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THE DEVELOPMENT OF A EXPERIMENTAL APPARATUS FOR ANALYSIS OF THE ZENER MODEL WITH NONLINEAR ELEMENTS

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Abstract: The reduction of noise and vibration in mechanical applications has been an important subject related to safety and comfort in various systems. Part of the problem involving vibration and noise is related to the concept of transmissibility. For the linear systems, its concepts can be the displacement and the force transmissibility. The first relates the amount of displacement (or velocity or acceleration) that is convert in displacement in a second system. A typical example of such concept is the linear mass-spring-damper system subject to base excitation. In this case, the spring damper set is defined as an isolator set. The objective of isolators is to filter the excitation from a source (base motion) from reaching a receiver (mass). It is know that viscoelastic isolators presents good isolation characteristics, especially for high frequencies. A model that is generally used to represent the viscoelastic behavior of systems is the Zener Model. This model, also known as the standard linear solid (SLS) model, is a simple and complete method of modeling the behavior of a viscoelastic material using a linear combination of springs and dashpots (dampers) to represent elastic and viscous components, respectively. Often, similar models, such as the Maxwell, which is defined by a spring in series with a damper, and the Kelvin-Voigt model, which is defined by a spring in parallel with a damper, are used. However, these models are often proved insufficient for the representation of materials which exhibit viscoelastic rheology. Thus, the Maxwell model does not describe creep or recovery, and the Kelvin-Voigt model does not describe stress relaxation. SLS is the simplest model that predicts both phenomena, which makes it also a complete model. This paper will investigate the linear Zener model and a experimental apparatus is developed to represent the behavior of the structure, in the experiment will be analyzed stiffness, damping and natural frequencies of the structure.

Keywords: Zener model, nonlinear spring

NOMENCLATURE (OPTIONAL)

Latin symbols

c: viscous damping
k: stiffness
m: mass
t: time
$x(t)$: displacement
$\dot{x}(t)$: velocity
$\ddot{x}(t)$: acceleration

Greek symbols

γ: ratio of stiffness coefficients
ξ_0: damping ratio
ω_0: natural frequency
Ω: ratio of frequency coefficients

Subscripts

knl: non-linear stiffnes

INTRODUCTION

There is the necessity to improve solutions for problems such as unwanted noise and vibration. Thus, the development of new materials to meet this demand is vast. In this context, the application of viscoelastic material can be found in the aeronautical, space, naval, civil, train, and automobile. The viscoelastic material is a class of materials which exhibit viscoelastic rheology, are materials that when deformed simultaneously elastic and viscous deformations. By applying this material in structures of any type are obtained significant gains in reducing vibration and noise.

According to (Rao, 2009) any movement to be repeated after an interval of time is called vibration or oscillation. The vibration theory and oscillation comes to study oscillatory movements of bodies and forces associated with them. The description of the behavior of viscoelastic materials is made through differential equations that combine three terms: the elastic deformation, the deformation rate viscous and inertial acceleration term. The total voltage is the sum of the partial voltage of each term.

Combinations of springs and dampers are of interest to the mate rials community as they are used to represent the behaviour of viscoelastic materials (Nashif, 1985). As (Masterson, 2002) even small vibrations induced in structures can can lead to severe degradation in the performance of high precision payloads. (Jazar, 2006) explores a model for a nonlinear one-degree of freedom passive vibration isolator system, known as a smart engine mount, are analyzed the frequency response of relative displacement for a hardening and softening spring.

Several studies were used to support knowledge concerning the vibration isolation behavior as well as their use in various systems arising from engineering and mathematics. As (Brennan, 2007) has a free and forced vibration analysis of a mass supported by a parallel combination of a spring and damper elastically supported *Zener* model. We also have to (Brennan, 2008) presented a paper on the frequencies of a Duffing oscillator with linear viscous damping systems with softening and hardening systems with stiffness variations.

Later (Blais, 2009) made an approach of impacted beam equations of motion solutions composed partially viscoelastic layer damping. A similar approach was also performed by (Granger and Ross, 2009), presenting experimental and numerical results. (Lee, 2014) made comparisons between two models of energy absorbers for low frequency, using the *Zener* model. (Silva 2014) developed a paper where was investigated the natural frequencies and the vibration modes in beams with different dampers, and use impulse input in the experimental tests. Another study using the model of *Zener* is the (Lee, 2015), who performed experimental analysis of vibration isolators designed for space application in orbit able to isolate micro-vibrations, and also support launch environment where there is severe vibration.

NUMERICAL MODELING

The system considered in this work is shown in figure 1(a) which comprises of a mass m supported by a linear spring with constant k in parallel with a viscous damper in series with another spring with constant Nk, where N is a positive real number. The damper constant is defined by c. An alternative representation is shown in figure 1(b).

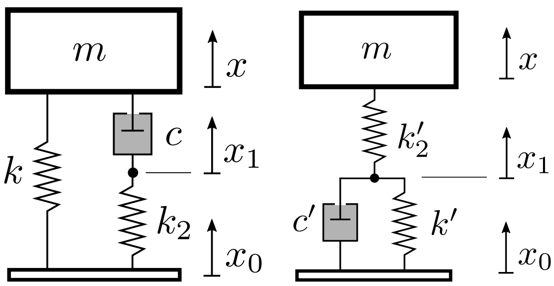


Figure 1 – Representation of viscoelastic models. (a) Zener Model, (b) Dynamically Equivalent to Zener Model.

The systems (a) and (b) shown in figure 1 are equivalent, considering that $k_2 = Nk$, the stiffness and damping equivalence is given by (Ruzicka and Derby, 1971)

$$k = \left(\frac{N}{N+1}\right)k', \quad c = \left(\frac{N}{N+1}\right)^2 c'$$

The equations of motion are obtained using the force equilibrium in the mass and in the series elements, such that $F_{Nk} = F_d$, where $F_{Nk} = NK(x_1 - x_0)$ and $F_d = c(\dot{x} - \dot{x}_1)$ are respectively, the forces on the spring with constant Nk and the damper. For the case of no external excitation and no base motion ($x_0 = 0$)

$$\ddot{x} = -\omega_0^2(x + Nx_1) \tag{1}$$

$$\dot{x}_1 = \dot{x} - \frac{N\omega_0}{2\xi_0}x_1 \tag{2}$$

where $\omega_0^2 = k/m$ and $\xi_0 = c/2m\omega_0$ are parameters of a linear spring-mass-damper oscillator defining the natural frequency and the damping ratio. Note that these parameters are not the natural frequency and damping ratio of the system shown in figure 1. The equilibrium positions can be obtained by reducing the order of of the system in terms of three first order ODEs, also neglecting the external excitations and considering the time derivatives are zero, such that

$$q_1 = x, \; \dot{q}_1 = \dot{x} = q_2 \tag{3}$$

$$q_2 = \dot{x}, \; \dot{q}_2 = \dot{x} = -\omega_0^2(q_1 + Nq_3) \tag{4}$$

$$q_3 = x_1, \; \dot{q}_3 = q_2 - \frac{N\omega_0}{2\xi_0}q_3 \tag{5}$$

The only equilibrium position in this case is $(x, x_1) = (0, 0)$. So the Jacobian matrix can be written for this position

$$J(0,0) = \begin{bmatrix} 0 & 1 & 0 \\ -\omega_0^2 & 0 & -N\omega_0^2 \\ 0 & 1 & -\frac{N\omega_0}{2\xi_0} \end{bmatrix} \tag{6}$$

According to (Brennan et al. 2008), there are three eigenvalues which two are a pair of complex conjugates and the third is purely real. It is shown that it is not possible to achieve critical damping of the complex roots unless the secondary spring is at least eight times greater than that of the main spring. Looking the in series elements, there are two limit conditions

- $N \to \infty$, the system behaves as a linear damped oscillator with natural frequency $\omega_0 = \sqrt{k/m}$ and damping ratio ξ_0.
- $C \to \infty$, the system behaves as a linear undamped oscillator with natural frequency $\omega_0 = \sqrt{\frac{k+Nk}{m}}$

The free motion response can be computed assuming harmonic motion of the response, such that

$$x(t) = A_1 e^{s_1} + A_2 e^{s_2} + A_3 e^{s_3} \tag{7}$$

where $s_{1,2,3}$ are obtained solving the eigenvalue problem using the Jacobian matrix (eq. 6). $A_{1,2,3}$ are constants to be obtained from the initial conditions of displacement, velocity and acceleration. Considering initial condition of displacement only, the time response of the system for different values of N for $\xi_0 = 1$ where computed and shown in figure 2. It can be seen that for $N \geq 8$ the system does not oscillate (as proved by (Brennan at al. 2008) the conditions for this system to achieve critical damping).

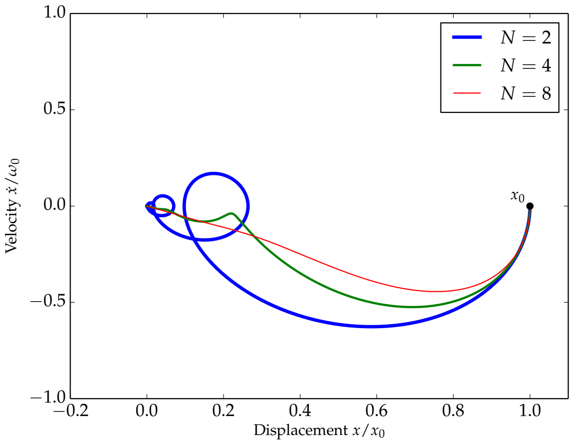


Figure 2 – Phase plane for different values of N for ξ_0 considering initial displacement.

Forced Harmonic Excitation

According to references (Ruzicka and Derby, 1971), eq. 1 and 2 can be simplified in terms of x only, by increasing the order of the differential equation, such that

$$\left(\frac{2\xi_0}{\omega_0 N}\right)\dddot{x} + \ddot{x} + 2\xi_0\omega_0\left(\frac{N+1}{N}\right)\dot{x} + \omega_0^2 x = \frac{f(t)}{m} + \frac{2\xi_0}{\omega_0 N}\frac{\dot{f(t)}}{m} \tag{8}$$

If harmonic excitation on the form $f(t) = Fe^{j\omega t}$ is assumed, and that the response $x(t) = Xe^{j\omega t}$, it is possible to define the transfer function in terms of the magnification factor

$$D(j\Omega) = \frac{1 + j\frac{2\xi_0}{N}\Omega}{1 - \Omega^2 + j\frac{2\xi_0}{N}\Omega(N+1-\Omega^2)} \tag{9}$$

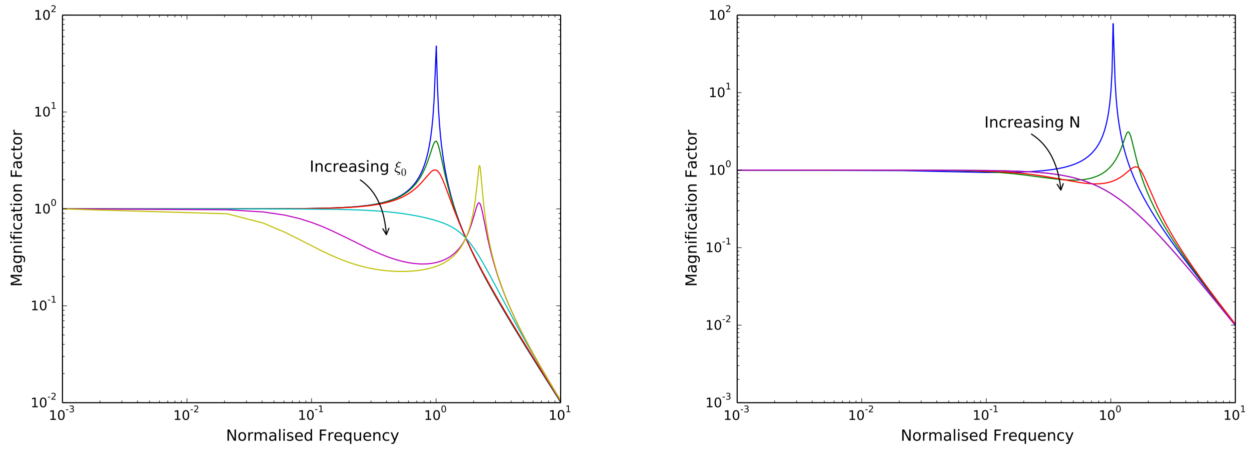
where $\Omega = \omega/\omega_0$ and $D(j\Omega) = X/(F/k)$. The expression of equation 9 is plotted in figures 3(a) and (b). It is possible to see the two different behavior of the resonance peak, which is a function of the parameters ξ_0 and N

NONLINEAR MAIN-SPRING

In the system shown in figure 1, the main support spring has restoring force defined by $f_k = (k + k_{nl}x^2)x$. The secondary spring has stiffness proportional to the linear term of the main spring by factor N. The equations of motion are obtained including the ratio of stiffness coefficients $\gamma = k_{nl}/k$ such that

$$\ddot{x} = -\omega_0^2(x - Nx_1) - \gamma\omega_0^2 x^3 + \frac{f(t)}{m} \tag{10}$$

$$\dot{x}_1 = \dot{x} - \frac{N\omega_0}{2\xi}x_1 \tag{11}$$



(a) (b)

Figure 3 – The Magnification factor as a function of the normalised frequency Ω. (a) varying the parameter ξ_0 for $N = 4$ and (b) varying the parameter N for $\xi_0 = 1$

This can be changed to a third order differential equation in terms of the variable x using the procedure discussed in the previous sections

$$\frac{mc}{Nk}\dddot{x} + m\ddot{x} + c\left(\frac{N+1}{N}\right)\dot{x} + \frac{3c\gamma}{N}x^2\dot{x} + \left(k + k_{nl}x^2\right)x = F\cos(\omega t) - \frac{Fc\omega}{Nk}\sin(\omega t) \tag{12}$$

Using the terms, $\tau = \omega_0 t$, $\alpha = N/2\xi_0$, the equation can be put in the form

$$x''' + \alpha x'' + \left(N + 1 + 3\gamma x^2\right)x' + \alpha\left(1 + \gamma x^2\right)x = \delta_0 \cos(\Omega\tau + \phi) \tag{13}$$

where

$$\delta_0 = \delta_{st}\left(\alpha^2 + \Omega^2\right)^{1/2} \phi = \operatorname{atan}(\Omega/\alpha)\delta_{st} = F/k \tag{14}$$

Harmonic Forced Excitation

The forced response of the system can be obtained considering the harmonic force defined previously such that

$$x''' + \alpha x'' + \left(N + 1 + \frac{3}{2}\gamma x^2\right)x' + \alpha\left(1 + \gamma x^2\right)x = \delta_0 \cos(\Omega\tau + \phi) \tag{15}$$

Assuming a harmonic response of the system in the form

$$x(t) = X\cos(\Omega\tau + \phi + \theta) = X\cos(\Delta) \tag{16}$$

Applying eq. 16 into eq. 15, and neglecting the cosine and sine terms associated with higher harmonics of Ω, it is possible to write

$$\begin{aligned}\Omega^3 X\sin z - \Omega^2\alpha\cos z - \Omega(N+1)X\sin z - \Omega 3\gamma X^3\cos^2 z\sin z + \alpha X\cos z + \alpha\gamma X^3\cos z = \\ \delta_0\cos(\Delta)\cos(\theta) + \delta_0\sin(\Delta)\sin(\theta)\end{aligned} \tag{17}$$

where $\delta_0 = F/k(\alpha^2 + \Omega^2)^{1/2}$

Equating the terms on the left hand side of equation 17 multiplying the $\cos(\Delta)$ and $\sin(\Delta)$ terms on the right hand side, it is possible to write

$$\delta_0\cos\theta = \alpha\left(1 - \Omega^2 + \frac{3}{4}\gamma X^2\right)X \tag{18}$$

$$\delta_0\sin\theta = -\Omega\left(N + 1 - \Omega^2 + \frac{3}{4}\gamma X^2\right)X \tag{19}$$

The combination of previous equations allow to write the expression for the frequency response as a function the the displacement amplitude (nonlinear equation).

$$\frac{X^2}{\delta_{st}^2} = \frac{\alpha^2 + \Omega^2}{\alpha^2\left(1 - \Omega^2 + \frac{3}{4}\gamma X^2\right)^2 + \Omega^2\left(N + 1 - \Omega^2 + \frac{3}{4}\gamma X^2\right)^2} \tag{20}$$

The results shown in figure 4 where obtained for low value of damping varying the parameter γ. Results show the influence of the nonlinear stiffness term.

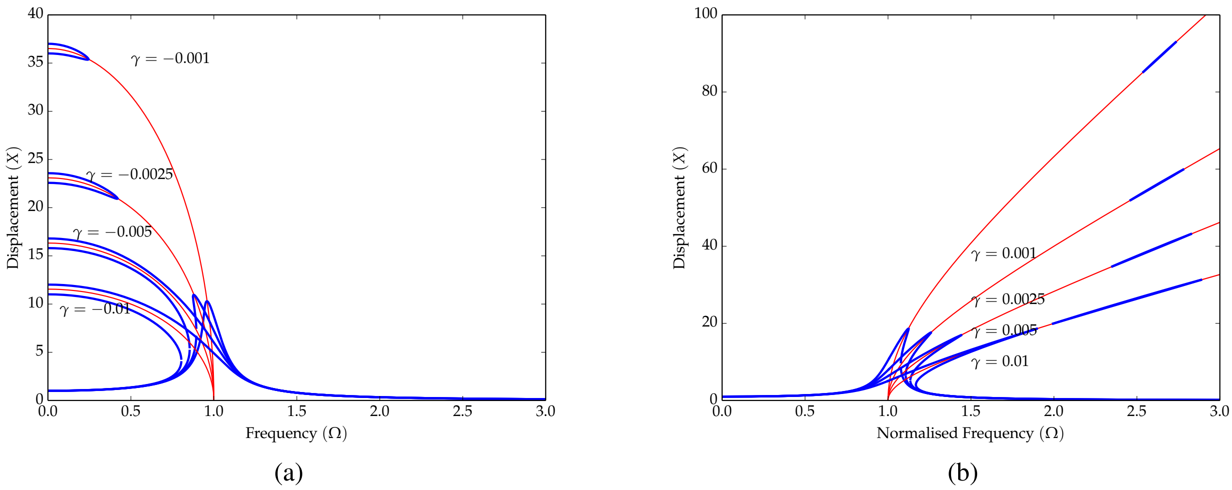


Figure 4 – The frequency response functions for the nonlinear system. (a) Softening spring $\xi = 0.05$, $N = 3$ and (b) Hardening Spring, $\xi = 0.025$, $N = 3$

DEVELOPMENT OF THE EXPERIMENTAL APPARATUS

An experimental apparatus was developed to represent the behavior of the zener model. For this, a structure in the shape of portal frame and a beam were used to represent the main and secondary springs, both of stainless steel 304 austenitic metal material according to ABNT(with Young's Modulus of 193 GPa e density of 8000 kg/m^3). The schematic illustration is shown in figure 5(a) and the photograph of the structure is shown in figure 5(b)

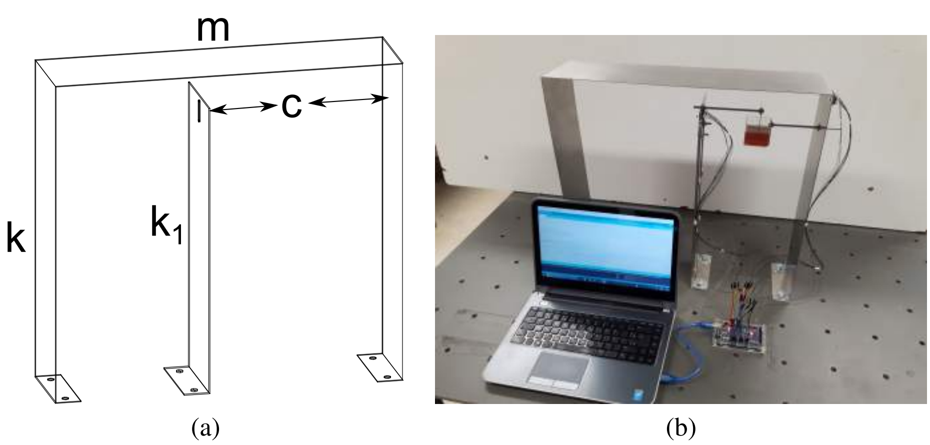


Figure 5 – Experimental apparatus developed to represent the behavior of the Zener Model.

Design using Finite Element Model

A finite element model of the structure was developed to help the design and construction of the system. The main objective of the finite element model was to investigate the natural frequencies and mode shapes of the structure. A mesh of the geometry was developed using four node shell element using the software *GMSH* which is and open source free software. Later, the finite element mesh was analyzed using the software *Calculix*, which is also open source and free software. The boudary conditions used for the nodes on the base of the structure where considered fixed, removing the motion of all degrees of freedom for these nodes. The modal analysis was performed to obtain the natural frequencies and mode shapes (eigenvalue-eigenvector analysis). The mode shapes obtained for the two parts of the structure are shown in figure 6.

The main interest in this analysis was the estimation of the stiffness for the two parts of the structure that could be calculated from the first natural frequencies for the portal frame and the secondary spring represented by the beam. These results were compared with the experimental tests described in the following sections.

Instrumentation of the test apparatus

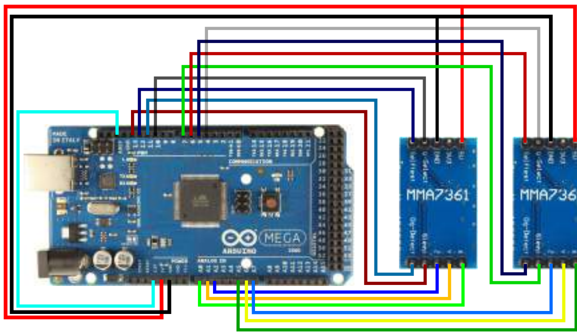
Experimental tests where conducted to verify the dynamic behavior of the structure. For these, a low cost acquisition system was used. The system is commonly known and the *Arduino* microcontroller. The microcontroller model used as the *Arduino MEGA 2560 R3* and two accelerometers (models *MMA 7361*). The experimental apparatus with the detail of the accelerometers are shown in figure 7(a) and the figure 7(b) shows the electrical diagram of how to assemble the

(a) 1^{st} Mode (b) 1^{st} Mode (c) 2^{nd} Mode (d) 2^{nd} Mode (e) 3^{rd} Mode (f) 3^{rd} Mode

Figure 6 – Three vibration modes in bending to the portal frame and beam

sensors in the microcontroller board.

(a) Instrumentation



(b) Assembly layout arduino

Figure 7 – Experimental apparatus

RESULTS OF THE EXPERIMENTAL TESTS

After the instrumentation, the system was subject to initial condition of displacement, so that the free motion of the system could be recorder using the microcontroller. This experimental analysis was repeated to estimate the main stiffness, the second stiffness and the damping ratio.

Estimation of the Main Stiffness - Portal frame

To estimate the main stiffness the structure (portal frame), a mass of 6.5 kg was positioned on the top of the portal frame, so it would behave similarly to one degree of freedom system. The free time response of the system was recorded and later transformed to the frequency domain using the Fast Fourier Transform (FFT) and the results is shown in figure 8.

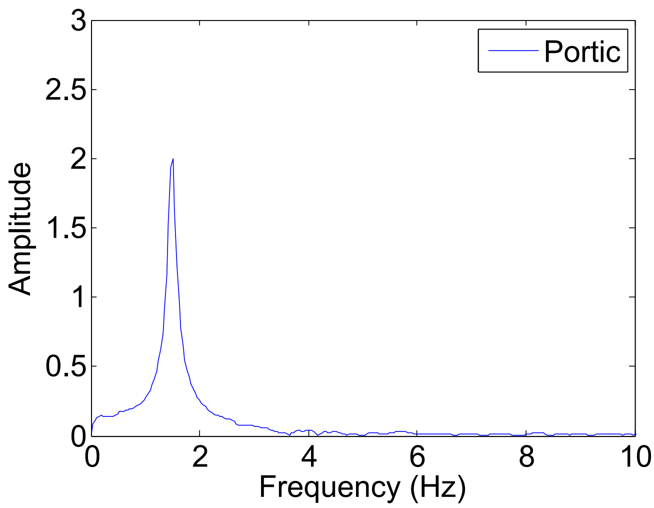


Figure 8 – Fast Fourier Transform of the free time response.

The frequency corresponding to the peak amplitude was estimated as $f_n = 1.5$ Hz. With this result, the stiffness was estimated to be $k = 290$ N/m. Using the values of Young's Modulus and considering the analytical stiffness of two clamped beams, $k = 2 \times 3EI/L^3$, where E is the Young's Modulus, I is the second moment of area and L is the length, the analytical value of the stiffness would be $k_a = 296$ N/m. Therefore there is a good agreement between the experiment and the theory.

Estimation of the Secondary Stiffness - Beam

The investigation of the second stiffness was conducted in a similar manner, using the same microcontroller and accelerometer from the previous case. The beam was subject to a condition of initial displacement and it's free time response was recorded. The second stiffness was varied with the use of other beams added in layers. The FFT of different beam configurations is shown in figure 9.

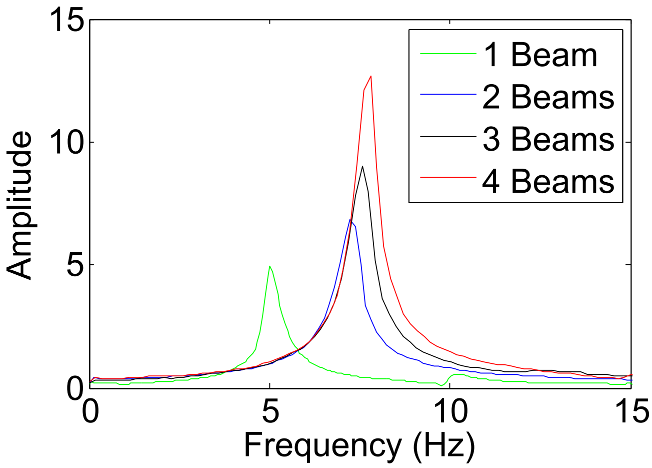


Figure 9 – Fast Fourier Transforms of the free time response for different beam configurations

The correspondent peaks in the results of figure 9 are shown in table 1

Table 1 – Natural frequency for different beam configurations

Number of Beams	1	2	3	4
Natural Frequency (Hz)	5.45	7.13	7.45	7.64

To estimate the stiffness, the expression for the natural frequency in bending of a clamped beam is used,

$$\omega_1 = \left(\frac{EI}{\rho S}\right)^{\frac{1}{2}} \frac{(\beta_1 L)^2}{L^2} \tag{21}$$

where ω_n is the natural frequency, E is the elasticity modulus of the material, I is the second moment of area, ρ is the density of the material, S is the cross section area, L is the length of the section and β_1 is the value of the root of a clamped free beam, for the first mode of vibration it is 1.875104. The estimation of the stiffness for each beam configuration is shown in table 2

Table 2 – Values of secondary the secondary stiffness with variation of number of beams

Number of Beams	1	2	3	4
Stiffness (N/m)	203	340	372	393

Estimation of Damping

The damping of the system was calculated using the method of logarithmic decrement. For this the free time response for initial condition of displacement was measured for two cases. The first case, only the structural damping was present. The second case, a viscous damper was introduced as shown in the detail of figure 7(a). The free time response for these two cases is show in figure 10

To estimate the damping using the method of logarithmic decrement it is necessary to measure the two amplitudes separated by a number n of periods (x_1 and x_{n+1}), for the first case, the these where $x_1 = 0.1437$ e $x_{n+1} = 0.038$, for the second case $x_1 = 0.1425$ and $x_{n+1} = 0.1045$, both with 17 periods between them, where n is the number of periods between them.

The decrement ratio is calculate as

$$\delta = \frac{1}{n} \log\left(\frac{x_1}{x_{n+1}}\right) \tag{22}$$

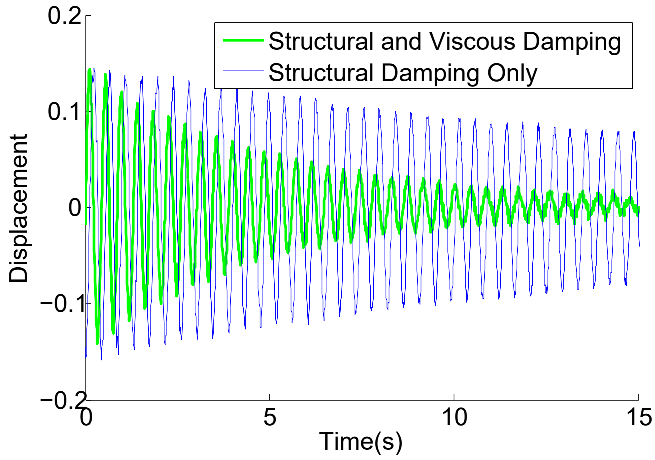


Figure 10 – Experimental damping result with *Arduino*

and it is related to the damping ratio as

$$\xi = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \tag{23}$$

To estimate the damping coefficient it is used

$$\xi = 2\xi m\omega_n \tag{24}$$

The results in table 3 shows the results of the estimation of damping ratio for the two situations.

Table 3 – Damping coefficient ξ

Structural Damping Only	0.0029
Structural and Viscous Damping	0.0125

The value of the viscous damping coefficient calculated for the second case was $c = 0.357Ns/m$

DISCUSSION OF EXPERIMENTAL RESULTS

In Table 4 shows all the experimental results of the natural frequencies of the portal frame and the secondary beams with and without the additional beams.

Table 4 – Analysis of experimental results (Hz)

	Portal frame with mass	Portal frame without mass	1 Beam	2 Beams	3 Beams	4 Beams
Arduino	1.505	4.538	5.450	7.133	7.452	7.641
Analytical	1.503	4.443	5.485	7.096	7.418	7.625
Calculix	1.454	3.464	5.280	7.112	7.430	7.621

With this table, it is seen that the results of the experimental, analytical and finite element are so close, and with this data is possible to estimate the stiffness and the damping of the structures and the results will be reliable.

CONCLUSION

This work has presented a discussion of the Zener model with nonlinear stiffness element of cubic type. The Zener model has been used in literature to represent the viscoelastic behavior of structures and materials. This work has presented an initial analysis using the harmonic balance method to calculate the frequency response function of this model with nonlinear stiffness.

Three different forms of analysis results are presented and compared between them, the first through the low cost device *Arduino*, the second by analytical methods and the third method is through the finite element software *GMSH* and the solver *Calculix*.

The second part of this work present the construction of an experimental apparatus to study and dynamic behavior of the structure. The experimental test apparatus was developed using low cost acquisition system and sensors and it was possible to investigate the values of the stiffness and damping of the elements of the Zener Model. Future work proposes the use of magnets to impose forces similar to the cubic stiffness, such that this behavior can be analyzed experimentally and compared to the numerical simulations.

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