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Lagrange's, Maggi's and Kane's equations applied to the dynamic modelling of serial manipulator

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Abstract: Robot Manipulators have been employed in many types of industries, such as pharmaceutical, chemical, automotive, aerospace, etc. A manipulator is a mechanism used to move an object along a given trajectory. Topologically, the mechanism can be constituted by parallel or serial chains. The serial kinematic chain is constituted by links connected sequentially by joints. The aim of this work is to obtain a qualitative comparison among three approaches typically applied to the modelling of multibody mechanical systems. The chosen system is a 5-DOF serial robot manipulator and the three approaches are based on the use of Lagrange's, Maggi's and Kane's equations. The purpose of the modelling is to obtain the equations of motion for this serial robotic manipulator. Some numerical simulations are performed to illustrate how the obtained models can be used to predict the dynamic behavior of the chosen system.

Keywords: Multibody Dynamics, Analytical Mechanics, Lagrangian formulation, Kane's formalism, Maggi's equations, Robotic Manipulators.

INTRODUCTION

The equations of motion for multibody systems can be obtained from different methodologies such as Newton-Euler equations, Lagrange equations, Kane's Method and Maggi's equations.

The main advantage of Newton-Euler Method (Tenenbaum, 2006) is that the equations of motion will always have the same fundamental form independently of the geometry, inertia or constraints of motion of a rigid body. On the other hand, the constraint forces or torques must be determined, which may lead to difficulties when the system is composed by many bodies.

Regarding to Lagrangian formalism (Gantmacher, 1970; Meirovitch, 2001), it allows obtain constraint-free ordinary differential equations, which is an important advantage when compared to Newton-Euler Method. Many commercial softwares for Multibody Dynamic Systems, such as ADAMS, DADS and DYMAC, apply Lagrangian formulation (Tsai, 1999).

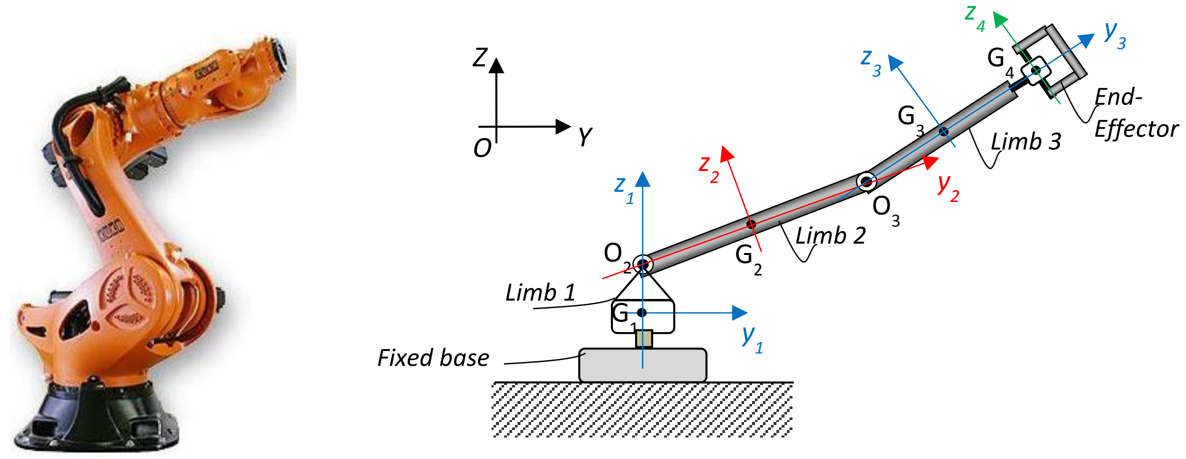
Another method used in commercial softwares for Multibody Dynamic Systems, such as SD-EXACT, NBOD2 and SD/FAST (Tsai, 1999), is Kane's. Developed at the time of the first applications of computational tools to the study of multibody systems, at the 1960s, Kane's method (Kane, 1985) is claimed by some authors not to be original as a theoretical formalism (Baruh, 1999; Desloge, 1987; Papastavridis, 1988), once it is based on the developments of Appell and Maggi dating from the early twentieth century. Kane's method, however, can be considered as a specialization of previouly developed methodologies for optimized computational analysis of multibody systems. As well as in Lagrangian formalism, Kane's approach allows obtain constraint-free ordinary differential equations.

Maggi's equations inspired not only Kane's developments but also are associated to several methodologies based on the use of orthogonal complement projections (Khan et al., 2005; Laulusa; Bauchau, 2008; Saha; Angeles, 1991). Such an approach allows the use of redundant variables in a given formulation without having to deal with the inconvenience of introducing undetermined multipliers. In its original form, Maggi's equations are an extension of the Lagrangian formalism, in which the application of a projection operator (orthogonal complement matrix) eliminates the terms containing the multipliers.

This work deals with a qualitative comparison of three approaches typically applied to the modeling of multibody mechanical systems. The chosen system is a serial robot manipulator and the three approaches are based on the use of Lagrange's, Maggi's and Kane's equations. The aim of modeling is to obtain the equations of motion for this serial robot manipulator. Moreover, the numerical simulations are performed with these models in order to analyze some typical motions performed by the system. Finally, a qualitative assessment of each approach is performed considering four features associated with these approaches.

DYNAMIC MODELING

The serial robot manipulator shown in Fig. 1 has 5-DOF. Figure 1a shows a typical industrial serial manipulator and Fig. 1b shows the serial manipulator model.



(a) (b)

Figure 1- (a) Typical Serial Manipulator Robot (KUKA, 2016); (b) Serial Manipulator Model.

Table 1 shows the robot movements and the generalized coordinates chosen in the Kane's, Lagrange's and Maggi's approaches. In order to simplify the dynamic modelling of this multibody mechanical system, the following hypotheses are adopted:

- All limbs of the robot are considered rigid bodies.
- The frictional torque at all joints is negligible.
- The following axes are ones of symmetry: x_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3. Consequently, the product of inertia of the bodies 1, 2 and 3 are nulls.
- The moments of inertia of body 4 (end-effector) are negligible. The mass of the object moved by the robot was added to the mass of end-effector.
- The O_4 point of the frame $O_4x_4y_4z_4$ represents the centre of mass of the end-effector.

Table 1- Movements of bodies of the manipulator robot.

Body Number	Robot Component	Movements (moving frames)	Generalized Coordinates
1	Limb 1	Rotation around the z_1 axis	γ_1
2	Limb 2	Rotation around the x_2 axis	α_2
3	Limb 3	Rotation around the x_3 axis	α_3
4	End-effector	Rotation around the y_3 and z_4 axes	β_4 e γ_4

Formulation based on Kane's method

The equations of motion of this serial manipulator robotic exposed in Fig. 1 can be obtained by the Kane's equations (Kane, 1985):

$$F_r + F_r^* = 0 \qquad (r = 1, 2, 3) \tag{1}$$

Where F_r is the generalized active forces associated to r-th coordinate and F_r* is the generalized inertia forces associated to r-th coordinate.

The generalized active forces F_r are obtained by the following equation:

$$F_r = \sum \mathbf{F_k} \cdot \frac{\partial \mathbf{v_{Gk}}}{\partial u_r} + \sum \mathbf{M_k} \cdot \frac{\partial \boldsymbol{\omega}_\mathbf{k}}{\partial u_r} \tag{2}$$

Where $\mathbf{F_k}$ and $\mathbf{M_k}$ are the forces and moments actuating on body k, respectively; $\mathbf{v_{Gk}}$ and $\boldsymbol{\omega}_\mathbf{k}$ are the velocity of the centre of mass and the angular velocity of body k, respectively; $\frac{\partial \mathbf{v_{Gk}}}{\partial u_r}$ and $\frac{\partial \boldsymbol{\omega}_\mathbf{k}}{\partial u_r}$ are the partial velocities and partial angular velocities of body *k*, respectively.

The generalized inertia forces F_r* are calculated by the Eq. (3). Notice that F_r* are obtained by dot-multiplying the inertia terms of Newton-Euler equations (Baruh, 1999) by the partial velocities and partial angular velocities.

$$F_r^* = -\sum\left[m_k \mathbf{a_{Gk}} \cdot \frac{\partial \mathbf{v_{Gk}}}{\partial u_r} + \left(J_k \dot{\boldsymbol{\omega}}_\mathbf{k} + \boldsymbol{\omega}_\mathbf{k} \times J_k \boldsymbol{\omega}_\mathbf{k} \right) \cdot \frac{\partial \boldsymbol{\omega}_\mathbf{Gk}}{\partial u_r} \right] \tag{3}$$

Where $\mathbf{a_{Gk}}$ is the acceleration of the centre of mass of body *k*. By applying Eq. (1), (2) and (3), we obtain the equations of motion for the serial mechanism exposed in Fig. 1:

$$\mathbf{F} = \mathbf{M}(\mathbf{q})\dot{\mathbf{u}} + \mathbf{V}(\mathbf{q},\mathbf{u}) \tag{4}$$

Where

$$\mathbf{F} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \qquad \dot{\mathbf{u}} = \begin{bmatrix} \ddot{\gamma}_1 \\ \ddot{\alpha}_2 \\ \ddot{\alpha}_3 \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{5}$$

$$\begin{cases} M_{11} = J_{z_1} + m_2 \frac{\ell_2^2}{4} cos^2 \alpha_2 + J_{y_2} sin^2 \alpha_2 + J_{z_2} cos^2 \alpha_2 + m_3 [\ell_2^2 cos^2 \alpha_2 + \frac{\ell_3^2}{4} cos^2(\alpha_2 + \alpha_3) + \ell_2 \ell_3 cos(\alpha_2 + \alpha_3) cos \alpha_2] \\ + J_{y_3} sin^2(\alpha_2 + \alpha_3) + J_{z_3} cos^2(\alpha_2 + \alpha_3) + m_4 [\ell_2^2 cos^2 \alpha_2 + 2_2 \ell_3 cos(\alpha_2 + \alpha_3) cos \alpha_2 + \ell_3^2 cos^2(\alpha_2 + \alpha_3)] \\ M_{12} = M_{13} = M_{21} = M_{31} = 0 \\ M_{22} = m_2 \frac{\ell_2^2}{4} + J_{x_2} + m_3(\ell_2^2 + \frac{\ell_3^2}{4} + \ell_2 \ell_3 cos \alpha_3) + J_{x_3} + m_4(\ell_2^2 + \ell_3^2 + 2\ell_2 \ell_3 cos \alpha_3) \\ M_{23} = m_3(\frac{\ell_3^2}{4} + \frac{\ell_2 \ell_3}{2} cos \alpha_3) + J_{x_3} + m_4(\ell_3^2 + \ell_2 \ell_3 cos \alpha_3) \\ M_{32} = M_{23} = m_3(\frac{\ell_3^2}{4} + \frac{\ell_2 \ell_3}{2} cos \alpha_3) + J_{x_3} + m_4(\ell_3^2 + \ell_2 \ell_3 cos \alpha_3) \\ M_{33} = m_3 \frac{\ell_3^2}{4} + J_{x_3} + m_4 \ell_3^2 \end{cases} \tag{6}$$

$$\begin{cases} V_1 = -m_2 \frac{\ell_2^2}{2} sin\alpha_2 cos \alpha_2 \dot{\gamma}_1 \dot{\alpha}_2 + 2(J_{y_2} - J_{z_2}) sin\alpha_2 cos \alpha_2 \dot{\gamma}_1 \dot{\alpha}_2 + 2J_{y_3} sin(\alpha_2 + \alpha_3) cos(\alpha_2 + \alpha_3)(\dot{\alpha}_2 + \dot{\alpha}_3)\dot{\gamma}_1 \\ - 2J_{z_3} sin(\alpha_2 + \alpha_3) cos(\alpha_2 + \alpha_3)(\dot{\alpha}_2 + \dot{\alpha}_3)\dot{\gamma}_1 + m_3[-2\ell_2^2 sin\alpha_2 cos \alpha_2 \dot{\alpha}_2 - \frac{\ell_3^2}{2} sin(\alpha_2 + \alpha_3) cos(\alpha_2 + \alpha_3)(\dot{\alpha}_2 + \dot{\alpha}_3) \\ - \ell_2 \ell_3 sin(\alpha_2 + \alpha_3)(\dot{\alpha}_2 + \dot{\alpha}_3) cos \alpha_2 - \ell_2 \ell_3 cos(\alpha_2 + \alpha_3)(\dot{\alpha}_2 + \dot{\alpha}_3) sin\alpha_2 \dot{\alpha}_2]\dot{\gamma}_1 \\ + m_4[-2\ell_2^2 sin\alpha_2 cos \alpha_2 \dot{\alpha}_2 - 2\ell_3^2 sin(\alpha_2 + \alpha_3) cos(\alpha_2 + \alpha_3)(\dot{\alpha}_2 + \dot{\alpha}_3) - 2\ell_2 \ell_3 sin(\alpha_2 + \alpha_3)(\dot{\alpha}_2 + \dot{\alpha}_3) cos \alpha_2 \\ - 2\ell_2 \ell_3 cos(\alpha_2 + \alpha_3) sin\alpha_2 \dot{\alpha}_2]\dot{\gamma}_1 \\ \\ V_2 = -m_3[\ell_2 \ell_3 sin\alpha_3(\dot{\alpha}_2 + \frac{\dot{\alpha}_3}{2})\dot{\alpha}_3] - m_4[\ell_2 \ell_3 sin\alpha_3(2\dot{\alpha}_2 + \dot{\alpha}_3)\dot{\alpha}_3] - \{ -m_2 \frac{\ell_2^2}{4} sin\alpha_2 cos \alpha_2 \dot{\gamma}_1^2 + (J_{y_2} - J_{z_2}) sin\alpha_2 cos \alpha_2 \dot{\gamma}_1^2 \\ - m_3[\ell_2^2 sin\alpha_2 cos \alpha_2 + \frac{\ell_3^2}{4} sin(\alpha_2 + \alpha_3) cos(\alpha_2 + \alpha_3) + \ell_2 \ell_3 sin(2\alpha_2 + \alpha_3)]\dot{\gamma}_1^2 + (J_{y_3} - J_{z_3}) sin(\alpha_2 + \alpha_3) cos(\alpha_2 + \alpha_3)\dot{\gamma}_1^2 \\ - m_4[-\ell_2^2 sin\alpha_2 cos \alpha_2 - \ell_2 \ell_3 sin(2\alpha_2 + \alpha_3) - \ell_3^2 sin(\alpha_2 + \alpha_3) cos(\alpha_2 + \alpha_3)]\dot{\gamma}_1^2 \} + m_2 g \frac{\ell_2}{2} cos \alpha_2 \\ + m_3 g[\ell_2 cos \alpha_2 + \frac{\ell_3}{2} cos(\alpha_2 + \alpha_3)] + m_4 g[\ell_2 cos \alpha_2 + \ell_3 cos(\alpha_2 + \alpha_3)] \\ \\ V_3 = -m_3 \frac{\ell_2 \ell_3}{2} sin\alpha_3 \dot{\alpha}_2 \dot{\alpha}_3 - m_4 \ell_2 \ell_3 sin\alpha_3 \dot{\alpha}_2 \dot{\alpha}_3 - \{ -m_3[\frac{\ell_3^2}{4} sin(\alpha_2 + \alpha_3) cos(\alpha_2 + \alpha_3)\dot{\gamma}_1^2 + \frac{\ell_2 \ell_3}{2} sin(\alpha_2 + \alpha_3) cos \alpha_2 \dot{\gamma}_1^2 \\ + \ell_2 \ell_3 sin\alpha_3(\dot{\alpha}_2 + \dot{\alpha}_3)\dot{\alpha}_2] + (J_{y_3} - J_{z_3}) sin(\alpha_2 + \alpha_3) cos(\alpha_2 + \alpha_3)\dot{\gamma}_1^2 \} - m_4[-\ell_2 \ell_3 sin(\alpha_2 + \alpha_3) cos \alpha_2 \\ - \ell_3^2 sin(\alpha_2 + \alpha_3) cos(\alpha_2 + \alpha_3)]\dot{\gamma}_1^2 - \ell_2 \ell_3 sin\alpha_3(\dot{\alpha}_2 + \dot{\alpha}_3)\dot{\alpha}_2] + m_3 g \frac{\ell_3}{2} cos(\alpha_2 + \alpha_3) + m_4 g \ell_3 cos(\alpha_2 + \alpha_3) \end{cases} \tag{7}$$

Where: $\mathbf{q}$ is the set of generalized coordinates; m_k is the mass of body k (k = 1, 2, 3, 4); Jx_k, Jy_k, Jz_k are the central inertia moments of body k (k = 1, 2, 3); ℓ_2, ℓ_3 are the length of bodies 2 and 3, respectively; T_1, T_2 and T_3 are the actuators torque applied to the bodies 1, 2 and 3, respectively.

Formulation based on the Lagrangian formalism

The Lagrange equations for open chain such as the serial mechanism as shown in Fig. 1 can be expressed as (Gantmacher, 1970):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=Q_{qi} \qquad (i = 1, 2, 3) \tag{8}$$

Where L is called Lagrangian, which is equal to the kinetic energy of system minus the potential energy; q_i represents the generalized coordinates of link i and $\dot{q}_i$ is the time derivative of the generalized coordinates. The Lagrange equations for the serial mechanism exposed in Fig. 1 are given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\gamma}_1}\right)-\frac{\partial L}{\partial \gamma_1}=Q_{\gamma_1} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}_2}\right)-\frac{\partial L}{\partial \alpha_2}=Q_{\alpha_2} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}_3}\right)-\frac{\partial L}{\partial \alpha_3}=Q_{\alpha_3} \tag{9}$$

By developing Eq. (9), we obtain the equations of motion for the serial mechanism shown in Fig. 1:

$$\mathbf{Q}=\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}+\mathbf{V}(\mathbf{q},\dot{\mathbf{q}}) \tag{10}$$

Where

$$\mathbf{Q}=\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix} \qquad \mathbf{M}=\begin{bmatrix}M_{11}&M_{12}&M_{13}\\M_{21}&M_{22}&M_{23}\\M_{31}&M_{32}&M_{33}\end{bmatrix} \qquad \ddot{\mathbf{q}}=\begin{bmatrix}\ddot{\gamma}_1\\\ddot{\alpha}_2\\\ddot{\alpha}_3\end{bmatrix} \qquad \mathbf{V}=\begin{bmatrix}V_1\\V_2\\V_3\end{bmatrix} \tag{11}$$

Where T_1, T_2 and T_3 are the actuators torque applied to the bodies 1, 2 and 3, respectively. The elements of the matrices **M** and **V** in eq. (11) were shown in eq. (6) and (7).

Formulation based on Maggi's Equations

After deriving the Lagrangian equations of motion for the mechanism, it can be noticed that most of the difficulties in this procedure arise from the complexity of the expressions of the energy functions involved. The hypothesis concerning the negligle moments of inertia of body 4 is rather fortunate, otherwise the algebraic complexity of the terms in these equations would be even greater. These difficulties could be overcome if the positions, orientations, velocities and angular velocities of the rigid-bodies could be described in terms of simpler expressions. The most straightforward alternative for doing so is to define redundant generalized coordinates and quasi-velocities so that the energy functions become as simple as possible. Such an approach has proven to be successful for the mathematical modeling of parallel mechanisms (Orsino et. al., 2015; Orsino and Hess-Coelho, 2013). In case of redundancy in the definition of generalized coordinates, the application of the Lagrangian formalism would require the use of undetermined multipliers. Maggi's equations, however, provide an alternative formalism in which one can still take advantage of a modeling based on redundant variables, without needing to use Lagrangian multipliers.

Let (q_i) denote a generic set of redundant generalized coordinates adopted for the description of the motion of a given multibody system. In order to this set of variables be able to represent configurations and states compatible with the constraints of the system, it must be required for them to satisfy some constraint equations. These equations might be written in one of the following forms:

$$h_k(t,q_i)=0 \tag{12}$$

$$n_k(t,q_i,\dot{q}_i)=0 \tag{13}$$

Holonomic constraints can be represented by expressions in both forms (12) and (13). Constraints which can only be represented by an expression in the form of (13) but not by any expression in the form of (12) are nonholonomic. Maggi's equations are applicable both to holonomic and nonholonomic systems whose constraints can at least be expressed in the form of (13). Both the time derivative of (12) and the second time derivative of (13) can be expressed as an affine equation in terms of $\ddot{q}_r$:

$$\sum_r A_{kr}\left(t,q_i,\dot{q}_i\right)\ddot{q}_r + b_k\left(t,q_i,\dot{q}_i\right) = 0 \tag{14}$$

In a given time instant t^*, consider that the state $(t^*; q_i^*; \dot{q}_i^*)$ is known, so that the variations of the coordinates and of its time derivatives can be assumed to be zero. In order for an infinitesimal variation of $\ddot{q}_r$ not to violate any constraint of the system, the following condition must be satisfied:

$$\sum_r A_{kr}\left(t,q_i^*,\dot{q}_i^*\right)\delta\ddot{q}_r = 0 \tag{15}$$

Therefore, in a time interval defined in a neighborhood of t^*, any virtual displacement must satisfy the condition (Orsino, 2016):

$$\sum_r A_{kr}\delta q_r = 0 \tag{16}$$

Let $A = [A_{kr}]$ denote the matrix constituted by the coefficients A_{kr}. A general solution for (16) involves finding a matrix $C = [C_{rs}]$ which is an orthogonal complement of A, i.e. a maximal rank matrix satisfying the condition $AC = 0$ (Orsino, 2016). Thus, the variations δq_r associated to the generalized coordinates q_r can be expressed as a linear combination of as much arbitrary variations (denoted by $\delta\theta_s$) as the number ν of degrees of freedom of the system, i.e.:

$$\delta q_r = \sum_s C_{rs}\delta\theta_s \tag{17}$$

Therefore, applying this result to the extended Hamilton's Principle, it can be stated that the Maggi's equations for this system are given by the following expression:

$$\sum_r C_{rs}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} - Q_{qr}\right] = \sum_r C_{rs}\Phi_{qr} = 0 \qquad s = 1,\ldots\nu \tag{18}$$

Analysing the foregoing derivations, it is convenient to define an extra angular generalized coordinate $\chi_3 = \alpha_2 + \alpha_3$, and $(x_j\,;\,y_j\,;\,z_j)$ representing the Cartesian coordinates of the centres of mass G_j of the limbs $j = 2$ and $j = 3$, and of the end-effector $j = 4$:

$$\begin{aligned} x_j &= -e_j\ell_2 \sin\gamma_1 \cos\alpha_2 - f_j\ell_3 \sin\gamma_1 \cos(\alpha_2+\alpha_3) \\ y_j &= e_j\ell_2 \cos\gamma_1 \cos\alpha_2 + f_j\ell_3 \cos\gamma_1 \cos(\alpha_2+\alpha_3) \\ z_j &= e_j\ell_2 \sin\alpha_2 + f_j\ell_3 \sin(\alpha_2+\alpha_3) \end{aligned} \tag{19}$$

With $e_2 = ½$, $f_2 = 0$, $e_3 = 1$, $f_3 = ½$, $e_4 = 1$ and $f_4 = 1$. In this case, taking $\delta q = (\delta\gamma_1, \delta\alpha_2, \delta\alpha_3, \delta\chi_3, \delta x_j, \delta y_j, \delta z_j)$ and $\delta\theta = (\delta\gamma_1, \delta\alpha_2, \delta\alpha_3)$ it can be stated that $\delta q = C\delta\theta$, with:

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ \hline -\cos\gamma_1\left(e_j\ell_2\cos\alpha_2 + f_j\ell_3\cos\chi_3\right) & -\sin\gamma_1\left(e_j\ell_2\sin\alpha_2 + f_j\ell_3\sin\chi_3\right) & -f_j\ell_3\sin\gamma_1\sin\chi_3 \\ -\sin\gamma_1\left(e_j\ell_2\cos\alpha_2 + f_j\ell_3\cos\chi_3\right) & -\cos\gamma_1\left(e_j\ell_2\sin\alpha_2 + f_j\ell_3\sin\chi_3\right) & -f_j\ell_3\cos\gamma_1\sin\chi_3 \\ 0 & e_j\ell_2\cos\alpha_2 + f_j\ell_2\cos\chi_3 & f_j\ell_3\cos\chi_3 \end{bmatrix} \tag{20}$$

Also:

$$\begin{aligned} L = &\frac{1}{2}J_{z_1}\dot{\gamma}_1^2 + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2\right) + \frac{1}{2}J_{x_2}\dot{\alpha}_2^2 + \frac{1}{2}J_{y_2}\left(sin\,\alpha_2\dot{\gamma}_1\right)^2 + \frac{1}{2}J_{z_2}\left(cos\,\alpha_2\dot{\gamma}_1\right)^2 \\ &+ \frac{1}{2}m_3\left(\dot{x}_3^2 + \dot{y}_3^2 + \dot{z}_3^2\right) + \frac{1}{2}J_{x_3}\dot{\chi}_3^2 + \frac{1}{2}J_{y_3}\left(sin\,\chi_3\dot{\gamma}_1\right)^2 + \frac{1}{2}J_{z_3}\left(cos\,\chi_3\dot{\gamma}_1\right)^2 \\ &+ \frac{1}{2}m_4\left(\dot{x}_4^2 + \dot{y}_4^2 + \dot{z}_4^2\right) + m_2 g z_2 + m_3 g z_3 + m_4 g z_4 \end{aligned} \tag{21}$$

Thus:

$$\begin{cases} \Phi_{\gamma_1} = \left[\left(J_{y_2} + J_{z_2}\right)\left(sin\,\alpha_2\right)^2 + \left(J_{y_3} + J_{z_3}\right)\left(sin\,\chi_3\right)^2 + J_{z_1}\right]\ddot{\gamma}_1 + \left(J_{y_2} + J_{z_2}\right)\left(sin\,2\alpha_2\right)\dot{\gamma}_1\dot{\alpha}_2 \\ \qquad + \left(J_{y_3} + J_{z_3}\right)\left(sin\,2\chi_3\right)\dot{\gamma}_1\dot{\chi}_3 - T_1 \\ \Phi_{\alpha_2} = J_{x_2}\ddot{\alpha}_2 - \left(J_{y_2} + J_{z_2}\right)\left(sin\,\alpha_2\,cos\,\alpha_2\right)\dot{\gamma}_1^2 - T_2 \\ \Phi_{\alpha_3} = -T_3 \\ \Phi_{\chi_3} = J_{x_3}\ddot{\chi}_3 - \left(J_{y_3} + J_{z_3}\right)\dot{\gamma}_1^2\,sin\,\chi_3\,cos\,\chi_3 \\ \Phi_{x_j} = m_j\ddot{x}_j \\ \Phi_{y_j} = m_j\ddot{y}_j \\ \Phi_{z_j} = m_j\left(\ddot{z}_j - g\right) \end{cases} \tag{22}$$

The system of Maggi's equations for the manipulator can be readily obtained by Eq. (18), using the expressions of C_{rs} from (20) and the expressions for Φ_{qr} from (22).

Numerical Simulations

The equations of motion obtained by Kane's, Lagrange's and Maggi's approaches are the same. In this section, some numerical simulations are performed to illustrate how the obtained models can be used to predict the dynamic behavior of the manipulator. Table 2 shows the robot parameters employed in the simulations.

Table 2- Robot parameters employed in the simulations.

Robot Parameters	Symbol	Value
Moment of inertia of the body 1with respect to the Z_1 axis [kgm^2]	J_{z1}	3
Mass of body 2 [kg]	m_2	3
Length of body 2 [m]	ℓ_2	1
Moment of inertia of the body 2 with respect to the x_2 axis [kgm^2]	J_{x2}	0.36
Moment of inertia of the body 2 with respect to the y_2 axis [kgm^2]	J_{y2}	0.07
Moment of inertia of the body 2 with respect to the z_2 axis [kgm^2]	J_{z2}	0.36
Mass of body 3 [kg]	m_3	3
Length of body 3 [m]	ℓ_3	1
Moment of inertia of the body 3 with respect to the x_3 axis [kgm^2]	J_{x3}	0.36
Moment of inertia of the body 3 with respect to the y_3 axis [kgm^2]	J_{y3}	0.07
Moment of inertia of the body 3 with respect to the $z_3$axis [kgm^2]	J_{z3}	0.36
Mass of body 4 [kg]	m_4	1

In order to perform numerical simulations, the expressions of the redundant variables χ_3, x_j , y_j and z_j (j = 2; 3; 4) and their time derivatives in terms of γ_1, α_2 and α_3 can be either replaced in Maggi's equations, leading to a system of ordinary differential equations (ODEs) with as much equations as the number of degrees of freedom of the manipulator (3, in this case), or can simply be taken along with the already obtained Maggi's equations, leading to an extended system of equations which are typically referred as differential-algebraic equations (DAEs). The former option, in this case, for the particular matrix C in Eq. (20), would lead to a system of equations identical to the foregoing Lagrangian equations of motion. Therefore, in the numerical simulations performed, the latter option is chosen.

Two inverse simulations and one forward simulation are performed. In the inverse ones, Maggi's equations are used to calculate the time histories of the torques provided by the actuators in order to perform a given prescribed motion. In the forward one, slight variations of the static values of the torques (the values that ensure the equilibrium of the system in the reference configuration in which the values of γ_1, α_2 and α_3 and their time derivatives are zero) are considered, and the output obtained are the time histories of the associated motion, provided by the integration of the corresponding equations of motion.

Both in the inverse and forward simulations, the interpolating curves use to define the time histories of the inputs respect two properties: their second time derivatives are sinusoidal and the extremum points are also inflection points (i.e. first and second time derivatives are simultaneously zero). In the case of the inverse simulations, two similar scenarios: in the second, the rates are 2.5 times faster than in the first. This allows to make an assessment of the influence of the inertial effects in the motion of this mechanism (once they must be much more influent in this second scenario). The results of these numerical simulations are shown in Figures 2–4.

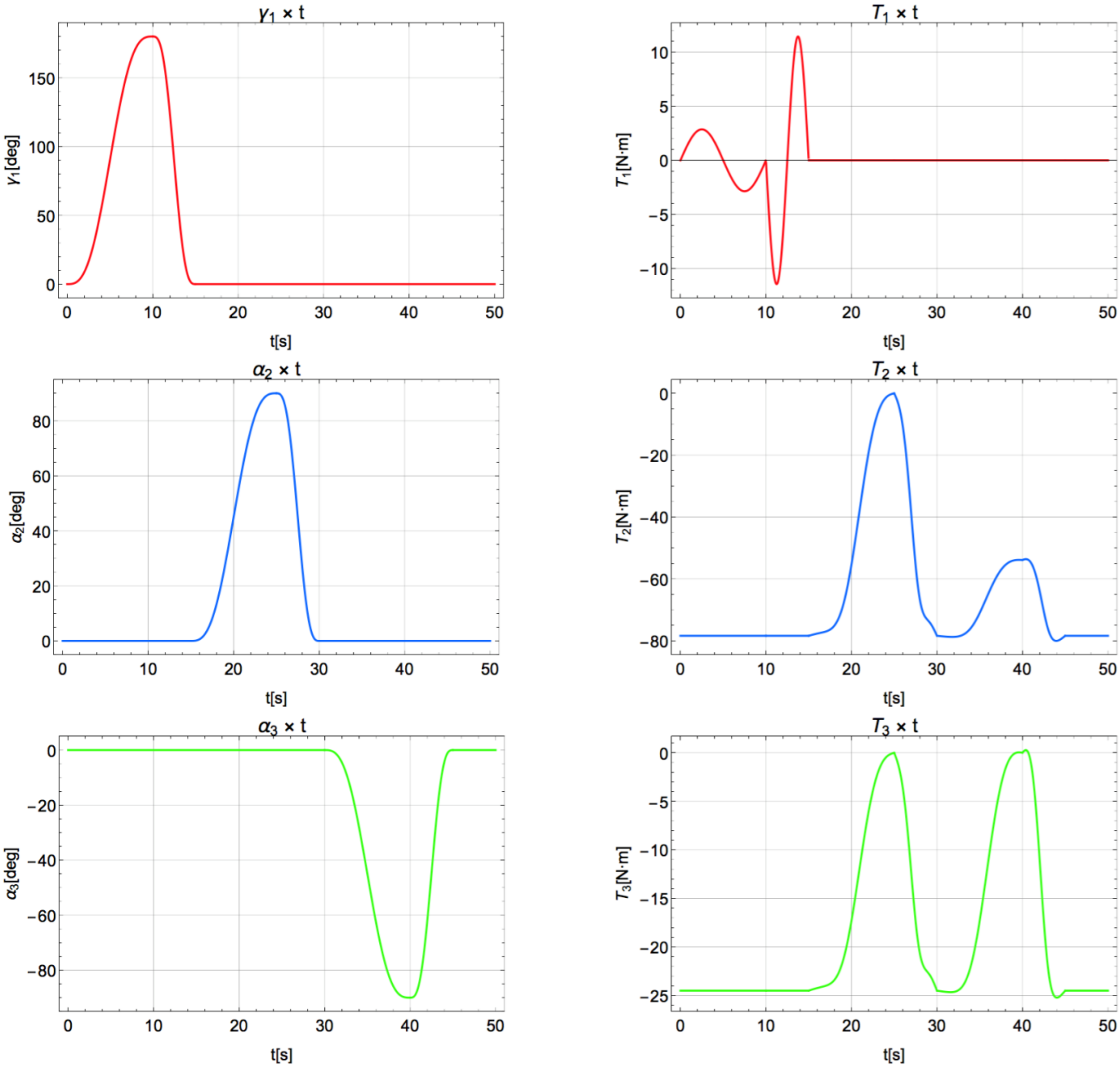


Figure 2 - First inverse simulation: slow prescribed motion.

The results seem to be consistent with the existing cylindrical symmetry of the mechanism (the motions with respect to the vertical plane passing through the centres of mass of 2, 3 and 4 is not influenced by the torque T_1) and reveal, as expected for a serial mechanism, a significant influence of the inertial effects in the torque imposed (being greater the closer the actuator is to the base of the mechanism).

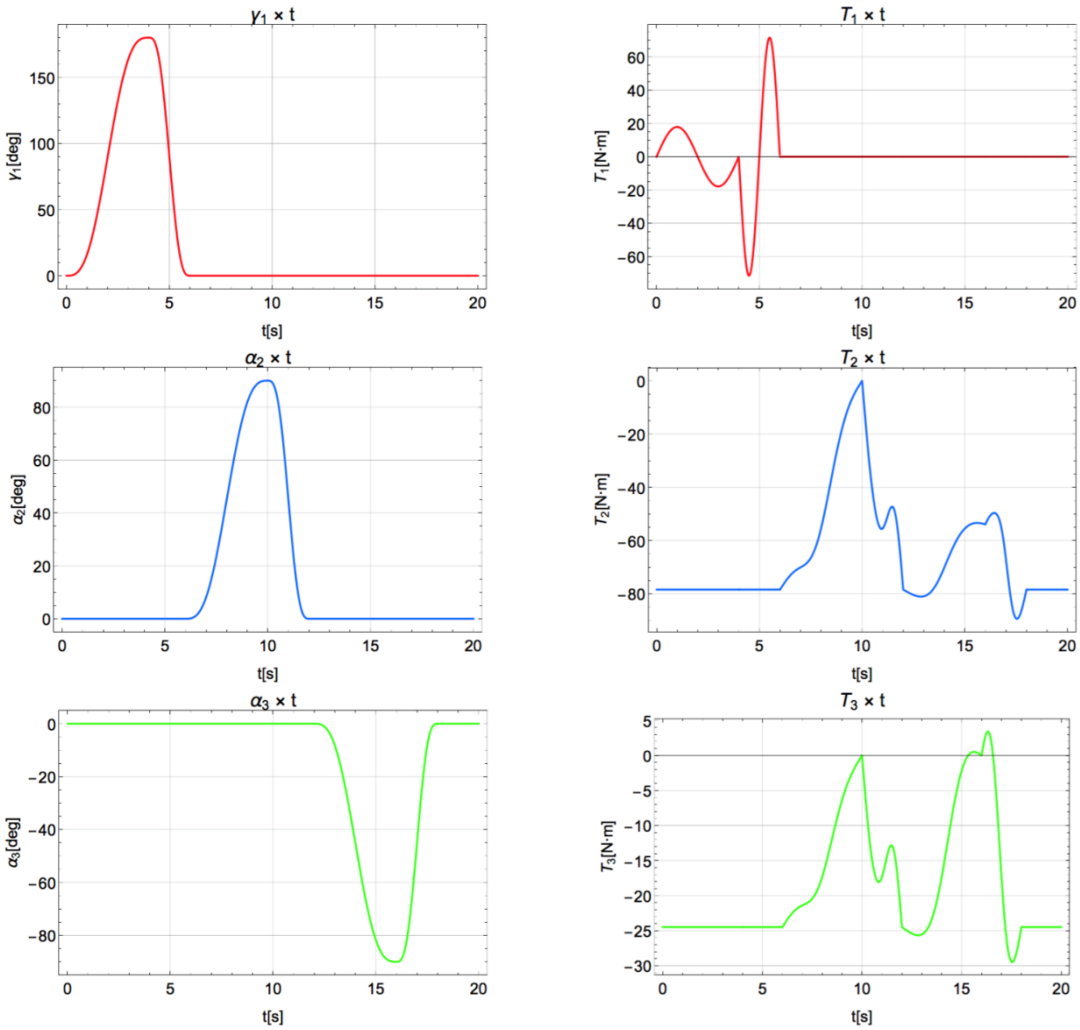


Figure 3 - First inverse simulation: faster prescribed motion.

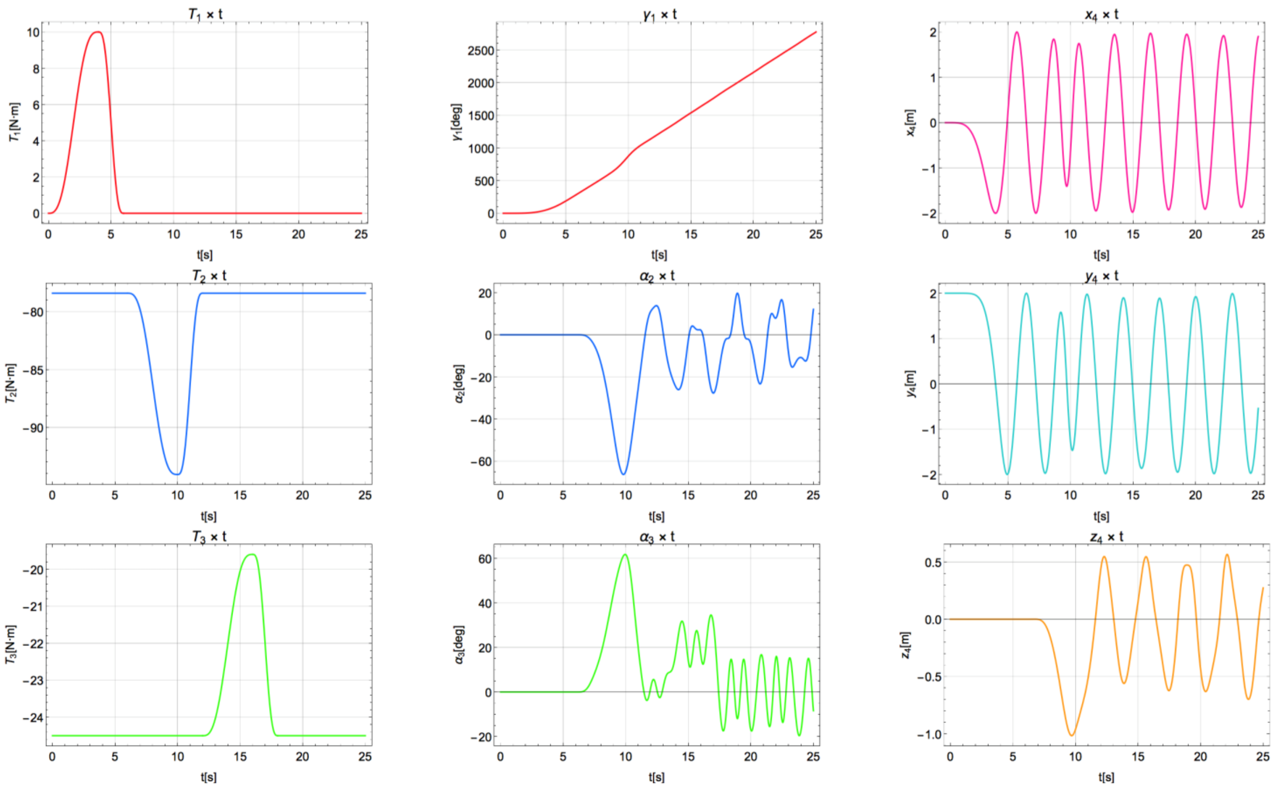


Figure 4 - Forward simulation.

Qualitative assessment of models

In order to conclude the discussion on the suitability of each analytical mechanics approach to model the chosen system, a qualitative assessment is performed based on the results presented in the previous sections. This assessment consists of comparing the following four characteristics associated with these approaches:

1. Derivation: the effort to obtain dynamic equations of the model in a form that is suitable to numerical simulations.
2. Interpretability: the ability to understand the meaning of each term of dynamic equations.
3. Modularity: how simple is the procedure of adding components in a model.
4. Constraint forces elimination: how simple is the procedure of obtaining equations of motion without terms related to constraint forces.

For each of these characteristics, Kane's, Lagrange's and Maggi's approach were ranked in order to guide the selection of an appropriate method to model other kinds of mechanical systems similar to the serial mechanism analyzed in this work. The three approaches receive a grade from 1 to 5, being awarded index equal to five to the method that stands out in a given characteristic in relation to others, as exposed in Table 3.

Table 3- Qualitative assessment of the models: Kane's, Lagrange's and Kane's approach.

Model	Derivation	Interpretability	Modularity	Constraint forces elimination
Kane	2	5	2	5
Lagrange	4	3	2	5
Maggi	5	2	5	5

In terms of derivation, Maggi's approach stand out due to the possibility of obtaining the generalized forces from the partial derivatives of scalar energy functions which, differently from the conventional Lagrangian formulation, do not need to be expressed in terms of a minimal set of variables and thus, can be further simplified by an adequate choice of redundant variables. Kane's equations, on the other hand, require the vector expressions of accelerations and angular accelerations to be obtained, representing the most complex derivation procedure.

Concerning interpretability, however, Kane's equations are the most physically insightful, once the interpretation of each term within the equations of motion follows automatically from their derivation. Such a natural interpretability is not immediate in Lagrangian formulation and is even more difficult when Maggi's equations are used, due to the use of redundant variables and of a projection operator (matrix C).

In terms of modularity, the fact that Maggi's equations allow the use of redundant variables makes it easier to include extra components in a model, once the dynamic description of them does not require the use of the same minimal set of variables adopted for the system itself, which characterizes Maggi's approach as a real modular one.

Finally, none of the methods applied presents any further complexity in terms of constraint force elimination.

CONCLUSIONS

In this work, a 5-DOF serial manipulator robot was modeled using Kane's Lagrange's Equations, and Maggi's equations. The equations of motion obtained by the three methods are the same. There was a certain difficulty related to the dynamic modelling due to the relative movements between the bodies of the chosen system.

Once we apply the Kane's approach it seems easy to work out a computational procedure, which allows to model a system methodically. However, to apply the Kane's approach it is necessary to calculate the acceleration of the centre of mass of each body, which is a disadvantage when compared to the Lagrangian formulation, in which only the velocities of centres of mass and angular velocity expressions of each body are required.

Maggi's method also requires the computation of partial derivatives of the Lagrangian of the system. However, it is possible to take advantage of the use of redundant coordinates to simplify the energy terms as much as possible, which can make the modeling procedure much simpler when compared to the conventional Lagrangian formalism.

Finally, the simulations results are consistent and reveal a significant influence of the inertial effects in the torque imposed as expected for a serial mechanism.

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