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Optimization of Fundamental Frequency with Bidirectional Evolutionary Structural Method of Continuum Structures

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Abstract: A hierarchical structure is a structure that can be described by different characteristic lengths, and is such that its layout in the smaller scale (microscale) affects its behavior in the bigger scale (macroscale). Each hierarchical level is treated as a continuous medium composed of one or more materials. The simultaneous design of multiphase composite structures aims at finding the optimal distribution of materials such that one or more structural parameter is maximized (or minimized). In this work, the Bi-directional Evolutionary Structural Optimization method, BESO hereinafter, is applied to the maximization of the fundamental frequency of a structure subjected to a constraint on the amount of materials used. Numerical experiments are made in order to validate the implementation and to confirm the efficacy of the method in optimizing the topology of the structure.

Keywords: topology optimization, hierarchical structure, BESO, homogenization

INTRODUCTION

The definition of hierarchical structures is related to structures that can be described with respect to different characteristics lengths, and where a structural element in a certain scale is composed by periodic substructures in a smaller scale. Each hierarchical level may be treated as a continuous medium made of a homogenized material with a certain microstructure.

The simultaneous design with multiple phases of material and composite structures is limited. The design of composite materials aims at finding the distribution of the base material in the material's microstructure while the topology optimization techniques is applied in order to reach desired material properties. For instance, on the design of composite structures, Huang and Xie (2009) use the BESO method to solve the problem of minimum compliance of structures with multiple materials, while Huang *et al*. (2010) study the problem of maximizing the fundamental frequency of multimaterial structures.

Zuo *et al*. (2013) highlight that fundamental frequency optimization is important in engineering, given that it is desirable to modify the natural frequencies spectrum distribution so as to avoid any resonance situation in a larger interval of excitation frequencies. As the natural frequency is associated to the performance of the macrostructure, microscale optimization is seldom studied.

In this context, the objective of this work is to develop an approach of simultaneous design, optimizing the fundamental frequency of the system through the BESO method. The design variables describe the distribution of material in both macro and micro levels, where the properties of both materials in the macro-level are composed via homogenization of microstructures composed of isotropic materials.

THEORETICAL BACKGROUND

BESO method for fundamental frequency optimization

The modal behavior of an undamped system can be analyzed by means of equation (1)

$$\left(\mathbf{K}-\omega^2\mathbf{M}\right)\mathbf{u}=\mathbf{0} \tag{1}$$

The k-th natural frequency ω_k and the corresponding mode associated to the displacement vector $\boldsymbol{u}_k$ are related through Rayleigh's quotient as given by (2):

$$\omega_k^2 = \frac{\mathbf{u}_k^T \mathbf{K} \mathbf{u}_k}{\mathbf{u}_k^T \mathbf{M} \mathbf{u}_k} \tag{2}$$

Consider the objective of maximizing the k-th natural frequency of a structure described by a discrete mesh using a predetermined amount of material. To every element in the meshed domain either a 0 or 1 value is assigned indicating the presence or absence of material, respectively. In this context, the problem may be presented as in (3) and (4) according to Huang and Xie (2010):

Maximize $\quad f(\mathbf{x})=\omega_k \qquad (3)$

Subject to: $\quad V^* - \sum_{i=1}^{N} x_i V_i = 0 \quad (x_i = x_{min} \text{ or } 1) \qquad (4)$

where V^* is the prescribed volume, meaning the fraction of the domain occupied by the structure. N is the number of elements in the domain. The binary design variable x_i indicates if the structure occupies the i-th element, and the small value x_{min} (e.g. 10^{-6}) corresponds to a region in the domain with no material.

Material properties interpolation scheme

For the solid-void design the material's density and Young modulus are functions of the design variable. Given that with the SIMP (Solid Isotropic Material with Penalization) method the ratio between elementary mass and stiffness is extremely high for small x_i values (e.g. after applying power law penalization to the stiffness and linear interpolation to the density), artificial localized modes appear in regions with low density. Huang and Xie (2010) proposed an alternative interpolation scheme where the mass/stiffness ratio is kept constant, as given by equation (5) and (6):

$$\rho(x_{min}) = x_{min}\, \rho^1 \tag{5}$$

$$E(x_{min}) = x_{min}\, E^1 \tag{6}$$

A better interpolation scheme takes the explicit form shown in Eq. (7), according to Huang and Xie (2010):

$$\rho(x_i) = x_i\, \rho^1 \tag{7a}$$

$$E(x_i) = \left[\frac{x_{min} - x_{min}^p}{1 - x_{min}^p} \left(1 - x_i^p\right) + x_i^p \right] E^1 \tag{7b}$$

where p is called the penalty exponent and x_i is 1 if the i-th element is composed of the corresponding material, 0 if otherwise. Further details may be found in Huang and Xie (2010).

Frequency optimization in hierarchical structures

The objective function for dynamic problems in multiscale structures can be written through Eqs. (8) to (13), according to Zuo *et al*. (2013):

Find: $\quad \mathbf{x} = \{\mathbf{x}^{mac}, \mathbf{x}^{mic,1}, \mathbf{x}^{mic,2}\} \qquad (x_i^{mac}, x_i^{mic,1}, x_i^{mic,2}: 0 \text{ or } 1)$

(8)

Maximize: $\quad f(\mathbf{x})=\omega_k \qquad (9)$

Subject to: $\quad \left(\mathbf{K} - \omega_k^2 \mathbf{M}\right)\mathbf{u}_k = \mathbf{0} \qquad (10)$

$$\sum_{i=1}^{M} x_i^{mac} V_i^{mac} = V^{mac} \tag{11}$$

$$\sum_{j=1}^{N} x_j^{mic,1} V_j^{mic,1} = V^{mic,1} \tag{12}$$

$$\sum_{j=1}^{N} x_j^{mic,2} V_j^{mic,2} = V^{mic,2} \tag{13}$$

where *V* is the volume, the superscripts *mac* represent the macromodel and *mic* represent the micromodel of the structure, respectively. The subscripts *i* and *j* correspond to the *i*-th and *j*-th element of the macromodel and the micromodel, respectively.

In this problem the vector $\mathbf{x}$ is composed by $\mathbf{x}^{mac}$, which describes the macromodel layout, and $\mathbf{x}^{mic,1}$ and $\mathbf{x}^{mic,2}$, which describe the micromodels corresponding to each phase present in the macromodel. This means that, $\mathbf{x}^{mic,1}$ describes the micromodel for the elements $x_i^{mac} = 0$, and $\mathbf{x}^{mic,2}$ the micromodel associated with the phase $x_i^{mac} = 1$.

For both macro and micromodels the design variable takes binary values corresponding to the presence or absence of a certain phase. As the design variables in the two micromodels vary over the same domain, from now on they will be called only $\mathbf{x}^{mic}$, requiring a distinction between $\mathbf{x}^{mic,1}$ or $\mathbf{x}^{mic,2}$ when necessary.

ω_k represents the *k*-th natural frequency associated with the structure. Equation (11) describes the volume constraint on the macromodel which controls the material distribution on the macroscale. V^{mac} correspond to the volume fraction of the predefined macro phase. Similarly, Eqs. (12) e (13) describe volume constraints on the microscale, where $V_j^{mic,1}$ is the volume of the *j*-th element in the first micromodel and $V^{mic,1}$ is the prescribed fraction of volume of the first microstructure in the micromodel; $V_j^{mic,2}$ and $V^{mic,2}$ have analogous meanings but for the second micromodel.

Sensitivity analysis

The element mass matrix $\mathbf{m}$ and the element stiffness matrix $\mathbf{k}$ of the macromodel and the material constitutive matrix $\mathbf{D}$ of the micromodel are defined by Eqs. (14), (15) and (16), Zuo *et al*. (2013), being necessary components for the sensitivity analysis. The penalized relations for the stiffness and constitutive matrix come from the SIMP model and are valid both for macro and micro levels of the structure,

$$\mathbf{m}\left(x_i^{mac}\right) = x_i^{mac}\mathbf{m}_i^1 + \left[1 - x_i^{mac}\right]\mathbf{m}_i^2 \tag{14}$$

$$\mathbf{k}\left(x_i^{mac}\right) = \left(x_i^{mac}\right)^p \mathbf{k}_i^1 + \left[1 - \left(x_i^{mac}\right)^p\right]\mathbf{k}_i^2 \tag{15}$$

$$\mathbf{D}\left(x_j^{mic}\right) = \left(x_j^{mic}\right)^p \mathbf{D}_j^1 + \left[1 - \left(x_j^{mic}\right)^p\right]\mathbf{D}_j^2 \tag{16}$$

The derivatives for the global mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$, and for the material constitutive matrix $\mathbf{D}$ of the micromodel are obtained through Eqs. (17), (18) and (19), respectively, Zuo *et al*. (2013):

$$\frac{\partial \mathbf{M}}{\partial x_i^{mac}} = \mathbf{m}_i^1 - \mathbf{m}_i^2 \tag{17}$$

$$\frac{\partial \mathbf{K}}{\partial x_i^{mac}} = p\left(x_i^{mac}\right)^{p-1}\left(\mathbf{k}_i^1 - \mathbf{k}_i^2\right) \tag{18}$$

$$\frac{\partial \mathbf{D}}{\partial x_j^{mic}} = p\left(x_j^{mic}\right)^{p-1}\left(\mathbf{D}_j^1 - \mathbf{D}_j^2\right) \tag{19}$$

Macroscale sensitivity analysis

Deriving the *k*-th natural frequency from Rayleigh's quotient (2) with respect to the *i*-th design variable in the macrolevel, normalizing the eigenvectors with respect to the mass matrix, and with help of the eigenvalues equation plus the interpolation equations, Eqs. (17) e (18), the sensitivity of the *k*-th natural frequency for the macromodel is obtained as Eq. (20), Zuo *et al*. (2013):

$$\alpha_i^{mac} = \frac{\partial \omega_k}{\partial x_i^{mac}} = \frac{1}{2\omega_k}\mathbf{u}_k^T\left[p\left(x_i^{mac}\right)^{p-1}\left(\mathbf{k}_i^1 - \mathbf{k}_i^2\right) - \omega_k^2\left(\mathbf{m}_i^1 - \mathbf{m}_i^2\right)\right]\mathbf{u}_k \tag{20}$$

Microscale sensitivity analysis

The micromodel describes the microstructure of the homogenized material which is taken into account to calculate the macromodel effective material properties. The homogenized matrix $\mathbf{D}^H$ is calculated according to Eq. (21), over the domain of the base cell described by the variable $\mathbf{x}^{mic}$. The procedure used to calculate $\mathbf{D}^H$ is explained in a series of papers by Hassani and Hinton (1998a, 1998b and 1998c), and a detailed computational implementation may be found, for instance, in Andreassen and Andreasen (2014).

$$\mathbf{D}^H = \frac{1}{|Y|}\int_Y \mathbf{D}\left(\mathbf{I}-\mathbf{bu}\right)dY \tag{21}$$

where,

- Y : material base cell domain
- $\mathbf{D}$: constitutive matrix of the material in the microstructure
- $\mathbf{I}$: identity matrix
- $\mathbf{b}$: strain matrix in the micromodel
- $\mathbf{u}$: displacement field

The stiffness matrix $\mathbf{k}_i$ may be calculated according to Eq. (22), assuming a 2D domain, by imposition of a periodic boundary condition, where the displacement fields are chosen so that the strain is uniform $[1,0,0]^T$, $[0,1,0]^T$ and $[0,0,1]^T$:

$$\mathbf{k}_i = \int_{V_i} \mathbf{B}^T \mathbf{D}^H \mathbf{B}\; dV_i \tag{22}$$

where,

- $\mathbf{B}$: strain matrix
- V_i : i-th element volume

Differentiating the k-th natural frequency from Eq. (2) with respect to the j-th variable of the micromodel, Eq. (23) is obtained:

$$\frac{\partial \omega_k}{\partial x_j^{mic}} = \frac{1}{2\omega_k}\left[\mathbf{u}_k^T\left(\frac{\partial \mathbf{K}}{\partial x_j^{mic}} - \omega_k^2\frac{\partial \mathbf{M}}{\partial x_j^{mic}}\right)\mathbf{u}_k\right] \tag{23}$$

As in a finite element analysis the global stiffness matrix **K** is composed from the element stiffness matrices, and the global mass matrix **M** is composed from the element mass matrices, the sensitivity of the frequency with respect to the microstructure design variable is developed by Zuo, *et al*. (2013), as given by Eq. (24):

$$\alpha_j^{mic} = \frac{\partial \omega_k}{\partial x_j^{mic}} = \frac{1}{2\omega_k}\sum_{i=1}^{M}\mathbf{u}_{k,i}^T\int_{V_i}\mathbf{B}^T\frac{\partial \mathbf{D}^H}{\partial x_j^{mic}}\mathbf{B}\,dV_i\,\mathbf{u}_{k,i} \tag{24}$$

The sensitivity of the homogenized matrix can be obtained from Eqs. (19) and (21), obtaining Eq. (25):

$$\begin{aligned}\frac{\partial \mathbf{D}^H}{\partial x_j^{mic}} &= \frac{1}{|Y|}\int_Y\left(\mathbf{I}-\mathbf{bu}\right)^T\frac{\partial \mathbf{D}}{\partial x_j^{mic}}\left(\mathbf{I}-\mathbf{bu}\right)dY \\ &= \frac{p\left(x_j^{mic}\right)^{p-1}}{|Y|}\int_Y\left(\mathbf{I}-\mathbf{bu}\right)^T\left(\mathbf{D}_j^1-\mathbf{D}_j^2\right)\left(\mathbf{I}-\mathbf{bu}\right)dY\end{aligned} \tag{25}$$

Thus, the sensitivity of the fundamental frequency with respect to the microstructure design variables is found substituting Eq. (24) in Eq. (25), producing Eq. (26):

$$\alpha_j^{mic} = \frac{p\left(x_j^{mic}\right)^{p-1}}{2\omega_k|Y|}\sum_{i=1}^{M}\mathbf{u}_{k,i}^T\left\{\int_{V_i}\mathbf{B}^T\left[\int_Y\left(\mathbf{I}-\mathbf{bu}\right)^T\left(\mathbf{D}^1-\mathbf{D}^2\right)\left(\mathbf{I}-\mathbf{bu}\right)dY\right]\mathbf{B}\,dV_i\right\}\mathbf{u}_{k,i}$$
(26)

In order to minimize the mesh dependency and checkerboard patterns in the final result, a filtering technique is applied. This in effect smooths the sensitivity values in time, helping convergence. The averaged sensitivity with respect to time is given in Eq. (27), where q is the iteration numerical index:

$$\tilde{\alpha} = \frac{1}{2}\left(\alpha_i^q + \alpha_i^{q-1}\right) \tag{27}$$

Convergence criteria

The evolution ratio ER is an algorithm parameter, and its importance resides in controlling the volume variation between iterations, given in Eq. (28) for the macromodel and according to Eq. (29) for the micromodel:

$$V^{mac,q} = V^{mac,q-1}\left(1 \pm ER^{mac}\right) \tag{28}$$

$$V^{mic,q} = V^{mic,q-1}\left(1 \pm ER^{mic}\right) \tag{29}$$

where $V^{mac,q}$ is the value of the volume in the q-th iteration, while $V^{mac,q-1}$ is defined in the previous iteration (q-1). $V^{mic,q}$ and $V^{mic,q-1}$ are defined in the same way.

Once the constraint volume is reached the material addition/removal process continues until the variation in the objective function for consecutive iterations reaches a certain threshold value. The considered variation of ω_k takes into account the last N iterations, shown in Eq. (30):

$$\tau = \frac{\sum_{i=1}^{N}\left(\omega_k^{q-i+1} - \omega_k^{q-N-i+1}\right)}{\sum_{i=1}^{N}\omega_k^{q-i+1}} \le \tau^* \tag{30}$$

where,

τ : objective function variation

τ^* : tolerance

N : predefined number of iterations

The flowchart in Fig. 1 shows the implementation of the BESO algorithm.

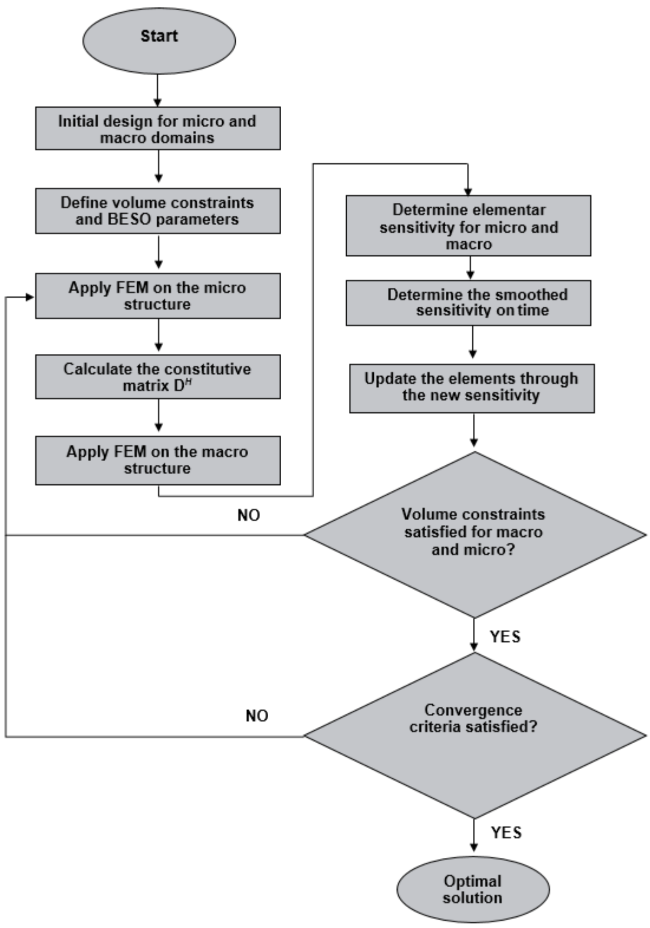

Figure 1 – General scheme of the BESO method applied.

RESULTS AND DISCUSSION

The program was tested in order to evaluate the accuracy of the implementation and to mitigate numerical instabilities. The four-node quadrilateral for plane strain was chosen for the finite element model, and results were compared through reference problems found in the literature.

The Problem 1 is a two-material beam clamped at both ends with a lumped mass $M = 1.4\times10^{-5}$ kg/cm³ on its center. The dimensions are 0.14 m × 0.02 m, and the domain is discretized with 280 × 40 elements. The objective is to maximize the fundamental frequency where each material occupies half of the domain. Table 1 shows properties of both materials.

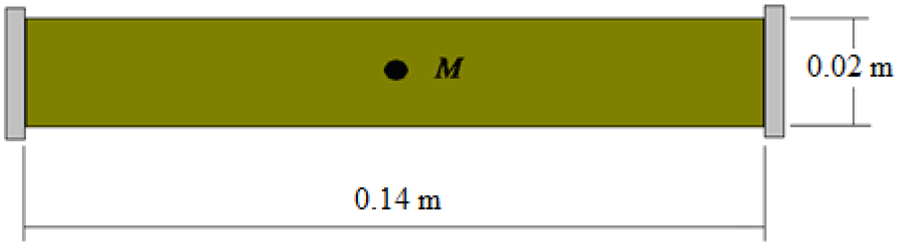

Figure 2. Problem 1, two-material beam with lumped mass and clamped at both ends.

Table 1 – Material properties for Problem 1.

Properties	Material 1	Material 2
Young Modulus (E)	100 N/cm²	20 N/cm²
Poisson ratio (ν)	0.3	0.3
Mass density (ρ)	10^{-6} kg/cm³	10^{-7} kg/cm³

The input data for the BESO method are the evolutionary rate ER = 2%, the maximum addition ratio AR_{max} = 2%, τ=0.1%, V^{mac} = 50%, $V^{mic,1}$ = 100% and $V^{mic,2}$ = 100%, filtering radius r_{min} = 0.0015 m and penalty factor p = 3. As shown in Fig. 3(a,b), the results of the BESO optimization were compared to those obtained by Huang *et al*. (2010). In both figures, the stiff material is shown in darker colors while the softer material has a lighter tone. Table 2 compares the fundamental frequencies.

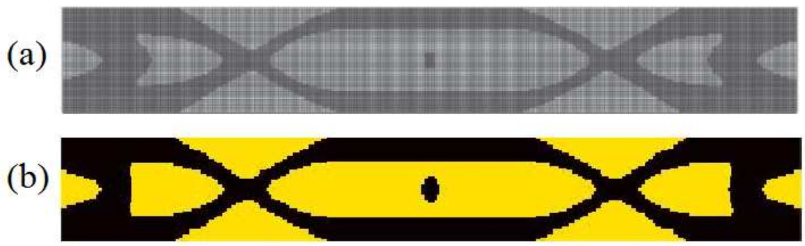

Figure 3. Problem 1, results obtained (a) by Huang *et al*. (2010) and (b) this work.

Table 2 – Natural frequencies for Problem 1 (rad/s).

Author	ω_1	ω_2
Huang *et al*. (2010)	37.1	-
This work	38.2	115

The Problem 2 considered the same beam shown in Fig. 2, but this time taking the softer material density and stiffness close to zero to emulate voids. The domain is discretized with 280 ×40 square elements with unitary sides, similar to the Problem 1. The beam is composed of the material whose properties are shown in Table 3. The input data is: evolutionary rate ER = 2%, addition ratio AR_{max} = 2%, τ = 0.01%, V^{mac} = 50%, $V^{mic,1}$ = 100%, $V^{mic,2}$ = 0%, filtering radius r_{min} = 0.0015 m and penalty factor p = 3.

Table 3 – Material properties for Problem 2.

Properties	Material
Young Modulus (E)	10^6 N/m²
Poisson ratio (ν)	0.3
Mass density (ρ)	1 kg/m³

Figure 4(a) shows the result found by Huang *et al*. (2010), where material is colored dark gray and void is white. Fig. 4(b) shows the result obtained in this work, where the coloration scheme is similar, with gray replaced by black. Table 4 compares the obtained frequencies.

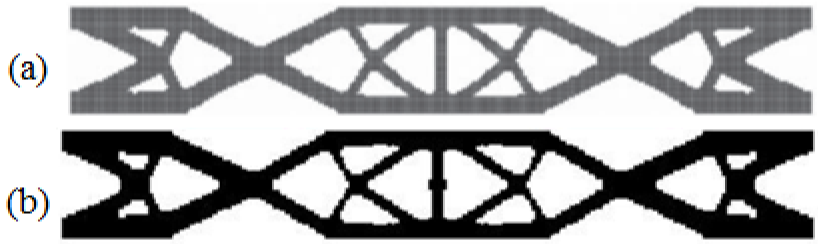

Figure 4. Problem 2, results obtained (a) by Huang *et al*. (2010) and (b) this work.

Table 4 – Natural frequencies for case 2 (rad/s).

Author	ω_1	ω_2	iterations
Huang *et al*. (2010)	33.7	-	55
This work	34.7	104.3	47

The Problem 3 solved, illustrated in Fig. 5, is a two-material beam clamped at both ends and elements of unitary dimension, the number of elements is 80 × 40. The objective is to maximize the fundamental frequency where each material occupies half of the domain. Table 5 shows properties of both materials.

Table 5 – Material properties for Problem 3.

Properties	**Material 1**	**Material 2**
Young Modulus (E)	1 N/m²	0.2 N/m²
Poisson ratio (ν)	0.3	0.3
Mass density (ρ)	1 kg/m³	2 kg/m³

The input data is: evolutionary rate ER = 1%, addition ratio AR_{max} = 1%, τ = 0.01%, V^{mac} = 50%, $V^{mic,1}$ = 100%, $V^{mic,2}$ = 100%, filtering radius r_{min} = 0.0015 and penalty factor p = 3.

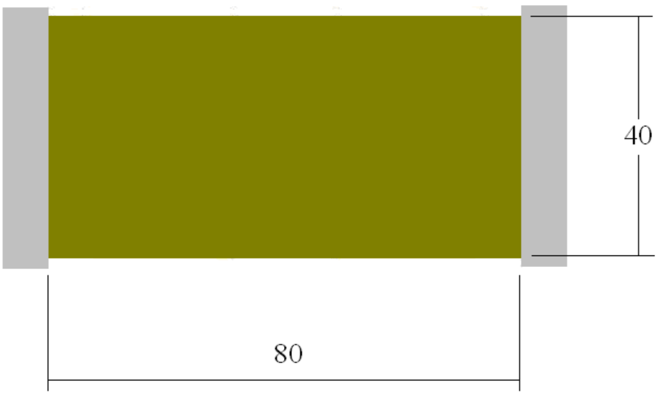

Figure 5. Problem 3, two-material beam clamped at both ends.

Figure 6(a) shows the result found by Zuo *et al*. (2013), where the stiff material is shown in darker color while the softer material has a lighter tone. Fig. 6(b) shows the result obtained in this work, where the coloration scheme is similar, the stiff material is shown in darker color while the softer material has a lighter tone. Table 6 compares the obtained frequencies.

Table 6 – Natural frequencies for Problem 3 (rad/s).

Author	ω_1	ω_2
Zuo *et al*. (2013)	0.011317	-
This work	0.012246	0.019632

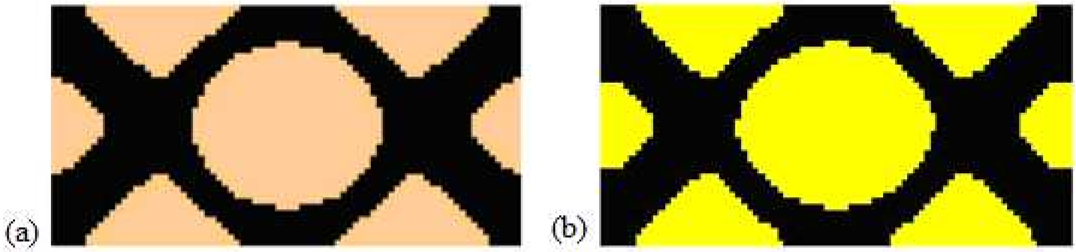

Figure 6. Problem 3, results obtained (a) by Zuo *et al*. (2010) and (b) this work.

CONCLUSION

The comparison shows that the BESO algorithm implemented in this work was capable of optimizing the fundamental natural frequency on different structures. The built-in algorithm make use of an own FEM module, thus being independent of commercial FEM solvers.

The averaged sensitivity used in this work enabled to optimize the fundamental natural frequency closed to most elaborated proposals of the literature, *e.g.* the fundamental natural frequency obtained in this work was higher in 2.96%, 2.97% and 8.21% in Problems 1, 2 and 3, if compared with values obtained by Zuo *et al*. (2013).

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