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An Application of Lurie Problem in Hopfield Neural Networks

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Abstract: The goal of this work is to present an application of Lurie problem results in Hopfield Neural Networks, aiming its analysis in relation to stability. The Lurie problem appeared in 1947, because of an aircraft control problem, and currently it is considered solved for the case of a single control, e.g. Popov criterion. However, for multiple controls the Lurie problem is not solved completely, with respect to generalization of the problem and the extent of the results of cases of a single control for multiple controls. The Hopfield network originated in 1984 due to J. J. Hopfield, it is a nonlinear network and there is interest to engineering or physics. The same proved to be initially useful to recovery patterns, however currently have shown to be relevant to the biological research area and aroused theoretical interest. In our case, more specifically, one may consider the Hopfield network as a particular case of type Lurie systems. In this sense, we use in this work the methodology results derived from the study of absolute stability of type Lurie systems to nonlinear network analysis. Numerical simulations are also presented.

Keywords: Lurie Problem, Hopfield Neural Network, Absolute Stability

INTRODUCTION

Traveling over the years since 1940, starting an automatic control system aircraft until reaching applications in complex areas that are targets of several studies, such as the human brain, known the Lurie problem (Lurie and Postnikov, 1945) mathematician A.I. Lurie has made its history. With a significant collaboration in Mathematics and Engineering in subareas as a theory of stability and control as well as laying the foundation for a new and important area, Robust Control.

Many names have worked in this problem, and certainly left significant contributions. Initially, between the 50s and 60s, we can mention Aizerman (1947), with the well known Aizerman conjecture, and Krasovskii (1953), Popov (1961), and Kalman (1963). In a way the research above the Lurie problem took a bigger leap from 1980, when it began to appear works that linked the problem to other areas and with other approaches such as neural networks (Forti *et al*, 1994) and (Kaskurewicz and Bhaya, 1995); chaos and chaos synchronization (Liao and Yu, 2008); and convex approach (Gapski and Geromel, 1994).

For solution of the problem usually are used the famous Lyapunov functions. Like this the Lurie problem although is solved for the case of a single control, it does not have complete solution in the case of multiple controls. Thus, there remains a need for more general cases results to bring necessary and sufficient conditions for absolute stability.

The Hopfield network proposed by J. J. Hopfield (Hopfield, 1984) is part of the area known as Neurodynamics. There is an important relationship between the Lurie problem and the Hopfield neural networks (Liao and Yu, 2006). The Hopfield network has been applied considerably in several areas. Currently the network has awakened great interest by theoretical, biological (Monteiro, 2006) and other applications such as optimization problems (Forti *et al*, 1994) and (Kaskurewicz and Bhaya, 1995).

Thus, in the following sections of this article we will proceed with a re-presentation of Lurie problem and neural network Hopfield, seeking to bring in a simpler approach, however broad, its main concepts. Also we will present an application of a result of Lurie problem theory with multiple controls on the Hopfield network for stability analysis.

THE LURIE PROBLEM AND ABSOLUTE STABILITY

Anatoly Isakovich Lurie (July 19, 1901 - February 12, 1980) was born in Mogilev (Belorussia), he worked in the field of mechanics and control theory, leaving a great legacy. In 1944, Lurie was imbued to solve a problem of stability of the automatic control system of an aircraft he proposed (Lurie and Postnikov, 1945). Based on this system and the analysis of the stability of the null solution of the system (that is the equilibrium point), one could get a control, seeking necessary and sufficient conditions to global asymptotic stability of the system.

The fixed-wing aircrafts have their moving surfaces, that are: the elevators, ailerons, and rudder. The elevators are responsible for the rise of movements or descent of the aircraft (pitch), the ailerons for rolling (roll), and rudders for direction movements (yaw) of the aircraft, as presented in figure 1.

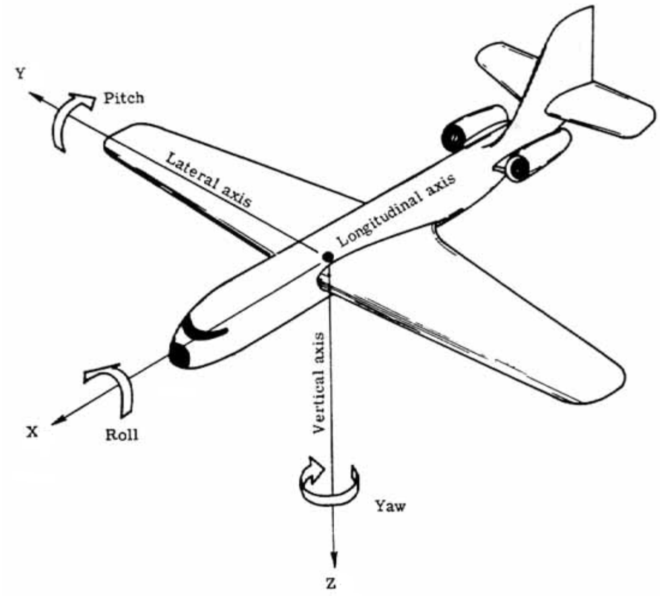

Figure 1 – The three axes of rotation of an aircraft

In control theory, the basic structure of a control system is generally based on the feedback, as presented in Figure 2 where $H_a(s)$ and $H_c(s)$ are linear systems represented by transfer functions, and $H_a(s)$ relating to the part to be controlled (the plant) and H_c represents the controller. A system transfer function is a mathematical model which is an operational method to express the ordinary differential equation that relates the output variable to the input variable (Ogata, 2003).

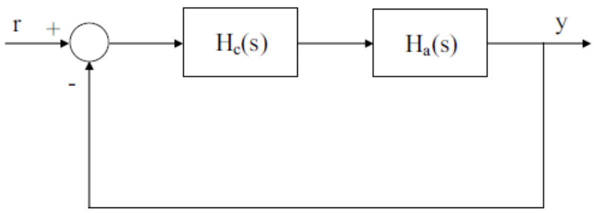

Figure 2 – Aircraft control system

Regarding the system of Figure 2, the role of the feedback control can be represented by the pilot to verify if the real route of the aircraft y matches the desired route r and then the rider himself, in the role H_c, operates in matche correcting the course of the aircraft through control surface, for example, the rudder.

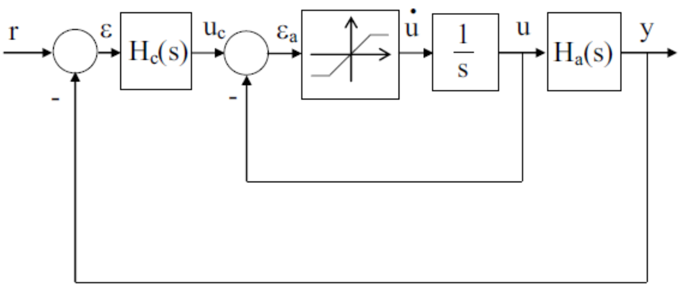

Figure 3 – Aircraft control system with rudder control

The feedback control can also be represented by the autopilot system of an aircraft, which has a input signal r that is the direction selected by the pilot, usually this input signal is related to frequency of a land station NDB, VOR [1] or even a radio station. This direction is constantly compared with the actual route of the aircraft y if there is any deviation, then control H_c runs the position correction, through of the acting on the control surface.

However, considering a situation nearest possible of the reality as shown in figure 3, it has been found that automatic control of the control surface may have a nonlinearity, for example, saturation of the actuator. As u represents the deflection of the rudder, the rudder control is intended to keep the deflection u_c determined by the controller H_c until it is fixed the deviation ε direction of the aircraft.

What Lurie did was separate the linear part of the nonlinear part of differential equations in state variables obtained in the process of mathematical modeling of the whole system or then from the transfer of functions $H_a(s)$ and $H_c(s)$ with nonlinearity (calculations can be checked in Rasvan (2001)).

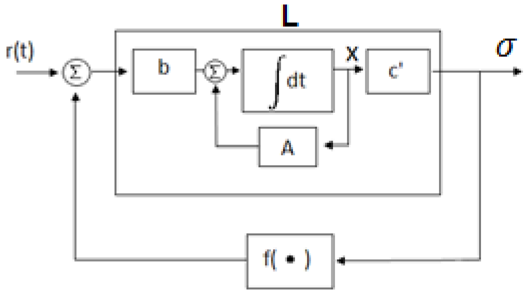

Figure 4 – Block diagram of the type Lurie system

We have in figure 4 a linear representation of the dynamics of the aircraft. Control $f(\sigma)$, in the feedback, represents a non-linearity in acting in one of the aircraft control surfaces (elevator, aileron or rudder). The nonlinearity $f(\sigma)$ is a continuous function restrict to third and fourth quadrants of the plane that can defined as:

$$F_{[0,k]} := \{f | f(0) = 0, 0 < \sigma f(\sigma) \leq k\sigma^2, \sigma \neq 0\},$$

$$F_{[0,k)} := \{f | f(0) = 0, 0 < \sigma f(\sigma) < k\sigma^2, \sigma \neq 0\},$$

$$F_{[k_1,k_2]} := \{f | f(0) = 0, k_1\sigma^2 \leq \sigma f(\sigma) \leq k_2\sigma^2, \sigma \neq 0\},$$

$$F_{\infty} := \{f | f(0) = 0, \sigma f(\sigma) > 0, \sigma \neq 0\}.$$

In Figure 5, we have some graphical representation of this kind of nonlinearity.

Considering $r(t) = 0$, the diagram in Figure 4 is expressed by the following differential equations system with output:

$$\left\{ \begin{array}{lllll} \dot{x} & = & Ax & + & bf(\sigma) \\ \sigma & = & c^T x & & \end{array} \right. \tag{1}$$

where $x \in R^n$, $b, c \in R^n$, $A \in R^{nxn}$ e $\sigma f(\sigma) > 0$.

Known as Lurie type systems, the Lurie problem can be summarized by the following question: **What are the necessary and sufficient conditions for the equilibrium point of the system (1) be globally asymptotically stable**. So Lurie defined absolute stability as the term used for global asymptotic stability of a Lurie type system.

Definition 1 *If the zero solution of the system (1) is globally asymptotically stable for $f(\sigma) \in F_{\infty}$, then we say that the zero solution of the system (1) is absolutely stable.*

Thus, considering the system (1) and the definition 1, we have the formulation for Lurie Problem with a single control (F_{∞}) and may also be formulated for $F_{[0,k]}$, $F_{[0,k)}$; $F_{[k_1,k_2]}$.

Finally, Lurie problem can be extended to the case of multiple controls (inputs), i.e. with m nonlinear controls:

$$\left\{ \begin{array}{lllll} \dot{x} & = & Ax & + & \sum_{j=1}^{m} b_j f_j(\sigma_j) \\ & & & & \\ \sigma_j & = & c_j^T x & = & \sum_{i=1}^{n} c_{ij} x_i, \qquad j = 1, \ldots, m, \end{array} \right. \tag{2}$$

[1]NDB or VOR are stations earth that send signals of radiofrequency of reference for aircraft navigation

where

$$A \in R^{nxn}, \quad x = (x_1, ..., x_n)^T, \quad b_j = (b_{1j}, ..., b_{nj})^T, \quad c_j = (c_{1j}, ..., c_{nj})^T,$$
$$f_j \in F_\infty := \{f : f(0) = 0, f(\sigma)\sigma > 0, \sigma \neq 0, f(\sigma) \in C[(-\infty, +\infty), R^1]\},$$
$$Re\lambda(A) \leq 0.$$

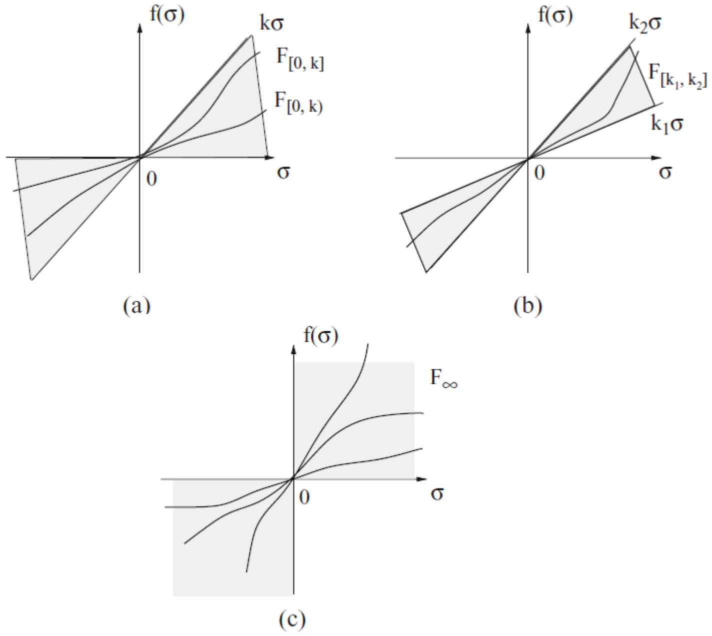

Figure 5 – (Liao and Yu, 2008) - Functions types f: a) $F_{[0,k]}$, $F_{[0,k)}$; b) $F_{[k_1,k_2]}$; c) F_∞

Therefore, with a nonlinear control $f(\sigma)$, many other practical problems can be placed into the form of the system (1) or system (2). Notice that f is not determined function, which part of its information can be obtained through experiments. This leads us to understand one of the reasons to say that the problem Lurie laid the foundations of the theory of Robust Control. On the other hand, we can also work with non-linear functions known to satisfy the characteristics here listed for f.

The difficulties of Lurie problem mainly happen because of the indeterminacy of the f, and also because the variables are not separated. To work around this problem we can make a transformation in (2), where we obtain new results for the case of multiple controls. This idea also applies to the case of a single control.

Without loss of generality we assume that $c_i = (c_{i1}, ..., c_{in})$ $(i = 1, 2, ..., m)$, are linearly independent, and we will consider the following transformation from (2):

$$y = G(g_{ij})x, \quad x, y \in R^n, G \in R^{nxn}$$

where

$$g_{ij} = \begin{cases} 1, & i = 1, ..., n-m, \\ c_j, & i = n-m+1, ..., n, \\ 0, & c.c. \end{cases}$$

In order to avoid confusion with indexes in function f if $m < n$, we take

$$\tilde{f}_{i+1} = f_i \quad i = 1, ..., n-1.$$

then, the system (2) can be transformed in the following:

$$\dot{y} = \tilde{A}y + \sum_{j=n-m+1}^{n} \tilde{b}_j \tilde{f}_j(y_j), \tag{3}$$

or:

$$\dot{y}_i = \sum_{j=1}^{n} \tilde{a}_{ij} y_j + \sum_{j=n-m+1}^{n} \tilde{b}_{ij} \tilde{f}_j(y_j). \tag{4}$$

HOPFIELD NEURAL NETWORK AND ITS RELATIONSHIP WITH LURIE PROBLEM

The Hopfield Neural Network (HNN) is a neural network model which was proposed by J. J. Hopfield in 1984 (Hopfield, 1984), which is part of an area known as Neurodynamics. This is the area that study Artificial Neural Networks (ANN) as nonlinear dynamical systems with an emphasis on stability problem. With the publication of Hopfield's article, research on ANN area got a jump where the Hopfield network was considered as an associative memory, with the main purpose to restore a pattern stored in response to the presentation of an incomplete or distorted version of this pattern. The HNN was applied in various areas, as presented in Braga *et al* (2007), and we can mention some applications: implementation of an identification systems of military target used in aircrafts model B-52; user authentication systems; oil exploration; prediction in the financial market; recognition of faces and autonomous navigation control ALVINN vehicles; currently have been used in optimization mainly when treat of its analysis in relation to absolute stability.

HNN can be represented by the following nonlinear differential equations

$$C_i \frac{du_i}{dt} = -\frac{u_i}{R_i} + \sum_{j=1}^{n} T_{ij} V_j + I_i, \qquad i = 1, 2, ..., n. \tag{5}$$

Where $u \in R^n$, $R \in R^n$, $T \in R^{nxn}$, $I \in R^n$, e $V_j = g(u_i)$, and $g : R \to [0,1]$, continuously differentiable and monotonically increasing, or $g_i'(u_i) > 0$.

The modeling process of the continuous model of HNN can be through an electric circuit where its realization can be observed by Figure 6a, (for more details see Pinheiro (2015)). First observe the model of artificial neuron RNH in Figure 6b. Resistance ρ_i in parallel with capacitor C_I form a low-pass filter, the application of this filter can be interpreted as a biological synapse, *i.e.* the output time constant of the ith biological neuron with ρ_i and C_I inherent amplifier g. The variable u_i represents an induced voltage at the input of a activation function $g(.)$ and v_i is the output voltage to be used for feedback. I_i is an external source of current applied which represents a bias, so to better adapt, by the neural network, the learning process. Basically modeling is made from the current flow in the circuit using the Kirchhoff law of current (see Figure 7). The current passing through C is given by

$$i = C_i \frac{du_i}{dt},$$

that is the way to add dynamics to the system (memory) that determines rate of variation of the input u_i.

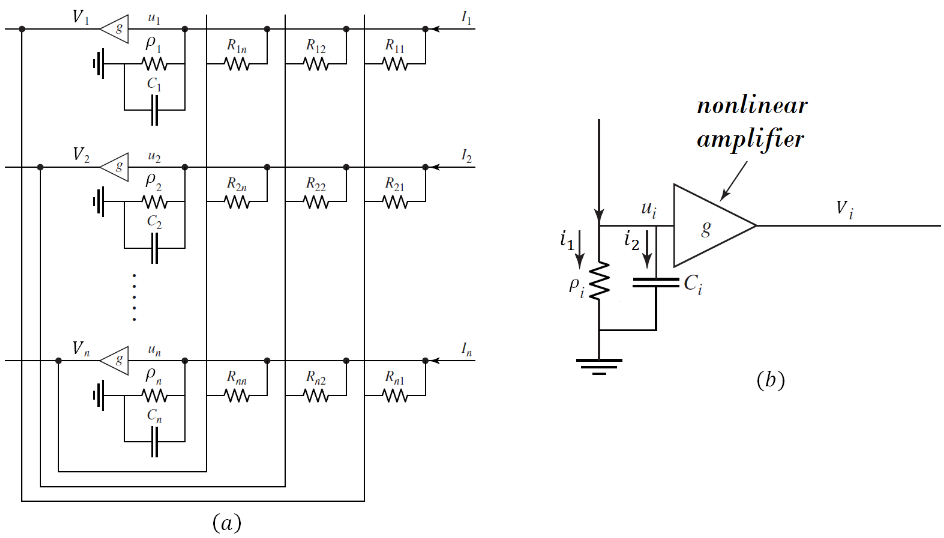

Figure 6 – (Zak, 2003) - (a) realization of the HNN (b) neuron of the HNN

We emphasize that R_i isn't the physical resistor in parallel with capacitor, but an equivalent resistance that make up the weights as can be seen in Figure 7. The values T_{ij} can be considered the synaptic weights, which is given by the inverse of resistance R_i, i.e. conductance, which receive the feedbacks representing the connection strength between the neurons. About V, we have which $V_i = g(u_i)$ and its inverse $g(V_i)^{-1} = u_i$ where $g : R \to [0,1]$ continuously differentiable and monotonically increasing, i.e., $g_i'(u_i) > 0$. The function g is called activation function, usually picks up as activation functions sigmoidal functions type. And observe this function g is compatible with Lurie problem nonlinear functions.

Now, based in Liao and Yu (2008), the HNN can be considered a special case of Lurie type systems with multiple controls. This fact opens doors for analysis using the theory of Lurie type systems, as well as the theory of absolute stability. Thus, considering the equation (5) after some changes of variables, proper consideration and handling, it can take the following form:

$$\frac{dx_i}{dt} = -d_i x_i + \sum_{j=1}^{n} b_{ij} f_j(x_j). \tag{6}$$

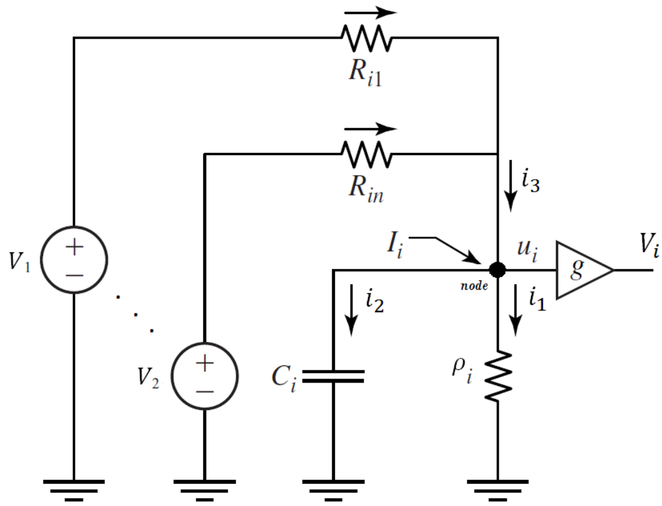

Figure 7 – (Zak (2003) - Representation of a node in realization circuit

In relation to the function f we have $f \in F_\infty$. Comparing the equation (6) with the equation (4)

$$\dot{y}_i = \sum_{j=1}^{n} \tilde{a}_{ij} y_j + \sum_{j=n-m+1}^{n} \tilde{b}_{ij} \tilde{f}_j(y_j).$$

We observe that the Hopfield neural network is a special case of a Lurie type system with multiple control, where $\tilde{a}_{ij} = 0$, $i \neq j$, $\tilde{a}_{ii} = d_i < 0$, e $m = n$.

Therefore, with this relationship between HNN and Lurie type systems the theory and methodology Lurie type systems can indeed promote the study of absolute stability of neural networks. So it makes sense to ask what are the necessary and sufficient conditions for a equilibrium point (6) to be globally asymptotically stable, that is, to be absolutely stable.

AN APLICATION OF THE LURIE PROBLEM TO HOPFIELD NEURAL NETWORK

We use a theorem due to Liao (1993), which is a result which provides a sufficient condition for absolute stability Lurie problem with multiple controls described by the equation (4). Then the same can be applied to the equation (6). For this, consider the following relationship:

$$\alpha_{ij} = \begin{cases} \tilde{a}_{ij}, & i = 1, \ldots n, \quad j = 1, \ldots, n-m, \\ \tilde{b}_{ij}, & i = 1, \ldots n, \quad j = n-m+1, \ldots, n. \end{cases}$$

Consider also the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Theorem 1 *The zero solution of the (4) or (6) is absolutely stable if the conditions of any of the following set are satisfied:*

1) $\tilde{a}_{ii} < 0$ for $i = 1, \ldots, n$, and the matrix $n_x n$ where $((-1)^{\delta_{ij}} |\tilde{a}_{ij}|)$ is Hurwitz.

2) There is a constant $k_l > 0$ for $l = n-m+1, \ldots, n$ such that

$$\begin{cases} k_l \tilde{b}_{ll} \leq \tilde{a}_{ll}, \quad k_l |\tilde{b}_{il}| < |\tilde{a}_{il}|, \quad para \quad l = n-m+1, \ldots, n, \quad i = 1, \ldots, n, \quad i \neq l, \\ \qquad\qquad or \\ k_l \tilde{b}_{ll} < \tilde{a}_{ll}, \quad k_l |\tilde{b}_{il}| \leq |\tilde{a}_{il}|, \quad para \quad l = n-m+1, \ldots, n, \quad i = 1, \ldots, n, \quad i \neq l, \end{cases}$$

or

1') $\tilde{a}_{ii} < 0$ for $i = 1, \ldots, n$, and the matrix $n_x n$ where $((-1)^{\delta_{ij}} |\alpha_{ij}|)$ is Hurwitz.

2') There is a constant $k_l > 0$ for $l = n-m+1, \ldots, n$ such that

$$\begin{cases} k_l\tilde{a}_{ll} \leq \tilde{b}_{ll}, \quad k_l|\tilde{a}_{il}| < |\tilde{b}_{il}|, \quad para \quad l = n-m+1,...,n, \quad i = 1,...,n, \quad i \neq l, \\ \qquad\qquad ou \\ k_l\tilde{a}_{ll} < \tilde{b}_{ll}, \quad k_l|\tilde{a}_{il}| \leq |\tilde{b}_{il}|, \quad for \quad l = n-m+1,...,n, \quad i = 1,...,n, \quad i \neq l, \end{cases}$$

Proof.: The condition 1) implies that there are constant $r_i > 0$, $i = 1,...,n$, such that

$$r_j\tilde{a}_{jj} + \sum_{i=1,i\neq j}^{n} r_i|\tilde{a}_{ij}| < 0, \quad j = 1,...,n.$$

The condition 1) implies that

$$r_l\tilde{b}_{ll} + \sum_{i=1,i\neq l}^{n} |r_i\tilde{b}_{il}| \leq \left[r_l\tilde{a}_{ll} + \sum_{i=1}^{n} r_i|\tilde{a}_{il}| \right] /k_l < 0, \quad l = n-m+1,...,n.$$

Constructing tha Lyapunov function

$$V = \sum_{i=1}^{n} r_i|y_i|,$$

and taking that D^+V is the Upper Dini Derivative (Khalil, 2002), then

$$D^+V(y)|_{(\ 4)} \leq \sum_{j=1}^{n} \left[r_j\tilde{a}_{jj} + \sum_{i=1,i\neq j}^{n} r_i|\tilde{a}_{ij}| \right] |y_j|$$

$$+ \sum_{l=n-m+1}^{n} \left[r_l\tilde{b}_{ll} + \sum_{i=1,i\neq l}^{n} r_i|\tilde{b}_{il}| \right] |\tilde{f}_l(y_l)| < 0 \quad para \quad y \neq 0,$$

Consequently, the zero solution (4) or (6) is absolutely stable. 1 ') and 2') can be proved analogously. ○

Example 1 *Let us consider the neural network with two neurons (Liao et al, 2007):*

$$\begin{cases} \dot{u}_1 &= -u_1 \quad -2g_1(u_1) + 1.5g_2(u_2) + I_1 \\ \dot{u}_2 &= -u_2 \quad +1.5g_1(u_1) - 2g_2(u_2) + I_2 \end{cases}$$

where $\quad g_i(u_i) = tanh(u_i), \quad R_i = C_i = 1, \quad I_i = 0, \quad i = 1,2, \quad T_{ii} = -2, \quad T_{12} = T_{21} = 1.5.$

$$\begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\alpha_{ij} = \begin{bmatrix} \tilde{b}_{11} & \tilde{b}_{12} \\ \tilde{b}_{21} & \tilde{b}_{22} \end{bmatrix} = \begin{bmatrix} -2 & 1.5 \\ 1.5 & -2 \end{bmatrix}.$$

In this example, it is easy to see that the null solution is equilibrium point. We will apply the theorem to see if this equilibrium point is absolutely stable. Applying the second set of conditions theorem, we have:

The condition 1' of the theorem is satisfied, because $\tilde{a}_{ii} < 0$ *and the matrix* $((-1)^{\delta_{ij}}|\alpha_{ij}|)$ *has eigenvalues equal to* -3.5 *and* -0.5 *so it is Hurwitz.*

In relation to condition 2', we have:

$$k\tilde{a}_{11} \leq \tilde{b}_{11} \rightarrow k_1(-1) \leq (-2), \quad k_1|\tilde{a}_{21}| < |\tilde{b}_{21}| \rightarrow k_1|0| < |1.5|, \quad para \quad l = 1, \quad i = 2;$$

$$k\tilde{a}_{22} \leq \tilde{b}_{22} \rightarrow k_2(-1) \leq (-2), \quad k_2|\tilde{a}_{12}| < |\tilde{b}_{12}| \rightarrow k_2|0| < |1.5|, \quad para \quad l = 2, \quad i = 1;$$

taking $k_1 = k_2 = 2$ *the condition 2' is satisfied, holding the equilibrium point is absolutely stable.* ○

The following is a simulation for example 1 using Matlab/Simulink. The figure 8 represents the Simulink diagram. In Figure 9, we have the time response of the states u_1 and u_2 with initial conditions $(u_1^0, u_2^0) = (5, -5)$ and $(u_1^1, u_2^1) = (-2, 3)$ respectively. The Figure shows that the solutions converge for zero solution. Figure 10 shows the phase portrait for another initial states.

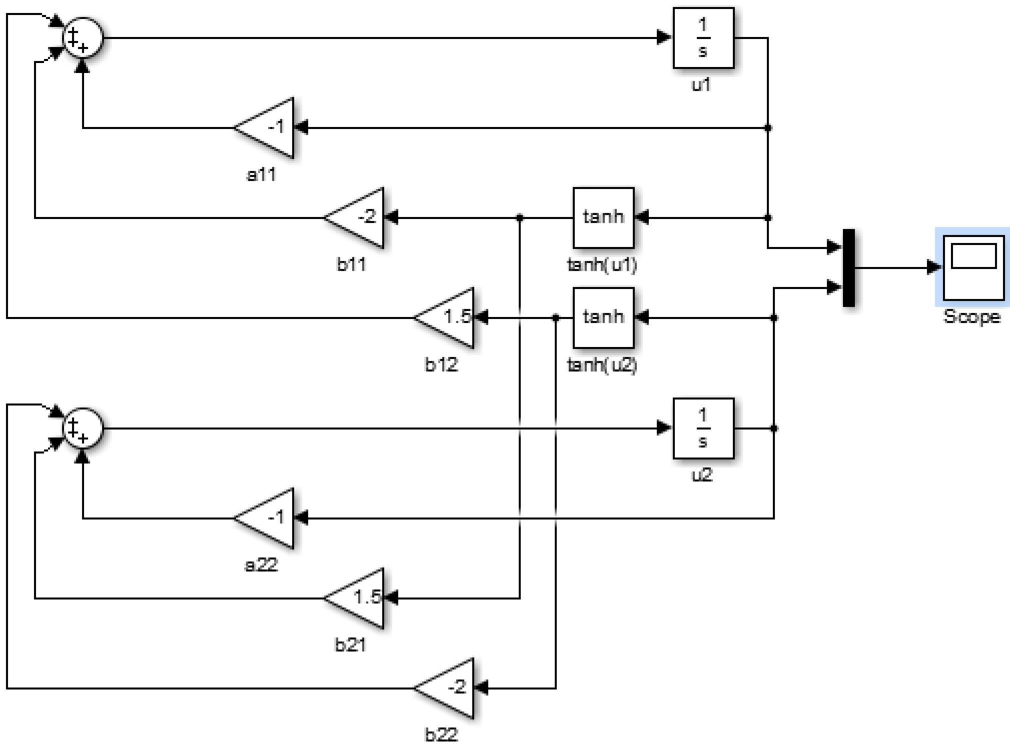

Figure 8 – Simulink diagram

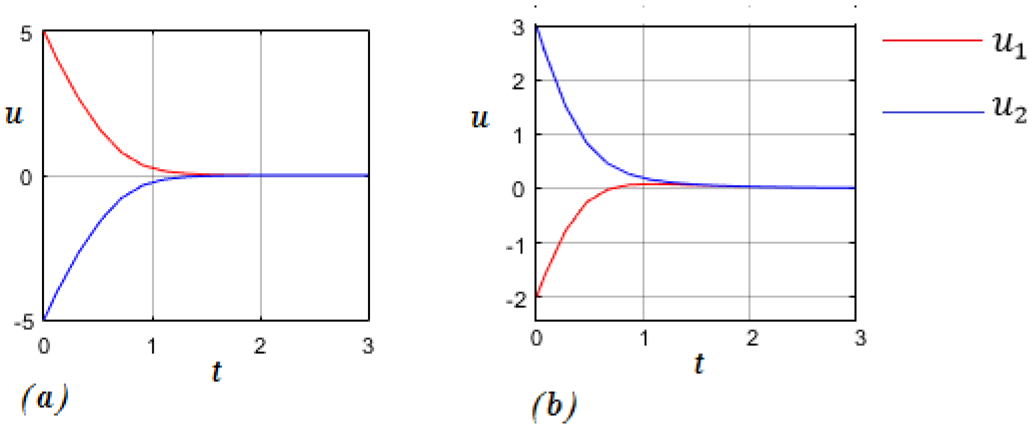

Figure 9 – Temporal response

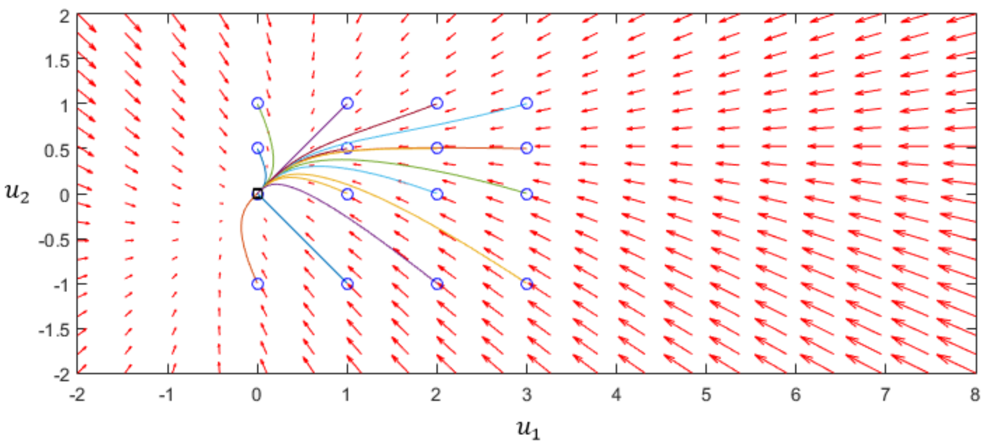

Figure 10 – Phase portrait

CONCLUSION

We conducted a presentation of Lurie problem in a simple and detailed language in relation to that normally found in the literature. An important fact is that although Lurie problem is considered solved for a single control, it remains open to multiple controls.

We verified that the Hopfield network is a particular case of a type Lurie system with multiple control. What presented leads to the theorem, where it became clear that it is feasible to use Lurie problem theory in neural networks Hopfield.

The stability analysis performed by Hopfield (Hopfield, 1984), he proves that, given certain conditions, there are attractors points in the network. However not necessarily known these points preliminarily. So the network may have a locator feature of equilibrium points (Liao and Yu, 2008). However when we look at the absolute stability of Hopfield network, we have at hand the possibility of discovering for a given equilibrium point, not only if it is stable, but mostly it is unique.

This was the case of the example shown in the application, as we had available a known equilibrium point, the null solution, and we did not know if there were other equilibrium points. It is important to emphasize, because isn't always an easy task to find equilibrium points. Thus, with the application of the theorem we see that the null solution is absolutely stable, or asymptotically stable (globally asymptotically stable).

Finally, we have observed in the literature that Lurie problem has provided tools to several problems. We can cite optimization problems, or control problems because their results may provide parameter settings for control. And another side we have another situation that is also open: is the search when there is no absolute stability for Lurie type system, so we will have questions such as: are there periodic solutions, are there limits cycles, can the system present chaos? what are the conditions for such situations?

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