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Dynamical Systems Identification with Smooth Decomposition

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Abstract: Smooth Decomposition (SD) is a multivariate data or statistical analysis method to find normal modes and natural frequencies in an spatial data field. The projection used for this method is made such as it keeps the maximum variance possible for the displacement vector and also as it keeps the smoothest motions along time. From this method we can get the "energy" participation in the response of each normal mode during the simulation or the experimental test which can be a relevant information to validate results concerning the identification process. This method of identification can be used for linear and nonlinear systems and uses only output data given that the excitation satisfies some properties normally met by a well chosen random excitation, as a white noise, for example. The objective of this method is to identify systems from their displacement field under ambient excitation which, in many cases, can be hard to compute or to describe. As the method is only based on the covariance matrices of the displacement field and the corresponding velocity field, it is needed no further considerations and approximations. In this point the method is a great tool for modal analysis and system identification. In this paper, the presentation of the method is firstly done which showing how we can interpret the results of SD for different systems and then the application of SD on simulated multi-DoF damped and undamped systems is performed and discussed to understand how SD can be a great tool for modal analysis. A discussion about the quality of the excitation is also performed.

Keywords: Smooth Decomposition (SD), System Identification, Operational Modal Analysis (OMA), Nonlinear Parameters Identification

INTRODUCTION

The Smooth Decomposition (SD) is a statistical analysis technique for finding structures in an ensemble of spatially distributed data such that the vector displacement not only keeps the maximum possible variance but also the motion, as the velocity field, is as smooth in time as possible. Closely related with the SD are the dual smooth modes used in the framework of oblique projection to expand a random response of a system. The concept of dual mode with the associated decomposition defines a tool that transforms the SD in an efficient modal analysis tool. This method of identification can be used for linear and nonlinear systems and uses only output data as soon as the excitation satisfies some properties normally met by a well chosen random excitation, as a white noise, for example.

The main properties of the SD are discussed and some optimality characteristics of the expansion are deduced. The parameters of the SD (using the dual smooth modes and the smooth values) give access to a modal parameters of a linear system in terms of mode shapes, resonance frequencies and modal energy participations. This part is a remarkable improvement with respect to the standard modal analysis methods. This novel modal analysis of a linear system is illustrated by examples.

One of the examples, to show the main features of the method, is a simple multi-DoF undamped system subject to a random excitation that is identified from the output signal. Then, more complex examples of a multi-DoF system are identified. A discussion concerning the difficulty to identify systems with high damping coefficient is made. We also study a case which can be a first step before considering continuous systems with the partially observed case. Finally we will discuss about the importance of the excitation quality for such a method.

It is interesting to say that this is a new method, not yet compared with the methods known in the literature as Operational Modal Analysis (OMA). So far the only association between SD and OMA is the fact that both methods use output signals for the identification and they require random excitation. However the theories are different. SD is a type of Karhunen-Loève Decomposition, using correlations and projections in the modes whereas OMA uses the controllability matrix and correlations of the measured signals that are not necessarily the state of the system.

DESCRIPTION OF THE SMOOTH DECOMPOSITION METHOD

First we will present the basis of this method (firstly presented by Zhou (2006) and also by Chelidze and Zhou (2006)) and its main objective. There is already another well known method called the "Karhunen-Loève Decomposition (KLD)" or the "Proper Orthogonal Decomposition (POD)" used to analyze random data. This method is not presented in this article. The main objective of KLD or POD consists in finding the basis that will be the best representation of the initial field.

KLD (or POD) and SD are based on the projection of the data field such as the displacement vector has the maximum variance in order to be sure that all the modes we are looking for are excited. Indeed, bigger will be the variance of the displacement vector, higher will be the probability of a mode being excited. The SD method is a bit different because we also consider the derivative of this displacement vector, the velocity field. The objective is to find the basis that gives the maximum variance for the displacement vector and the minimum variance for the velocity vector (in order to keep the displacement as smooth as possible in time).

Decomposition Principle

First, let us describe the data field used in this method. We consider the sampled scalar field $\mathbf{X}(t)$ formed of random values (in the matrix form) as a function of the time t ($t \in \mathbb{R}$). This field can be described as:

$$\mathbf{X}(t) \in \mathbb{R}^{n\times m}, \tag{1}$$

where n represents the different instants and m represents the spacial points where we measure the information. Each measurement for each spacial position is done with a time interval Δt. For experimental measurements, Δt is directly linked to the acquisition frequency of the measuring system. Now, we have to define some properties of the data field $\mathbf{X}(t)$. It has to be a second order and stationary ergodic process with zero mean value and also has to admit a time derivative. If the data field has not a zero mean value we have to remove the mean value from the entire field to get a zero mean valued process.

The central point of this method is to find a linear projection such as:

$$\boldsymbol{q} = \mathbf{X}(t)\boldsymbol{\phi}, \tag{2}$$

where $\boldsymbol{q} \in \mathbb{R}^{n\times 1}$ (representing the modal coordinates) and $\boldsymbol{\phi} \in \mathbb{R}^{m\times 1}$ (representing a projection vector). Note that this projection is made in only one mode. Now, the objective of this method is to find this projection such as it keeps the maximum variance for the original field $\mathbf{X}(t)$ (the displacement field) and its smoothest variation in time. Later we will see that its variation in time is expressed with the derivative of the original field, the velocity one.

If $\mathbf{X}(t)$ has the previous properties we can define $\dot{\mathbf{X}}(t)$, which is also a second order and stationary ergodic process. In order to find the time derivative we can define a differential operador $\mathbf{D}$ such as:

$$\mathbf{D} = \frac{1}{\Delta t}\begin{bmatrix} -1 & 1 & 0 & ... & 0 \\ 0 & -1 & 1 & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \tag{3}$$

From this operator we can express the time derivative of the original field as $\dot{\mathbf{X}} \simeq \mathbf{D}\mathbf{X}$. Note that the velocity field $\dot{\mathbf{X}}(t) \in \mathbb{R}^{(n-1)\times m}$. At this step, it is quite interesting to note that $\ddot{\mathbf{X}} \simeq -\mathbf{D}^T\mathbf{D}\mathbf{X}$ (which corresponds to the acceleration field). For more convenience we will consider the derivation step (using the $\mathbf{D}$-operator) as a "true" operation and thus we will not use "$\simeq$" anymore, but "$=$".

Keeping in mind that the projection has to keep the maximum variance for the displacement field and the minimum one for the velocity field, we can write these two propositions with $\mathbf{X}(t)$ and $\dot{\mathbf{X}}(t)$ which leads to $\max_{\boldsymbol{\phi}} ||\mathbf{X}(t)\boldsymbol{\phi}||^2$ and $\min_{\boldsymbol{\phi}} ||\dot{\mathbf{X}}(t)\boldsymbol{\phi}||^2$ with respect to these two propositions they are equivalent to finding the maximum of the function $f(\boldsymbol{\phi})$ defined as:

$$f(\boldsymbol{\phi}) = \frac{||\mathbf{X}(t)\boldsymbol{\phi}||^2}{||\dot{\mathbf{X}}(t)\boldsymbol{\phi}||^2}. \tag{4}$$

Now we can simplify this ratio using the covariance matrices $\mathbf{R}_{\mathbf{X}}(0)$ and $\mathbf{R}_{\dot{\mathbf{X}}}(0)$, respectively, for the displacement field ($\mathbf{X}(t) \in \mathbb{R}^{n\times m}$) and the velocity field ($\dot{\mathbf{X}}(t) \in \mathbb{R}^{(n-1)\times m}$). Indeed, we can easily write:

$$||\mathbf{X}\boldsymbol{\phi}||^2 = (\mathbf{X}\boldsymbol{\phi})^T\mathbf{X}\boldsymbol{\phi} = \boldsymbol{\phi}^T\left(\mathbf{X}^T\mathbf{X}\right)\boldsymbol{\phi} = n\boldsymbol{\phi}^T\mathbf{R}_{\mathbf{X}}\boldsymbol{\phi}. \tag{5}$$

Similarly, we can simplify the denominator of this ratio as:

$$||\dot{\mathbf{X}}\boldsymbol{\phi}||^2 = ||\left(\mathbf{D}\mathbf{X}\right)\boldsymbol{\phi}||^2 = (\mathbf{D}\mathbf{X}\boldsymbol{\phi})^T\mathbf{D}\mathbf{X}\boldsymbol{\phi} = \boldsymbol{\phi}^T\left(\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X}\right)\boldsymbol{\phi} = (n-1)\boldsymbol{\phi}^T\mathbf{R}_{\dot{\mathbf{X}}}\boldsymbol{\phi}. \tag{6}$$

Finally we get this new expression for $f(\boldsymbol{\phi})$ (keeping in mind that n is the number of time samples which is rather big and thus can be simplified in the ratio) and we want to find:

$$\max_{\boldsymbol{\phi}}\left\{f(\boldsymbol{\phi}) = \frac{\boldsymbol{\phi}^T\mathbf{R}_{\mathbf{X}}\boldsymbol{\phi}}{\boldsymbol{\phi}^T\mathbf{R}_{\dot{\mathbf{X}}}\boldsymbol{\phi}}\right\}. \tag{7}$$

In order to find the maximum of $f(\boldsymbol{\phi})$ we can express its derivative with respect to $\boldsymbol{\phi}$, called $\nabla f(\boldsymbol{\phi})$, such as:

$$\nabla f(\boldsymbol{\phi}) = \frac{\partial f(\boldsymbol{\phi})}{\partial \boldsymbol{\phi}} = \frac{2\left(\boldsymbol{\phi}^T \mathbf{R}_{\dot{\mathbf{X}}} \boldsymbol{\phi}\right) \mathbf{R}_{\mathbf{X}} \boldsymbol{\phi} - 2\left(\boldsymbol{\phi}^T \mathbf{R}_{\mathbf{X}} \boldsymbol{\phi}\right) \mathbf{R}_{\dot{\mathbf{X}}} \boldsymbol{\phi}}{\left(\boldsymbol{\phi}^T \mathbf{R}_{\dot{\mathbf{X}}} \boldsymbol{\phi}\right)^2}, \tag{8}$$

and then find when $\nabla f(\boldsymbol{\phi})$ vanishes. We can also find this maximum using Lagrange multiples. In both cases we will find the following eigenvalue problem as the expression of the two initial propositions. The problem is equivalent to:

$$\mathbf{R}_{\mathbf{X}} \boldsymbol{\phi}_k = \lambda_k \mathbf{R}_{\dot{\mathbf{X}}} \boldsymbol{\phi}_k, \qquad k = 1, ..., m. \tag{9}$$

Solving this eigenvalue problem we get the eigenvalues, the λ_k's, and the eigenvectors, the $\boldsymbol{\phi}_k$'s, such as the λ_k's are in ascending order ($\lambda_1 > \lambda_2 > ... > \lambda_m$) and the $\boldsymbol{\phi}_k$'s are organized in columns. Finally, using $\boldsymbol{\phi}_k$ we can find the $\boldsymbol{\psi}_k$ such as:

$$\boldsymbol{\psi}_k = \mathbf{R}_{\dot{\mathbf{X}}} \boldsymbol{\phi}_k. \tag{10}$$

At this step we are able to find several parameters from a displacement field of a mechanical system. We can identify the λ_k (the Smooth Value - SV), the $\boldsymbol{\phi}_k$ (Smooth Mode - SM) and the $\boldsymbol{\psi}_k$ (Dual Smooth Mode - DSM). From Eq.(2), we can get the $\boldsymbol{q}_k$ (Modal Coordinates - MC) thanks to the projection of the signal onto the SM's. Depending on the characteristics of the system, we can interpret these parameters differently, as seen in the contribution of Belizzi and Sampaio (2015).

Expansion Principle

From this decomposition we do have two different bases, the smooth basis called $\boldsymbol{\Phi}$, formed with the $\boldsymbol{\phi}_k$'s, and the smooth dual basis, called $\boldsymbol{\Psi}$, formed with the $\boldsymbol{\psi}_k$'s (for $k = 1, ..., m$ with m as the number of measuring points). Now we propose to use these two bases to find the smooth expansion of $\mathbf{X}(t)$ and its dual smooth expansion.

Smooth expansion in the $\boldsymbol{\Phi}$-basis

Considering the expansion of $\mathbf{X}(t)$ in the $\boldsymbol{\Phi}$-basis we can write $\mathbf{X}(t) = \sum_{k=1}^{m} \xi_k(t) \boldsymbol{\phi}_k$, which can be simplified using the projection $\boldsymbol{\Pi}_1 = \boldsymbol{\Phi}\boldsymbol{\Psi}^T$ then we get $\boldsymbol{\Pi}_1 \mathbf{X}(t) = \sum_{k=1}^{m} \boldsymbol{\psi}_k^T \mathbf{X}(t) \boldsymbol{\phi}_k$. From these two expressions with can find the Dual Smooth Components (DSC) $\xi_k(t)$:

$$\xi_k(t) = \boldsymbol{\psi}_k^T \mathbf{X}(t). \tag{11}$$

Dual smooth expansion in the $\boldsymbol{\Psi}$-basis

Let us consider the $\boldsymbol{\Psi}$-basis to express $\mathbf{X}(t)$. The dual smooth expansion of this field into this basis can be written as $\mathbf{X}(t) = \sum_{k=1}^{m} \chi_k(t) \boldsymbol{\psi}_k$. Using the oblique projection $\boldsymbol{\Pi}_2 = \boldsymbol{\Psi}\boldsymbol{\Phi}^T$, we get $\boldsymbol{\Pi}_2 \mathbf{X}(t) = \sum_{k=1}^{m} \boldsymbol{\phi}_k^T \mathbf{X}(t) \boldsymbol{\psi}_k$. Then from these equations we get the following expression for the Smooth Components (SC) $\chi_k(t)$:

$$\chi_k(t) = \boldsymbol{\phi}_k^T \mathbf{X}(t). \tag{12}$$

At this step we can notice an interesting property for the Smooth Components. Let us consider the square of it and develop to the following form:

$$\chi_k^2(t) = \left(\boldsymbol{\phi}_k^T \mathbf{X}(t)\right)^2 = \boldsymbol{\phi}_k^T \mathbf{X}(t) \left(\boldsymbol{\phi}_k^T \mathbf{X}(t)\right)^T = \boldsymbol{\phi}_k^T \mathbf{X}(t) \mathbf{X}(t)^T \boldsymbol{\phi}_k = \boldsymbol{\phi}_k^T \mathbf{R}_{\mathbf{X}} \boldsymbol{\phi}_k. \tag{13}$$

Now considering the original eigenvalue problem formulated in Eq.(9) we can write:

$$\boldsymbol{\phi}_k^T \mathbf{R}_{\mathbf{X}} \boldsymbol{\phi}_k = \lambda_k \mathbf{R}_{\dot{\mathbf{X}}}, \tag{14}$$

then considering the mean value of each part we get:

$$\mathbb{E}\left(\boldsymbol{\phi}_k^T \mathbf{R}_{\mathbf{X}} \boldsymbol{\phi}_k\right) = \mathbb{E}\left(\lambda_k \mathbf{R}_{\dot{\mathbf{X}}}\right), \tag{15}$$

which considering the Eq.(13) and the properties of the covariance matrices leads to:

$$\mathbb{E}\left(\chi_k^2(t)\right) = \mathbb{E}\left(\lambda_k \mathbf{R}_{\dot{\mathbf{X}}}\right) \Rightarrow \mathbb{E}\left(\chi_k^2(t)\right) = \lambda_k. \tag{16}$$

Energetic point of view

An interesting thing with SD is the energetic study that can be made with this method. Let us call the "energy" of the displacement field $\mathbf{X}(t)$ the expression $\mathbb{E}\left(||\mathbf{X}(t)||^2\right)$ that can be reduced (using the dual smooth expansion). From the dual smooth expansion we get:

$$\mathbf{X}(t) = \sum_{k=1}^{m} \chi_k(t) \boldsymbol{\psi}_k \Rightarrow ||\mathbf{X}(t)||^2 = \sum_{k=1}^{m} ||\chi_k(t) \boldsymbol{\psi}_k||^2. \tag{17}$$

The expression of the "energy" can then be simplified using the previous formulation and we get:

$$\mathbb{E}\left(||\mathbf{X}(t)||^2\right) = \mathbb{E}\left(\sum_{k=1}^{m} ||\chi_k(t)\boldsymbol{\psi}_k||^2\right) = \sum_{k=1}^{m}\left[\mathbb{E}\left(\chi_k^2(t)\right)\mathbb{E}\left(||\boldsymbol{\psi}_k||^2\right)\right]. \tag{18}$$

Simplifying using Eq.(16) we can find the final expression for the "energy" of $\mathbf{X}(t)$ as:

$$\mathbb{E}\left(||\mathbf{X}(t)||^2\right) = \sum_{k=1}^{m} \lambda_k \, ||\boldsymbol{\psi}_k||^2. \tag{19}$$

Note that, from this formula it is quite easy to find the energy captured in each mode during the simulation since the expression:

$$\frac{\lambda_i \, ||\boldsymbol{\psi}_i||^2}{\sum_{k=1}^{m} \lambda_k \, ||\boldsymbol{\psi}_k||^2}, \tag{20}$$

represents the fraction of energy captured by the mode i during the simulation. This value can be a really good way to verify if a mode has been well excited during a simulation and then if the estimation of its frequency and mode shape can be validated.

Modal Assurance Criterion - MAC

In order to evaluate the modes basis $\mathbf{\Psi}$ found from SD ($\mathbf{\Psi}_{SD}$) with respect to the expected ones we will use the Modal Assurance Criterion called MAC representation. According to Allemang (2003), this tool is a good way to verify if modes found from one method (SD in our case) correspond to modes from another one (from the initial eigenvalue problem for us, defined as the modes shapes base $\mathbf{\Psi}_{EIG}$). The formulation of this criteria is:

$$MAC(\mathbf{\Psi}_{SD}, \mathbf{\Psi}_{EIG}) = \frac{\left|\mathbf{\Psi}_{SD}^H \mathbf{\Psi}_{EIG}\right|^2}{\left(\mathbf{\Psi}_{SD}^H \mathbf{\Psi}_{SD}\right)\left(\mathbf{\Psi}_{EIG}^H \mathbf{\Psi}_{EIG}\right)}, \tag{21}$$

where the term with the H-exponent denotes the complex conjugate transpose of the quantity. Let us note that for identical modes from two different methods the MAC should give one.

SMOOTH DECOMPOSITION FOR MODAL ANALYSIS

From the formulation of the initial projection (Eq.(2)), we can easily generalize this expression for m modes (the $\mathbf{\Phi}$-matrix, such as $\mathbf{\Phi} \in \mathbb{R}^{m\times m}$) and get the following equation with the matrix form with $\mathbf{Q}$ as the SC-matrix ($\mathbf{Q} \in \mathbb{R}^{n\times m}$):

$$\mathbf{X} = \mathbf{Q}\mathbf{\Phi}^T. \tag{22}$$

Multi-DoF Undamped Free Vibration System

Interpretation of the method

Any multi-DoF undamped system can be written in a matrix form of the classical dynamic equation, where $\mathbf{M}$ is the mass matrix, $\mathbf{K}$ is the stiffness, $\boldsymbol{x}$ and $\ddot{\boldsymbol{x}}$ represent respectively the displacement and the acceleration fields of the system, $\mathbf{X}$ and $\ddot{\mathbf{X}}$ represent their matrices:

$$\mathbf{M}\ddot{\boldsymbol{x}}_k + \mathbf{K}\boldsymbol{x}_k = 0 \Rightarrow \mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} = 0, \qquad k = 1,...,m. \tag{23}$$

From Eq.(23), it is quite easy to find an expression for the acceleration of the system and generalize it to the matrix form (if the mass matrix $\mathbf{M}$ is not singular):

$$\ddot{\boldsymbol{x}}_k = -\mathbf{M}^{-1}\mathbf{K}\boldsymbol{x}_k \Rightarrow \ddot{\mathbf{X}} = -\mathbf{X}\mathbf{K}^T\mathbf{M}^{-T}, \qquad k = 1,...,m. \tag{24}$$

Apply SD to this kind of mechanical system means maximize the function $f(\boldsymbol{\phi})$. Considering the original form of $f(\boldsymbol{\phi})$ defined in Eq.(4) and reminding that the acceleration can be written as $\ddot{\mathbf{X}} = -\mathbf{D}^T\mathbf{D}\mathbf{X}$, we get the following equivalence:

$$\max_{\boldsymbol{\phi}}\{f(\boldsymbol{\phi})\} = \max_{\boldsymbol{\phi}}\left\{\frac{\boldsymbol{\phi}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\phi}}{\boldsymbol{\phi}^T\mathbf{X}^T\mathbf{D}^T\mathbf{D}\mathbf{X}\boldsymbol{\phi}}\right\} = \max_{\boldsymbol{\phi}}\left\{\frac{\boldsymbol{\phi}^T\mathbf{X}^T\mathbf{X}\boldsymbol{\phi}}{\boldsymbol{\phi}^T\mathbf{X}^T\mathbf{X}\mathbf{K}^T\mathbf{M}^{-T}\boldsymbol{\phi}}\right\}. \tag{25}$$

As we have shown before, solving this maximization problem is equivalent to solve the following eigenvalue problem. When we write it in the matrix form we have to be very careful for the eigenvalues. Indeed, the $\mathbf{\Lambda}$-matrix is a diagonal of the inverses of the λ_k's, thus we get (if the product $\mathbf{X}^T\mathbf{X}$ is invertible):

$$\mathbf{X}^T\mathbf{X}\boldsymbol{\phi}_k = \lambda_k \mathbf{X}^T\mathbf{X}\mathbf{K}^T\mathbf{M}^{-T}\boldsymbol{\phi}_k \Rightarrow \mathbf{K}\boldsymbol{\Phi}^{-T} = \mathbf{M}\boldsymbol{\Phi}^{-T}\boldsymbol{\Lambda}. \qquad k = 1,\dots,m. \tag{26}$$

Now, let us consider the initial mechanical system defined with Eq.(23). The associated eigenvalue problem is written with $\boldsymbol{\Omega}$ which is a diagonal matrix formed with the squares of the natural frequencies of the mechanical system (the ω_k^2's) associated to each column of $\boldsymbol{\Psi}$:

$$\mathbf{K}\boldsymbol{\Psi} = \mathbf{M}\boldsymbol{\Psi}\boldsymbol{\Omega}. \tag{27}$$

Then from the two expressions we have two equivalent eigenvalue problems (Eqs.(26) and (27)) which lead to equivalences between the modal quantities for this kind of mechanical systems:

$$\boldsymbol{\Psi} = \boldsymbol{\Phi}^{-T} \tag{28}$$

$$\boldsymbol{\Omega} = \boldsymbol{\Lambda} \Rightarrow \lambda_k = \frac{1}{\omega_k^2} \qquad k = 1,\dots,m. \tag{29}$$

Application of the method

Let us apply this method to a numerical simulated case. To illustrate this theory we can observe the five degrees of freedom undamped system presented in Fig.1. This system has got the following mechanical properties ($m_i = 1$ kg and $k_i = 1000$ N/m for $i = 1,\dots,5$) as it is represented in the following figure.

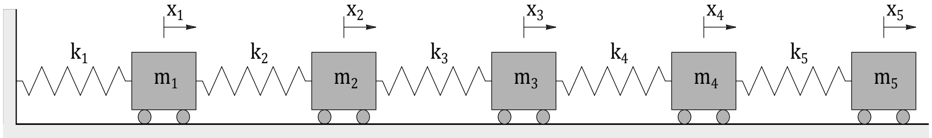

Figure 1 – Five degrees of freedom undamped system

From these mechanical characteristics we can easily find the natural frequencies and the modes of the system using the Eq.(27). As it is well known, this eigenvalue problem can be easily solved and, for our specific case, we get the following results (see Table 1) which will be our references for further comparison.

Table 1 – Reference values for natural frequencies and modes of the system presented in Fig.1.

N°	Natural Frequences (Hz)	Normalized Modes				
i	λ_i	x_1	x_2	x_3	x_4	x_5
1	1.4325	0.1699	0.3260	0.4557	0.5485	0.5969
2	4.1815	-0.4557	-0.5969	-0.3260	0.1699	0.5485
3	6.5917	0.5969	0.1699	-0.5485	-0.3260	0.4557
4	8.4679	0.5485	-0.4557	-0.1699	0.5969	-0.3260
5	9.6581	-0.3260	0.5485	-0.5969	0.4557	-0.1699

Let us now simulate the dynamic response of this system for a given time. Our simulation is 50 seconds long with an acquisition frequency of 1000 Hz (which means that the time interval $\Delta t = 0.001$ sec). Now, as this method does not implies the computation of the input signal, we will consider the forcing as random generated by the simulation system. We decide to exercise this kind of force on all the degrees of freedom of our system. Simulating it numerically we get the following displacement field (see Fig.2) that will constitute our starting point for applying the SD.

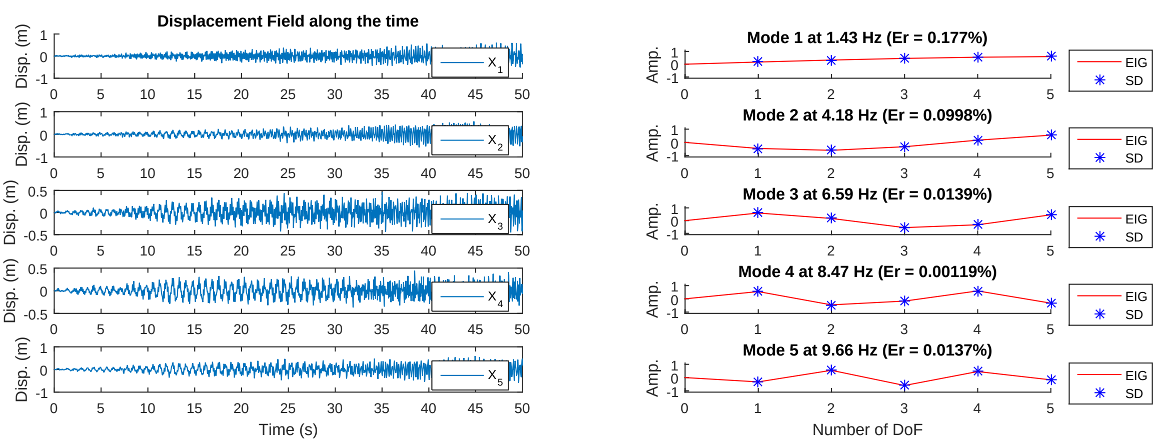

Figure 2 – Left: Displacement Field along time for each DoF / Right: Modes shapes graphic with the analytical solution (EIG for Eigenvalue problem resolution) and the Smooth Decomposition (SD) from the Disp. Field

As we have seen, the response of the system seems to be completely random and quite hard to characterize without any information on the input signal. Let us apply the SD method to such data to extract the SV, the SM, and the DSM of this mechanical system and then compare them to the analytical frequencies and modes calculated via the solved eigenvalue problem (Eq.(27)). Also, an important tool to verify our results is the MAC graphic which shows the really good approximation of modes estimated for this numerical case (see Fig.3).

From the Fig.2 we can see that the mode shapes of the system are well evaluated by the Smooth Decomposition and the natural frequencies too. From the Smooth Decomposition we can also have an evaluation of the energy captured in the modes during the simulation (see Fig.3).

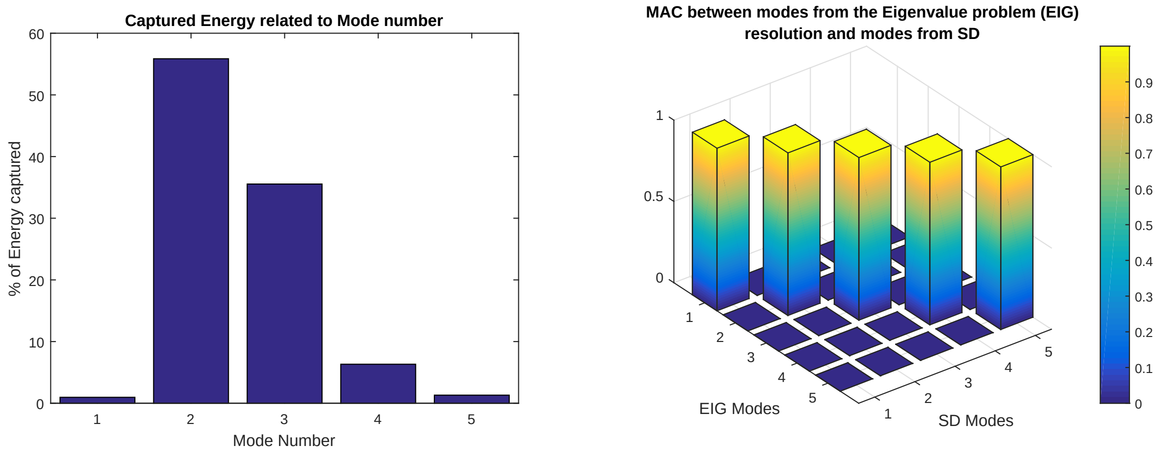

Figure 3 – Left: Captured energy in each mode during the simulation / Right: MAC between the EIG resolution modes and the SD ones

This section has shown the force of SD for linear undamped systems as a modal tool to identify modal parameters such as natural frequencies and normal modes. We have also shown that we do have several tools to evaluate our approximation and our results. In the following section the influence of the modal damping factor is discussed.

General Mechanical System

Interpretation of the method

In this part we will consider a multi-DoF damped system (see Fig. 4). This consideration is closer to real mechanical systems which can be formulated thanks to the dynamics equation:

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} + \mathbf{A}\mathbf{X} = \mathbf{F}, \tag{30}$$

where $\mathbf{M}$, $\mathbf{K}$ e $\mathbf{C}$ are the mass, the stiffness and the damping matrices of our system. $\mathbf{F}$ is the forcing vector which, in our specific case is not monitored (unknown excitation, characteristic of the output only methods).The term called $\mathbf{A}$ represents the nonlinearity of our mechanical system.

As it was shown in the literature (Belizzi and Sampaio, 2015), the interpretation for those cases is not as simple as for the linear ones. Indeed, we cannot find the simple equivalence shown with Eqs.(28) and (29). These considerations provide from the statistical linearization method thus they give results for a linear system. If we apply these equivalences (Eqs.(28) and (29)) to non-linear systems we actually get the modal parameters for the linear equivalent system but not for the non-linear one.

If we consider a damped system with the $\mathbf{C}$-matrix as a linear combination of the $\mathbf{M}$ and $\mathbf{K}$ ones (i.e. $\mathbf{C} = \alpha\mathbf{C} + \beta\mathbf{K}$) we can reach a similar interpretation as it was done for undamped systems. From this method we do have access to the normal modes of the systems. For example, if we consider the $\mathbf{C}$-matrix as a combination of the $\mathbf{M}$-matrix only, we observed that for an α rather small ($0 \leq \alpha \leq 1$) the results still acceptable. This shows that the theory works perfectly for undamped systems but for damped ones we have to be careful. Similar conclusions have been found by Farooq and Feeny (2008).

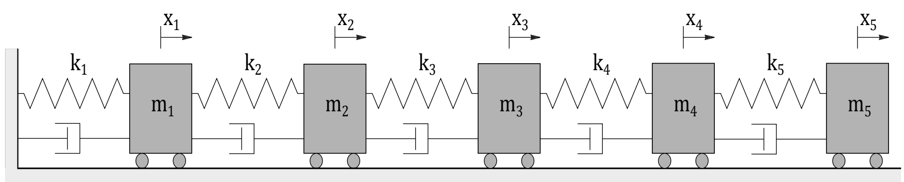

Figure 4 – Five degrees of freedom damped system

Indeed, for damped systems, the modes have an imaginary part which cannot be expressed with normal modes and thus which cannot be found with SD (as the method was developed for undamped systems). The following application will investigate this aspect of SD and show that for small damping factors the results are still acceptable.

Application of the method

As for the first part, we will apply the SD method to a specific simulated case, which now is a multi-DoF damped system presented in Fig.4. This system has got the following mechanical properties ($m_i = 1$ kg and $k_i = 1000$ N/m for $i = 1, ..., 5$) as it is represented in the following figure. The damping of the system is now represented with the modal coefficients, the ζ's.

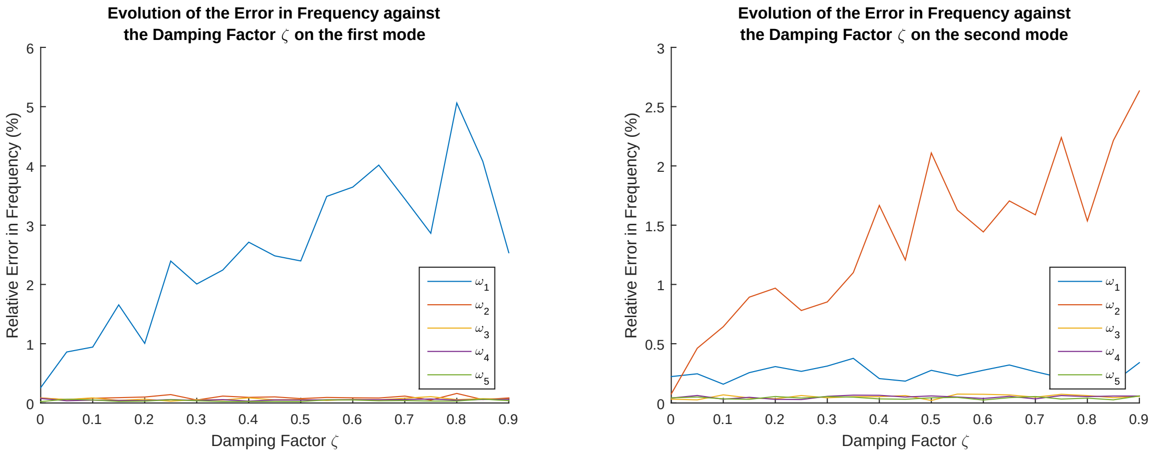

Figure 5 – Evolution of the relative error in the natural frequencies against the Damping Factor ζ in each mode (10 simulations for each value of ζ)

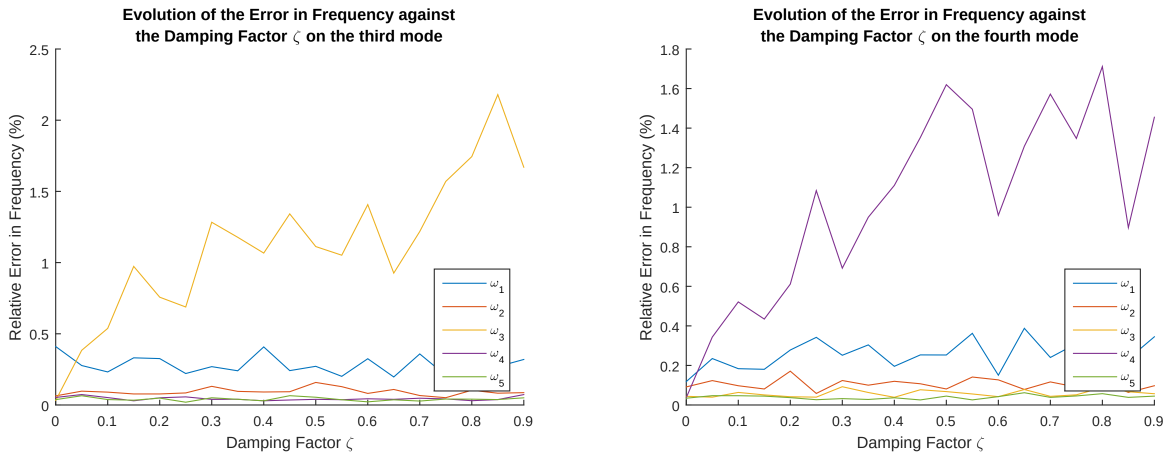

Figure 6 – Evolution of the relative error in the natural frequencies against the Damping Factor ζ in each mode (10 simulations for each value of ζ)

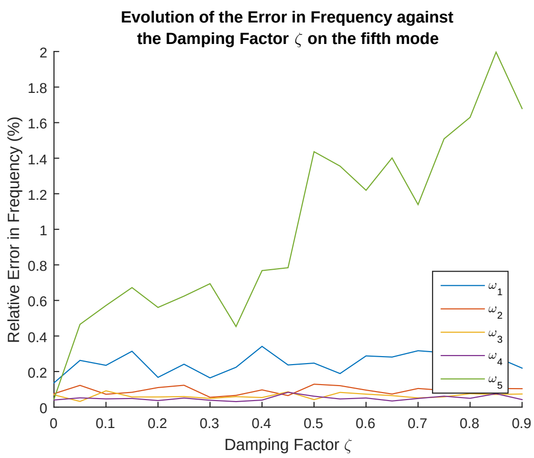

Figure 7 – Evolution of the relative error in the natural frequencies against the Damping Factor ζ in each mode (10 simulations for each value of ζ)

This application will investigate the influence of the modal damping factor on the evaluated frequencies from SD. To discuss this we will observe the relative error in each natural frequency for different values of the modal factor. For this discussion we will perform the simulation 10 times (each time it is going to be a different simulation since the excitation is random) for each value of the modal damping factor ζ_i in order to get the mean value of natural frequencies for each value.

The previous figures (Figures 5, 6 and 7) show that increasing the modal damping factor for each mode the relative error will be bigger for the corresponding natural frequency found with SD. We can see that for damping factor smaller than 0.2 the relative error in the concerned frequency is lower than 1% which still a good approximation. Unfortunately, when the damping factor is bigger than 0.2 the relative error starts to increase quite fast to reach 2%-6%. This proves that for high modal damping factors this method is not adequate. As SD gives the evaluation of the normal modes, it was expected for high damped systems an inadequate evaluation of them and their frequencies. All those conclusions are illustrated with the previous Figures 5, 6 and 7. Concerning modes, the conclusions are similar. For small damping factors the approximations are quite good but for bigger ones we observe some mode shapes that do not correspond to the expected ones.

Partially observed mechanical system

In this part we will discuss the SD results for a new kind of systems, a partially observed system. Let us consider the first undamped mechanical system presented in Fig.1 but now with a complex that does not allow the measurement of the third DoF displacement (see Fig.8). This kind of constrain can appear for mechanical problems. With this approach we can also expect the application of SD to continuous systems since this approach can be similar in some points to the instrumentalization of continuous systems. As we do not know the displacement of the third DoF, the output field (the displacement field) used for the SD process is only composed of the four others signals (Fig.9).

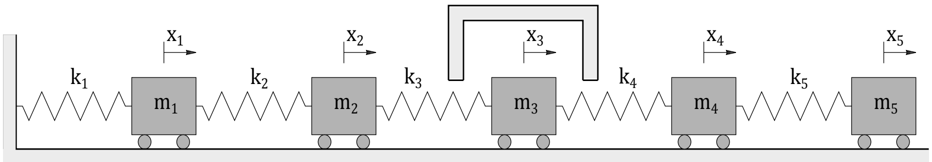

Figure 8 – Five degrees of freedom undamped system with measuring difficulties on the third DoF

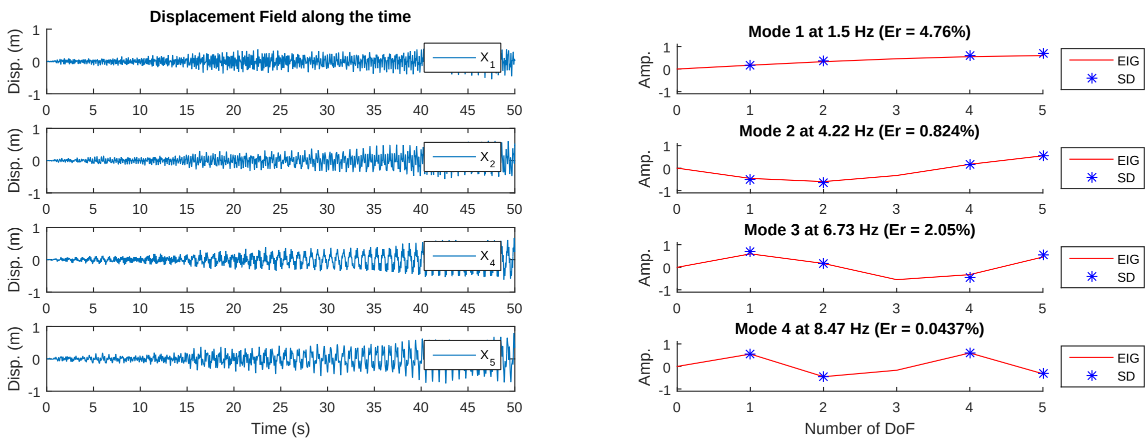

Figure 9 – Left: Displacement Field along time for each DoF / Right: Modes shapes graphic with the analytical solution (EIG for Eigenvalue problem resolution) and the Smooth Decomposition (SD) from the Disp. Field

After performing the SD process we can observe some interesting results about the partially observed system. First, as we only have four signals as output, we are only able to approximate four natural frequencies and four normal modes. As we can see on Fig.9 the approximation of mode shapes are quite good but unfortunately the evaluation of natural frequencies is partially good only. These results can be explained by the low level of energy captured by the first mode during the simulation (see Fig10). Also, as the first frequency is quite low, then a small numerical approximation during the SD process can perturb the estimation of it. The MAC shows us that the evaluation of the modes is quite good. We do have some residual correspondences between different modes showing that the results are only an approximation and they need to be improved. However this is a first step before continuous systems that is quite promising.

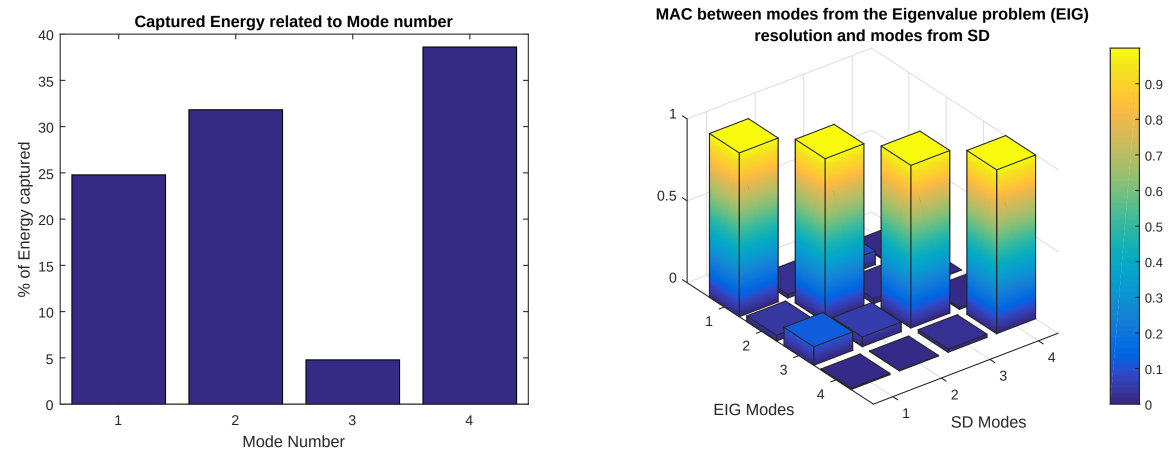

Figure 10 – Left: Captured energy in each mode during the simulation / Right: MAC between the EIG resolution modes and the SD ones

IMPORTANCE OF THE EXCITATION QUALITY

In this section we will show that the quality of the excitation is one of the primary issues of the SD and the parameters identification success. In order to prove it we will consider a specific frequency for the excitation and show that when the force is periodic the SD and identification process do not find the expected results with a good approximation. The forcing in this part consists of a unique discrete periodic force on the fifth DoF such as the force frequency is equal to 2.5 Hz for instance. With this excitation we get the following results after the SD process (Figures 11 and 12).

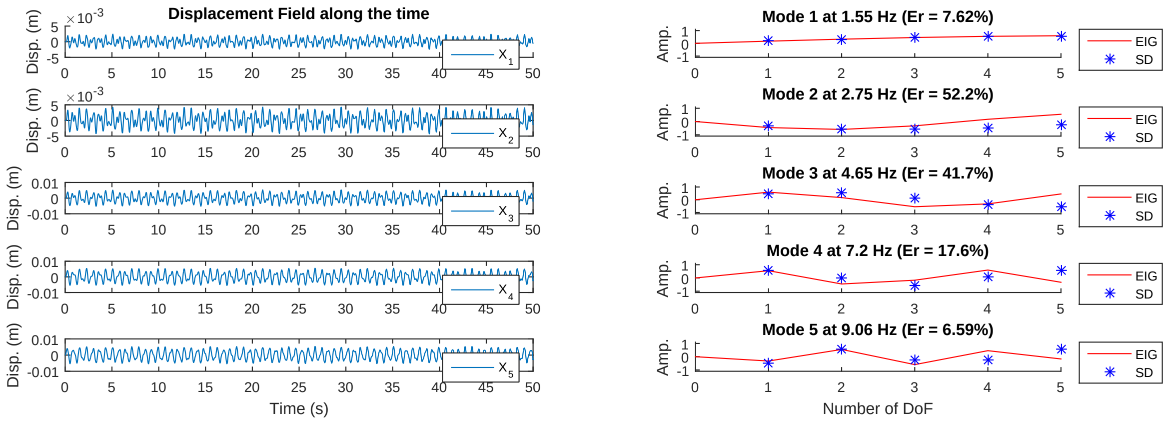

Figure 11 – Left: Displacement Field along time for each DoF / Right: Modes shapes graphic with the analytical solution (EIG for Eigenvalue problem resolution) and the Smooth Decomposition (SD) from the Disp. Field

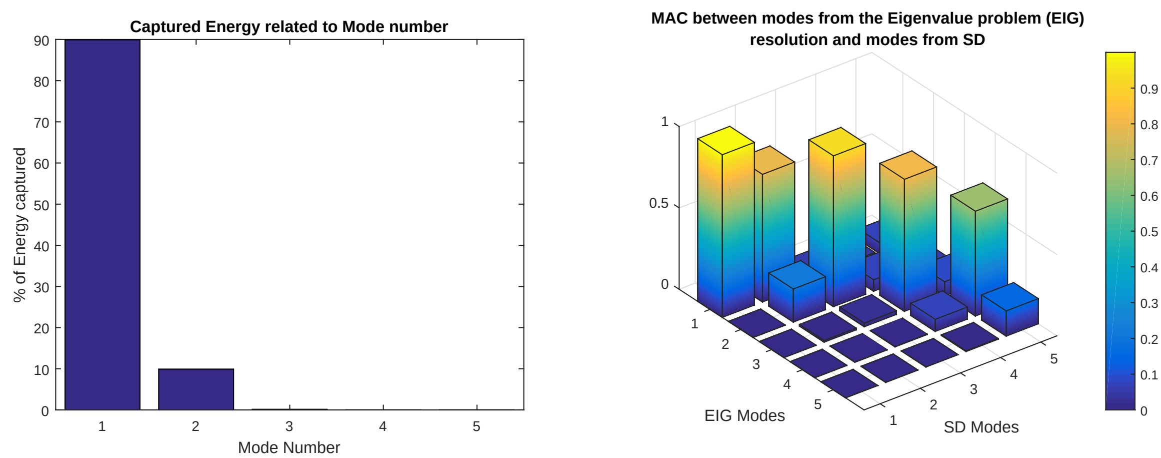

Figure 12 – Left: Captured energy in each mode during the simulation / Right: MAC between the EIG resolution modes and the SD ones

As we can see on Fig.11, the SD process identifies a natural frequency at 2.75 Hz, very closed to the forcing frequency (10% higher) which, according to the MAC (see Fig.12) corresponds more or less to the first mode shape expected. But, as we know the system, we can say that this frequency cannot be a natural one. The SD process has identified the forcing frequency with an approximation. The "mode shape" identified is directly related to the energy captured by the first mode during the simulation. Concerning all the others frequencies identified, we can see that they are all $\sim 10\%$ higher that the expected ones. If we have a closer look on the MAC we can see that the mode shape estimations are not that bad but cannot be satisfying for further considerations.

Repeating this kind of simulation with several excitation frequencies we observe exactly the same behavior. First, the SD process identifies the excitation frequency ($\sim 10\%$ higher) and attributes the most excited mode during the simulation as its mode shape. Secondly, all the other frequencies are evaluated as $\sim 10\%$ higher than the expected ones. Finally all the mode shapes are quite well evaluated but should be improved for further considerations. This kind of results has been observed also when we force the system with one of its natural frequencies. All those observations show that the quality of the excitation is a crucial point and a precondition for the success of the identification process.

CONCLUSION

In this article we have presented the concept of Smooth Decomposition and explained the method. Its properties were exposed and the concept of Dual Smooth Modes were presented and then used to expand a displacement field or any random process obtained from a excitation that satisfies the presented properties. Then the objective was to use SD as a modal tool for different mechanical systems in order to identify modal parameters.

As we have seen in the first part, this method works perfectly for undamped linear systems which makes sense as soon as this method was developed exactly for this kind of systems. We have seen in the second part that SD also works for low-damped systems. Unfortunately we have shown that for high-damped systems SD is not really adapted and should be improved. Indeed, as SD gives the normal modes associated to a given displacement field, this method does not really work for finding modes with an imaginary part which is the case for damped systems.

In this article we have also introduced an interesting part which can be a first step for using SD as a modal tool for continuous systems. Indeed, we have shown that SD can also be adapted for partially observed systems and that the estimation is partially good but must be improved for further considerations.

The last part of this article focused on a crucial point of the SD process. The excitation of the system has to respect some conditions to get a good approximation and estimation of the modal parameters. If the excitation does not satisfy the properties the results are affected and this has to be taken into account for the interpretation.

Finally, we can say that SD is a great tool for modal analysis and that we have some good indicators for its application on more complex cases for the future. Another important aspect of SD is the energy indicator which gives us a first idea of the quality of the results. In order to validate the results we can also use the MAC to compare, with another modal analysis method, the modes and then validate the identified parameters.

REFERENCES

Allemang, R. J., 2003, “The Modal Assurance Criterion - Twenty Years of Use and Abuse”, Sound and Vibration, August 2003, 14-21.

Bellizzi, S., Sampaio, R., 2015, “The Smooth Decomposition as a nonlinear modal analysis tool”, Elsevier, Mechanical Systems and Signal Processing 64-65 (2015) 245-256.

Chelidze, D., Zhou, W., 2006, “Smooth orthogonal decomposition-based vibration mode identification”, Elsevier, Journal of Sound and Vibration 292 (2006) 461-473.

Farooq, U., Feeny, B.F., 2008, “Smooth orthogonal decomposition for modal analysis of randomly excited systems”, Elsevier, Journal of Sound and Vibration 316 (2008) 137-146.

Zhou, W., 2006, “Multivariate analysis in vibration modal parameter identification”, Ph.D Thesis, University of Rhode Island, United States of America.

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