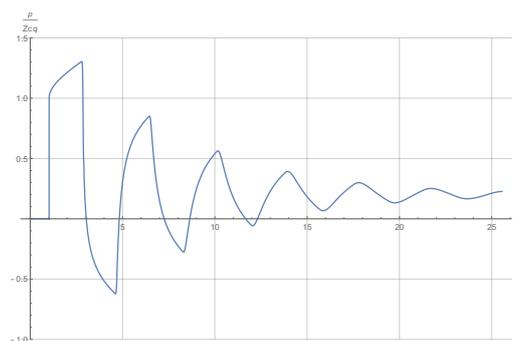

Figure 9 – Step response of hydraulic hose. Inverse transformation from the frequency domain and convolution with a unit input step at time=1 sec.

1 MODELLING OF VISCO-ELASTIC WALLS IN THE FREQUENCY DOMAIN

Hoses differs from steel pipes in that they have more complience and also more damping especially at high frequencies. See Johansson and Nyström (2004) . Consider the case where there is some impedance $G'_w(s)$ at each line increment. Fig (10) shows an increment of a line with wall dynamics. The prime " ' "indicates that the entity is per line increment. This model is not completely general since no axial propagation in the wall can occur. It is, however, capable of dealing with the capacitance and energy loss in the wall in a very consistent way, and these are the only effects of some importance to hydraulic systems.
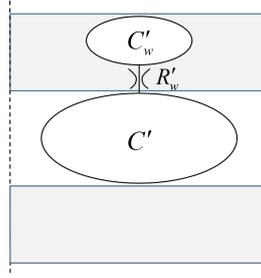
Figure 10 – Increment of a transmission line with wall dynamics

$G'_w(s)$ is defined as

$$G'_w(s) = P'/Q'_w \tag{25}$$

Where Q'_w is the flow communicated through the wall. To establish a model the original Telegrapher's equation developed by Oliver Heaviside (1892) where an equivalent model for electrical lines is used.

$$\begin{pmatrix} A_L & B_L \\ C_L & D_L \end{pmatrix} \times \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} = \begin{pmatrix} -Q_2 \\ P_2 \end{pmatrix} \tag{26}$$

$$A_L = \cosh\gamma l \tag{27}$$

$$B_L = -\frac{1}{Z_{c0}}\sinh\gamma l \tag{28}$$

$$C_L = -Z_{c0}\sinh\gamma l \tag{29}$$

$$D_L = \cosh\gamma l \tag{30}$$

Here

$$\gamma = \sqrt{(sL'+R')(sC'+G'_w(s))} \tag{31}$$

and

$$Z_{c0} = \sqrt{\frac{sL'+R'}{sC'+G'_w(s)}} \tag{32}$$

where $L' = L/l$, $R' = R/l$, $C' = C/l$, $G'_w(s) = G_w(s)/l$. This yelds

$$\gamma = \frac{1}{l}\sqrt{(sL+R)(sC+G_w(s))} \tag{33}$$

and

$$Z_{c0} = \sqrt{\frac{sL+R}{sC+G_w(s)}} \tag{34}$$

An alternative representation is:

$$A_L = = \cosh Ts\sqrt{N_1} \tag{35}$$

$$B_L = -\frac{1}{Z_c\sqrt{N_2}}\sinh Ts\sqrt{N_1} \tag{36}$$

$$C_L = -Z_c\sqrt{N_2}\sinh Ts\sqrt{N_1} \tag{37}$$

$$D_L = = \cosh Ts\sqrt{N_1} \tag{38}$$

This representation has the advantage that the lossless transmission line is obtained by setting $N_1 = N_2 = 1$. Compared to equation (4) N has been replaced by N_1 and N_2 in order to handle also the wall dynamics.

Equation (33) can be written as:

$$\gamma = \frac{s}{l}\sqrt{\frac{(1+\frac{R}{Ls})(1+\frac{G_w(s)}{Cs})}{LC}} \tag{39}$$

Equation (34) can be written as:

$$Z_{c0} = \sqrt{\frac{L}{C}} \sqrt{\frac{1 + \frac{R}{Ls}}{1 + \frac{G_w(s)}{Cs}}} \tag{40}$$

Identification with equation (35) to (38) yields

$$T = 1/\sqrt{LC} \tag{41}$$

$$Z_c = \sqrt{L/C} \tag{42}$$

$$N_1 = (1 + \frac{R}{Ls})(1 + \frac{G_w(s)}{Cs}) \tag{43}$$

$$N_2 = \frac{1 + \frac{R}{Ls}}{1 + \frac{G_w(s)}{Cs}} \tag{44}$$

Introducing

$$N_R = 1 + \frac{R}{Ls} \tag{45}$$

$$N_w = 1 + \frac{G_w(s)}{Cs}) \tag{46}$$

This yelds

$$N_1 = N_R N_w \tag{47}$$

and

$$N_2 = N_R/N_w \tag{48}$$

$$A_L = \cosh T s \sqrt{N_R N_w} \tag{49}$$

$$B_L = -\frac{1}{Z_c \sqrt{\frac{N_R}{N_w}}} \sinh T s \sqrt{N_R N_w} \tag{50}$$

$$C_L = -Z_c \sqrt{\frac{N_R}{N_w}} \sinh T s \sqrt{N_R N_w} \tag{51}$$

$$D_L = \cosh T s \sqrt{N_R N_w} \tag{52}$$

where

$$N_R = 1 + \frac{R}{Ls} \tag{53}$$

$$N_w = 1 + \frac{G_w(s)}{Cs} \tag{54}$$

A first order approximation of the wall behaviour would be to model the capacitance and the energy loss in the wall. This can be done by introducing a wall resistance R_w and a wall capacitance C_w. This is shown in Fig (10).

For this case

$$G_w(s) = \frac{C_w s}{\tau_w s + 1} \tag{55}$$

where

$$\tau_w = R_w C_w \tag{56}$$

and

$$\kappa = C/(C+C_w) \tag{57}$$

A comparison of the resulting dimensionless transfer functions is shown in Fig. 7. In the examples the following data is used: $\tau_w = 0.03T$, $\kappa = 0.5$, and $\alpha = 0.2/T$, This means that the quotient $R/Z_c = 0.2$. In Johansson and Nyström 2004, the time constant for a typical high-pressure hydraulic hose was reported to be in around $0.05 - 0.1$ ms.
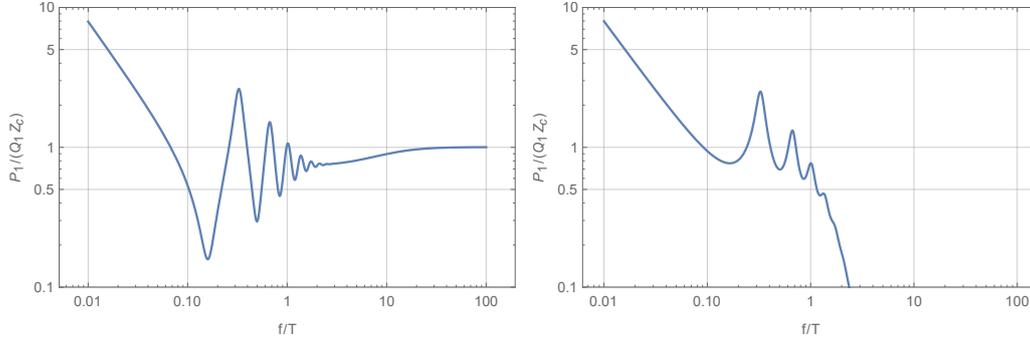

Figure 11 – The frequency spectra of the pressure at the inlet (left) and outlet (right) of a transmission line with a blocked outlet in Log-Log scales.

It can be seen that here the resonance peaks get progressively more damped with the frequency.
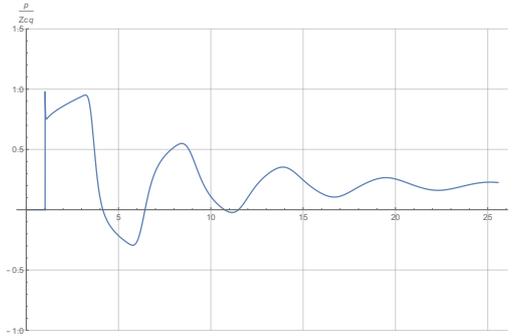

Figure 12 – Step response of hydraulic hose. Inverse transformation from the frequency domain.

2   APPROXIMATE MODELS FOR THE TIME DOMAIN

In a transmission line with ideally elastic lossless walls, the capacitance of the walls can simply be added to the fluid capacitance. This gives a longer, total delay time which is referred to her as T_t. This is defined as:

$$T_t = 1/\sqrt{L(C+C_w)} = T/\sqrt{\kappa} \tag{58}$$

The block diagram can be defined using this delay time instead of T. N_w must then be redefined as

$$N_w = \kappa(1 + \frac{G_w(s)}{Cs}) \tag{59}$$

Equation (55) in (59) yields

$$N_w = \frac{\kappa\tau_w s + 1}{\kappa(\tau_w s + 1)} \tag{60}$$

At the same time the total effective characteristic impedance Z_{ct} is introduced as:

$$Z_{ct} = \sqrt{L/(C+C_w)} = Z_c\sqrt{\kappa} \tag{61}$$

The part of the $G_f(s)$ filter that is dependent of the wall can be written as:

$$G_{fw}(s) = e^{-s(T\sqrt{\frac{\kappa\tau_w s+1}{\kappa(\tau_w s+1)}}-T_e)} \tag{62}$$

Here T_e is an effective time delay in the transmission line. In a transmission line with elastic walls two delay time can be defined. The delay time T correspond to the wave propagation time in a line with stiff walls. Equation (62) can now be written as:

$$G_{fw}(s) = e^{-s(T_t \sqrt{\frac{s\kappa\tau_w+1}{s\tau_w+1}}-T_e)} \tag{63}$$

Inverting and Taylor expansion around $s = 0$ yields

$$G_w(s)^{-1} = \frac{1}{2}[T_e^2 - 2T_t T_e - (1-\kappa)T_t\tau_w]s^2 + [T_t - T_e]s + 1 + O(s^3) \tag{64}$$

Chose T_e so that

$$T_e^2 - 2T_t T_e - (1-\kappa)T_t\tau_w = 0 \tag{65}$$

The only meaningful root for this is:

$$T_e = T_t - \sqrt{(1-\kappa)\tau_w T_t} \tag{66}$$

This yields

$$G_w(s)^{-1} = [\sqrt{(1-\kappa)\tau_w T_t}]s + 1 + O(s^3) \tag{67}$$

An approximate expression for $G_{fw}(s)$ is therefore

$$G_{fwa}(s) = \frac{1}{\tau_{we}s+1} \tag{68}$$

where $\tau_{we} = \sqrt{(1-\kappa)\tau_w T_t}$. This approximation can, however, only be used if. $T_t > (1-\kappa)\tau_w$. In other cases a lumped parameter model is better. The filter $H_1(s)$ can be written as

$$H_1(s) = Z_{ct}(\sqrt{\frac{(1+\frac{R}{Ls})(R_w C_w s+1)}{\frac{R_w C_w}{C+C_w}+1}} - 1) \tag{69}$$

Equations (56) and (57) in (69) yields the H_2 filter as:

$$H_2(s) = Z_{ct}\sqrt{\frac{(1+\alpha\tau_w+\frac{\alpha}{s}+\tau_w s)}{\tau_w\kappa s+1}} - 1) \tag{70}$$

Obviously the main modifications compared to the original transfer functions is that high frequency range is different. There is, however a great deal of uncertainty concerning the real behaviour of vessels in the high frequency range. Furthermore, excitation is usually limited in frequency range and for system simulation it is usually adequate with to have a good representation of the lower frequency range. The main contribution of the visco-elastic walls for such systems are the damping effect, and that can be handled very well with just the $G_{fw}(s)$ filter.

Approximate expressions for both $H_1(s)$ and $H_2(s)$ can be found these filters can be found. And they are shown here for completeness.

$$H_1(s) = Z_{ct}\frac{\frac{\tau_w}{\alpha}(1-\kappa)s2+\tau_w s+1}{(\frac{\tau_w\kappa}{2}s+1)(\frac{2}{\alpha}s+1)} \tag{71}$$

$$H_2(s) = Z_{ct}\frac{\omega_{21}}{\omega_{22}}\frac{(\frac{s}{\omega_{21}}+1)(\tau_w k_1 s+1)}{(\frac{s}{\omega_{21}}+1)(\tau_w k_1\sqrt{\kappa}s+1)} \tag{72}$$

2.1 Example

Using the approximate expressions for $H_1(s)$, $H_2(s)$ and $G_f(s)$ the approximate time domain model can be built. The result is shown in Fig.13. This should be compared to Fig. 12. The spike in the beginning of the step is missing, although this can be included with Eqn. 71 and 72. However, to excite this a very sharp input signal is needed, and it has to the authors knowledge never been observed. Furthermore, there could be other effects that are not considered here, that can

have a greater effect. Therefore, it can be omitted for all practical purposes, since the model capture the effect of increased damping at high frequencies that is known.
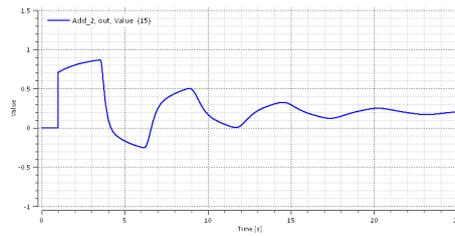

Figure 13 – Pressure response of a transmission line with visco-elastic walls

DISCUSSIONS

The model for hose dynamics is based on analytical models of assumed mechanisms and effects in a hose. There is no experimental results yet, and this is a topic for further research to validated the model. However, the general approach used to derive models. That is, to make models first in frequency domain and then use them to produce approximate models that can be inverse transformed analytically into the time domain is likely to be valid in any case. The same approach was earlier used to derive the model for a transmission line with distributed resistance. Since the damping of the walls in a hose is so dominant, it is usually sufficient to combine it with the simple model of distributed friction.

CONCLUSIONS

A simplified frequency dependent friction model was introduced in this paper. This is an analytical function that can be used instead of the expression with Bessel functions that represents the exact solution in the frequency domain. This can substantially speed up simulation in the frequency domain, which can be important for system optimization. Practical models for simulation of flexible hoses has also been derived, both for simulation in the frequency domain and also as an approximate model for simulation in the time domain. This model display higher damping of high frequencies that a pure pipe does, which is consistent with experience of real hoses.

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