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# Stochastic parametric analysis: a new approach to stick-slip oscillations

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*Abstract: In this paper a parametric analysis of a sample of responses of a dry-friction oscillator is performed in order to construct a statistical model. The system consists of a simple oscillator (mass-spring) moving on a base with a rough surface. Due to this roughness, the mass is subject to a dry-frictional force modeled as a Coulomb friction. It is considered that the base has an imposed stochastic bang-bang motion which excites the system in a stochastic way. The base velocity is modeled by a Poisson process for which a probabilistic model is fully specified. The non-smooth behavior of the dry-frictional force associated with the non-smooth stochastic base motion induces in the system stochastic stick-slip oscillations. The system response is composed by a random sequence alternating stick and slip-modes. With realizations of the system, a statistical model is constructed for this sequence. In this statistical model, the variables of interest of the sequence are modeled as random variables, as for example, the number of time intervals in which stick or slip occur, the instants at which they begin, and their duration. Samples of the system response are computed by integration of the dynamic equation of the system using independent samples of the base motion. Statistics and histograms of the random variables which characterize the stick-slip process are estimated for the generated samples. The objective of the paper is to analyze how these estimated statistics and histograms vary with the system parameters, i.e., to make a parametric analysis of the statistical model of the stick-slip process.*

***Keywords: stick-slip, dry-friction oscillator, stick duration, statistical model, parametric analysis.***

## INTRODUCTION

The analysis of a stick-slip dynamics caused by dry-friction is not a new subject. The literature dealing with the problem is vast (Berger, 2002, Feeny *et al.*, 1998, Awrejcewicz and Olejnik, 2007 and Kang *et al.*, 2009) and, reflects the economic interest in understanding the dynamic behavior of this type of non-smooth vibration (Fidlin, 2006, Vande Vandre *et al.*, 1999, Galvanetto, 1999).

Dry-friction appears in several situations, as in drilling process and in mechanical gear systems (Tonazzia *et al.*, 2013 and Claeys *et al.*, 2016 and Wagner *et al.*, 2007). It could be the source of dynamic instability, noise, and reduction of performance (Ritto *et al.*, 2013, Ritto and Sampaio, 2013). In drilling, stick-slip oscillations happen due to the dry-friction force that exists between the bit and the rock. This friction force could be big enough to stuck the bit during some time intervals. As a constant velocity is imposed at the top of the drillstring and during the stick the bit at the bottom does not move, the drillstring is twisted. It accumulates energy in terms of torsion up to the instant that it is suddenly released. This phenomenon generates torsional vibrations and instabilities in the system dynamics. If not controlled, it can be harmful to the drilling process and causes waste of energy. Furthermore, when the bit is stuck, there is no penetration. Due to all these reasons, in this case the stick-slip oscillations are undesirable and should be avoided.

Despite the great number of papers in the area, few of them address the problem with a stochastic approach. The majority of the references that characterizes the dynamics with dry-friction only make it with a deterministic approach. They do not discuss or quantify the uncertainties that are involved in the dynamics, although the dry friction force itself presents an inherent random behavior (Feng, 2003). The influence of ambient conditions in the properties of contact surfaces (Bengisu and Akay, 1999 and Worden *et al.*, 2007) and the dependency on the relative velocity of the bodies in contact make the dry friction force uncertain. Beyond this, dry-friction appears in mechanical systems in which uncertainties play an important role. For example, in drilling some sources of uncertainties are the bit-rock interaction, the presence of impacts, and the fluid-structure interaction (Ritto and Sampaio, 2012, Ritto *et al.*, 2010 and Ritto *et al.*, 2013). Randomness arises also from manufacturing, assembly errors, and random load. Therefore, a stochastic approach is the ideal way to address problems with dry-friction (Lima and Sampaio, 2014, Lima and Sampaio, 2015a and Lima and Sampaio, 2016).

In this paper, we analyze the dynamics of a dry-friction oscillator which moves over a base with a rough surface. The base has an imposed stochastic bang-bang motion which excites the system in a stochastic way. The non-smooth behavior of the dry-friction force (Lima and Sampaio, 2015b) associated with the non-smooth stochastic base motion induces in the system stochastic stick-slip oscillations. The system response is composed by a random sequence alternating stick and slip-modes. To characterize it, we construct a statistical model, in which the variables of interest of the random sequence are modeled as random variables (Lima and Sampaio, 2017). Examples of these variables are the number of time intervals in which stick or slip occur, the instants at which they begin, and their duration.

To estimate statistics and histograms of the system responses, samples of the random sequence of stick and slip-modes are computed by the integration of the dynamic equation of the system using independent samples of the base motion. The objective of the paper is to analyze how the estimated statistics and histograms vary with the system parameters, i.e., to make a parametric analysis of the statistical model of the stick-slip process. Such detailed analysis is new in the literature and it goes far beyond the usual curve of the mean response and an envelope of uncertainty, that are based only in two moments and, of course, cannot characterize the complete statistics.

A parametric analysis of a nonlinear system can be computed by the numerical integration of the dynamic equation of the system for different combinations of the values of the system parameters. In the traditional deterministic parametric analysis, for each combination of the system parameters, a single simulation have to be computed. The obtained responses represent how the system behaves. In a stochastic parametric analysis, for each combination of the system parameters, a large number of integrations have to be computed in order to get samples of the system response. Each realization is just one possibility of system response. From a set with a large number of realizations, statistics and histograms are estimated.

The non-smooth behavior of dry-friction oscillator analyzed in this paper turns the numerical integration of its dynamic very time consuming. To reduce the simulation time, we adopted the strategy of parallelization of the simulations. However, even using a cluster composed of sixteen computers, the computation time necessary to perform the integrations was $55,5$ days.

This paper is organized as follows. Section “Dynamics of the stick-slip oscillator” describes the stick-slip oscillator analyzed. A probabilistic model to the base motion is construct in Section “Construction of a probabilistic model to the base motion”. The definition of the random variables which characterizes the variables of interest in the stick-slip process is made in Section “Construction of a statistical model of the stick-slip process”. An explanation about the choice of the parameters considered in the stochastic parametric analysis is given in Section “Strategy to compute the parametric analysis of the statistical model of the stick-slip process”. In this Section it is also discussed the computational cost required to make the stochastic parametric analysis, and it is presented the strategy adopted to reduce it. The influence of $\lambda$ in the statistics and histograms is discussed in Section “Influence of $\lambda$ in the statistical model of the stick-slip process”, and the influence of $\mu$ is discussed in Section “Influence of the friction coefficient on the statistical model of the stick-slip process”.

## DYNAMICS OF THE STICK-SLIP OSCILLATOR

The system analyzed is composed by a simple oscillator (mass-spring) moving on a rough surface, as shown in Fig. 1. The roughness induces a dry-frictional force between the mass and the base which is modeled as a Coulomb friction.

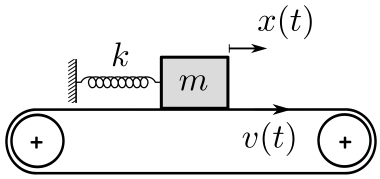



**Figure 1 – Stick-slip oscillator.**

Due to this friction model, the resulting motion of the mass can be characterized in two qualitatively different modes: the stick-mode (in which the mass and base have the same velocity during an open time interval) and the slip-mode, in which mass and base have different velocities (Den Hartog, 1931, Luo and Gegg, 2006, Hundal, 1979, Shaw, 1986 and Hagedorn, 1988). The position of the mass over the base is represented by $x$ and its equation of motion is

$$m\,\ddot{x}(t) + k\,x(t) = f(t)\,, \tag{1}$$

where $m$ is the mass, $x$ is the position of the mass over the base, $k$ is the spring stiffness and $f$ is the frictional force between mass and base. During the slip-mode, it is assumed that $f$ depends on the slip velocity, i.e, depends on $v - \dot{x}$, where $v$ is the base speed. We have

$$f(t) = mg\mu\,\mathrm{sgn}(v(t) - \dot{x}(t))\,, \tag{2}$$

where $g$ is the acceleration of gravity, considered to be 9.8 m/s$^2$, and $\mu$ is the friction coefficient (Jordan and Smith, 2007, Hagedorn and DasGupta, 2007). Since we use the simplest friction model, the friction coefficient is assumed to be constant. Thus, during the slip-mode, the value of the frictional force, $f$, is known. Its absolute value is constant and equal to the maximum friction force, $f_{\max} = \mu mg$. During the stick-mode, the mass velocity is equal to the base speed, i.e. $\dot{x} = v$, and the mass acceleration is equal to the base acceleration, i.e. $\ddot{x} = \dot{v}$. Thus, the equation of motion Eq. (1) can be rewritten as

$$m\dot{v} + k\,x(t) = f(t)\,. \tag{3}$$

The value of the frictional force during the stick-mode varies and it is confined to the interval $-f_{\max} \leq f \leq f_{\max}$. When the base speed is constant in time, the system dynamics is very simple and has an analytical solution. The literature dealing with this configuration is vast (Thomsen and Fidlin, 2003). During the stick-mode we have

$$\begin{array}{rcccl} -f_{\max} & \leq & k\,x(t) & \leq & f_{\max} \\ -mg\mu & \leq & k\,x(t) & \leq & mg\mu\,. \end{array} \tag{4}$$

Then, once in a stick-mode, the mass stays moving with the base until $x(t) = \frac{mg\mu}{k}$ in case of positive base velocity, or until $x(t) = -\frac{mg\mu}{k}$ in case of negative base velocity. Observe that during the stick-mode, the modulus of the elastic force increases up to the limit value $|f_{\max}|$, i.e. the modulus of maximum friction force. Since this maximal value cannot be exceeded, the stick-mode ends and the mass will start to slip. Because of this, considering that base speed is constant in time, knowing the mass position when a stick starts, it is possible to predict its duration. Remark that the duration of the stick-mode is bounded and its maximum value is $d_{\max} = 2\frac{mg\mu}{kv}$. For the slip one can make no prediction, in principle it can last forever. When the base speed is not constant in time, the system dynamics is richer.

## CONSTRUCTION OF A PROBABILISTIC MODEL TO THE BASE MOTION

Considering that the dry-friction oscillator has an imposed stochastic bang-bang motion, we propose to model its velocity as a Poisson process (Cox and Isham, 1980), with constant rate $\lambda$, represented by $\mathcal{V}$. We consider that $\mathcal{V}$ is constant by parts and assumes only two values: $1,0$ m/s and $-1,0$ m/s. A realization of such stochastic process consists of point events in time which represents the instants in which occur changes of the velocity sign of the base motion. The parameter $\lambda$ represents the expected value of number of changes per unit of time. As $\mathcal{V}$ is modeled as a Poisson process, the instants of change are given by random variables which can be ordered as

$$0 < Y_1 < Y_2 < Y_3 < \dots. \tag{5}$$

The structure of the random sequence given in Eq. (5) is well known (Kingman, 2002). From this sequence, it is possible to define the independent random variables

$$W_1 = Y_1, \qquad W_j = Y_j - Y_{j-1} \qquad (j \geq 2). \tag{6}$$

Each of them has exponential probability density function

$$p_{W_j}(t) = \mathbb{1}_{[0,\infty)}(t)\lambda\, e^{-\lambda t}, \tag{7}$$

with mean $1/\lambda$. The random variable $W_j$, with $j \geq 2$, indicates the waiting time between two consecutive change of the velocity sign of the base motion. Observe that a higher $\lambda$ corresponds to a smaller average waiting time. The first change is at $W_1$, the second at $W_1 + W_2$, et cetera. In general, the $j$th change, $Y_j$, is at:

$$Y_j = \sum_{i=1}^{j} W_j\,, \tag{8}$$

The theory of sums of independent random variables can be used to determine the probability density function of $Y_j$, which is

$$p_{Y_j}(t) = \mathbb{1}_{[0,\infty)}(t)\frac{\lambda^j t^{(j-1)} e^{-\lambda t}}{(j-1)!}\,. \tag{9}$$

The imposed discontinuities in the base velocity makes the system dynamics very rich. Due to the bang-bang base motion, if the mass is in the stick-mode in the instant just before the discontinuity on the base velocity, it must be in the slip-mode in the instant just after the discontinuity. Thus, the stick is interrupted by the discontinuities on the base velocity, as if the dynamics were reinitialized; all previous information lost. Remark that with the bang-bang base motion, just knowing the mass position when a stick starts, it is not possible to predict its duration, although the duration of the stick-mode is limited and its maximum value is $d_{\max} = 2\frac{mg\mu}{kv}$. On the other hand, the slip is not interrupted by the discontinuities on the base velocity, and the duration of the slip-mode may be unlimited.

## CONSTRUCTION OF A STATISTICAL MODEL OF THE STICK-SLIP PROCESS

As it was assumed that the base motion is uncertain, the equation of motion of the system, Eq. 1, became a stochastic differential equation. Thus, the response of the stochastic stick-slip oscillator is a random process which presents a sequence alternating stick and slip-modes. We are interested in the stochastic characterization of these sequences. Defined a time interval for analysis, the variables of interest are the number of time intervals in which stick or slip occur, the instants at which they start, and their duration. These variables are modeled as stochastic objects in order to allow the stochastic characterization the dynamics of the oscillator. Thus we have the

- number of time intervals in which stick occurs represented by the discrete random variable $S_T$;
- number of time intervals in which slip occurs represented by the discrete random variable $S_L$;
- instants at which the sticks begin represented by a discrete random process $T_1, \cdots, T_{S_T}$, where the subscripts $1, \cdots, S_T$ indicate the order that they occur, i.e., the instant in which starts the first stick, the second, and so on up to the $S_T$-th stick;
- duration of the sticks represented by a discrete random process $D_1, \cdots, D_{S_T}$, where again the subscripts $1, \cdots, S_T$ indicate the order that they occur;
- instants at which the slips begin represented by a discrete random process $L_1, \cdots, L_{S_L}$, where $1, \cdots, S_L$ indicate the order that they occur;
- duration of the sticks represented by a discrete random process $H_1, \cdots, H_{S_L}$, where $1, \cdots, S_L$ indicate the order that they occur.

Figure 2 shows a sketch of the sequence of sticks and slips in the system response. Observe that we count the first slip just after the first stick, i.e., we have $L_1 > T_1$. Besides, if the system response ends during a slip, the number of sticks is equal or the number of slips, i.e. $S_T = S_L$. If the system response ends during a stick, then $S_T = S_L + 1$ .

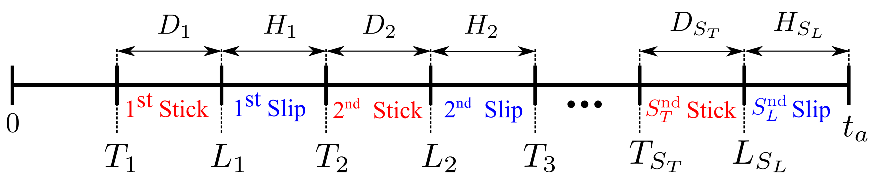



**Figure 2 – Sketch of the sequence of sticks and slips in the system response for the case in which $S_T = S_L$.**

## STRATEGY TO COMPUTE THE PARAMETRIC ANALYSIS OF THE STATISTICAL MODEL OF THE STICK-SLIP PROCESS

As explained in the introduction, the objective of the paper is to analyze how estimated statistics and histograms of the variables of interest which characterize the stick-slip process vary with the system parameters, i.e., make a parametric analysis of the statistical model of the stick-slip process. Two system parameters are considered in this parametric analysis. One of them is related to the probabilistic model of the base motion $\lambda$ and the other is the friction coefficient $\mu$ of the friction force. These parameters were chosen because they have a big influence on the system response.

In Sections 2 and 3, it is explained that the instant at which a stick ends is determined by the occurrence of a discontinuity of the base velocity, or by the growth of the spring force up to the maximum friction force. This way, the values of the parameters $\lambda$ and $\mu$ (respectively the mean of the number of discontinuities on the base velocity per unit of time and friction coefficient) are closely connect to the behavior of the system response. When $\lambda$ is small, meaning that the mean of the number of discontinuities on the base velocity is small, we expect that the majority of the sticks end due to the growth of the spring force up to the maximum friction force, which gave us longer sticks in average. On the other hand, as $\lambda$ grows, we expect that the majority of the sticks end due to occurrence of a discontinuity of the base velocity. This way, the majority of the sticks end before that the spring force reaches the maximum friction force, i.e., in average the sticks are shorter. To the parameter $\mu$ we expect the opposite. When $\mu$ is small, the maximum friction force is small. Consequently the spring force will quickly reach this upper limit and than in average the sticks are shorter. As $\mu$ grows, the maximum friction force grows too and then, in average, the sticks are longer.

To quantify the influence of $\mu$ and $\lambda$ and in the statistics of the stick duration, we performed numerical simulations combining different values of these parameters. To $\lambda$, 40 values were selected nonuniformly in the interval $[0.1, 30, 0]$. To $\mu$, 8 values were selected nonuniformly in the interval $[0.5, 7, 0]$. For each combination of $\lambda$ and $\mu$, the dynamics equations were integrated 2,000 times using independent realizations of the base movement. A previous convergence study was developed to determine the acceptable number of realizations. In total, $640,000$ integrations were performed.

As explained in the introduction, the non-smooth behavior of dry-friction oscillator analyzed in this paper turns the numerical integration of its dynamic very time consuming. Furthermore, as we compute the integrations for high values of $\lambda$, this effort increases even more. One should note that as $\lambda$ represents the expected value of number of changes of the velocity sign of the base motion per second, the increase $\lambda$ implies in a more non-smooth dynamics. Then, small time steps, and relative and absolute tolerances are required. In order to compute the $640,000$ integrations, we adopted the strategy of parallelization of the simulations. Using a cluster composed of sixteen computers, as shown in Fig. 3, the computation time necessary to perform the $640,000$ integrations was approximately $55,5$ days. Without the parallelization, it would be need approximately $2,5$ years to compute the integrations, which is infeasible.

For computation, duration $t_a$ was chosen as 50 seconds. For the integration, it was used the function *ode45* of the *Matlab* software, which applies the Runge-Kutta 4th/5th-order method as time-integration scheme with a varying time-step algorithm. The maximal step size is equal to $10^{-4}$ seconds, and the relative and absolute tolerance are equal to $10^{-9}$.

The values of the parameters used in all simulations were 1.0 kg for the mass, 4.0 N/m for the spring stiffness, 0.51 for the constant friction coefficient, and $v_0 = 1.0$ m/s for the base speed. The initial conditions of the system were modeled as two independent random variables, $X_0$ and $\dot{X}_0$, uniformed distributed over $[-1, 1]$.

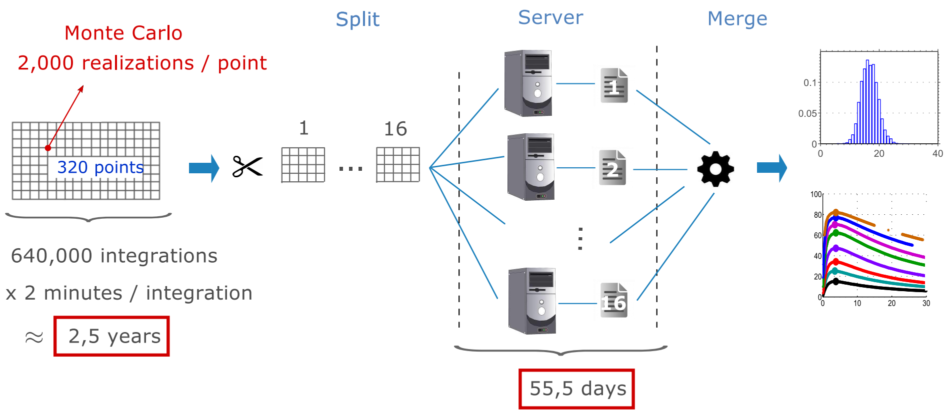



**Figure 3 – Parallelization of the simulations in the parametric analysis.**

## INFLUENCE OF $\lambda$ IN THE STATISTICAL MODEL OF THE STICK-SLIP PROCESS

First, we analyzed the influence of $\lambda$ in the statistical model of the stick-slip process for a fixed value of $\mu$. We took $\mu = 5.0$.

We started comparing the normalized histograms of the durations of the first six sticks for two values of $\lambda$, which are $\lambda = 0.1$ 1/s and $\lambda = 10.0$ 1/s. The graphs are shown in in Figs. 4 and 5. For $\lambda = 0.1$ 1/s, we verify that the duration of sticks are identically distributed. All of them have estimated mean value equal to 1.53 s and variance equal to 0.42 s$^2$. Besides, they have one pick near to the maximum stick duration, which is $d_{\max} = 2\frac{mg\mu}{kv} = 2.5$ s. For $\lambda = 10.0$ 1/s, we also verify that the duration of sticks are identically distributed. However, the normalized histograms change considerably if compared with the histograms obtained for $\lambda = 0.1$ 1/s. Their maximum value occurs near the origin. The mean value is 0.10 s and variance $9.72 \times 10^{-3}$ s$^2$. The similarity between the normalized histograms shown in Figs. 4 and 5 suggests that for a fixed value of $\lambda$, the duration of sticks are identically distributed. Then, we take $D_1, \cdots, D_{S_T}$ as identically distributed random variables and, we call them as $D$.

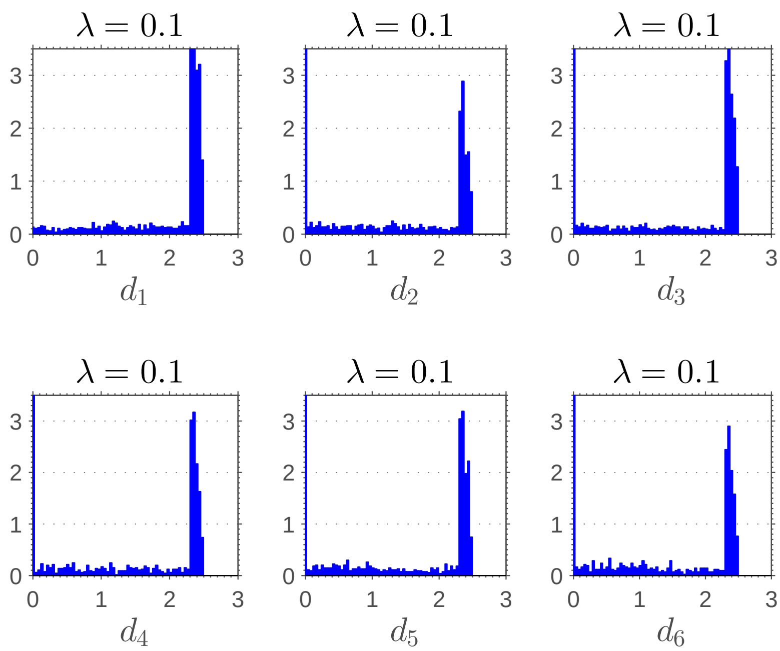



**Figure 4 – Normalized histograms constructed with $2{,}000$ samples of the duration of the first six sticks, i.e., random variables $D_1, \cdots, D_6$, for $\lambda = 0.1$ 1/s.**

Observing Figs. 4 and 5, we verify that the normalized histograms of the stick duration are sensitive to variations on $\lambda$. To better understand the influence of this parameter, we investigated the variation of the normalized histograms for

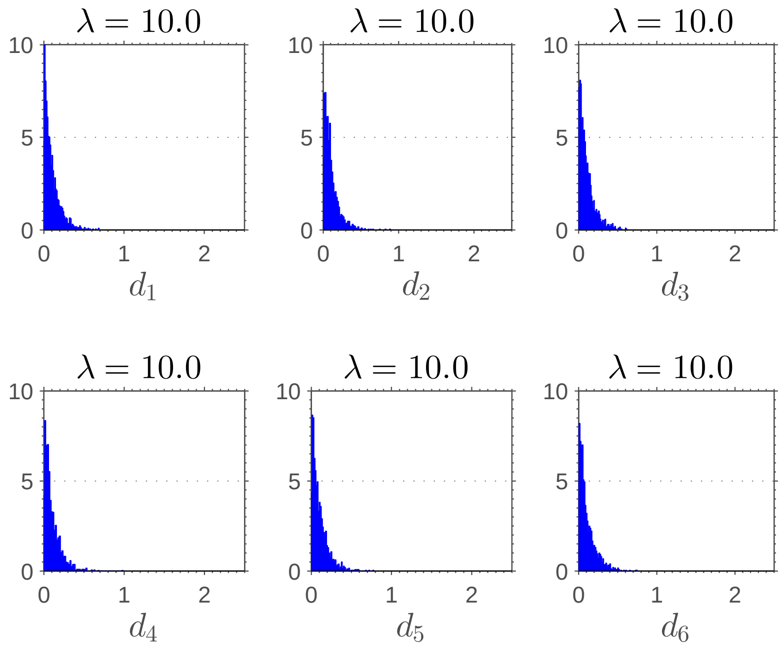



**Figure 5 – Normalized histograms constructed with $2,000$ samples of the duration of the first six sticks, i.e., random variables $D_1, \cdots, D_6$, for $\lambda = 10.0$ 1/s.**

intermediate values of $\lambda$. The results are shown in Fig. 6. In this figure, we verify that as $\lambda$ grows, the peak near to the maximum stick duration, which is $d_{\max} = 2\frac{mg\mu}{kv} = 2.5$ s, disappears, and the support of the normalized histogram is reduced. We conclude that the estimated mean of the stick duration decreases as $\lambda$ grows. To quantify this decay, we plotted the estimated mean of the stick duration $\hat{\mu}_D$ as a function of $\lambda$, as shown in Fig. 7. This graph confirm that our expectation about the reduction of the stick duration as $\lambda$ grows.

In the beginning of the Section, we explained that we expect that when $\lambda$ is low, the majority of the sticks end due to growth of the spring force up to the maximum friction force and as $\lambda$ grows, the majority of the sticks end due to occurrence of a discontinuity on the base velocity. These two expectations are confirmed by the graphs of Figs. 8(a) and 8(b). We modeled the number of sticks that end due to the spring and number of sticks that end due to the change of base motion as discrete random variables, $S_{Ts}$ and $S_{Tb}$ respectively, and in Figs. 8(a) and 8(b), we show respectively the estimated mean of $S_{Ts}$ and $S_{Tb}$ as a function of $\lambda$. In these graphs, the estimated means are divided by the duration $t_a$, in order to inform the rate of the number of sticks per unit of time.

Adding $S_{Ts}$ and $S_{Tb}$ we get $S_T$, the total number of sticks. The graph of the estimated mean of this random variable divided by the duration $t_a$, as a function of $\lambda$ is shown in Fig. 9(a). Observing it, we verify that the mean of the total number of sticks increases as $\lambda$ grows. As we known from Fig. 7 that the stick duration decreases as $\lambda$ grows, we conclude that as $\lambda$ grows, the system response presents on average a higher number of sticks, but these sticks have on average a lower duration. Given that, we may ask what happens with the total time of stick as $\lambda$ grows. Computing the sum of the duration of all sticks, and dividing it by the duration $t_a$, we get the total time of stick in relation to the time interval analyzed. We call this random variable $R$. The question is what happens with the mean of $R$ when $\lambda$ grows. Is it possible that, on average, a higher number of sticks with lower duration give us a higher total time of stick than a lower number of sticks with a higher duration? To answer this question, we plotted the estimated mean of $R$ as function of $\lambda$. The obtained graph is shown in Fig. 9(b). Observing it, we verify that the mean of total time of stick reaches a maximum value, $\hat{\mu}_R^{\star}$, which is almost 80% and occurs at $\lambda = 3.8$ 1/s. The conclusion is that for $\lambda \in [0.5, 3.8]$ 1/s, the increase of the number of sticks causes the increase of the total time of stick, even though the sticks have a lower duration. The larger number of sticks compensates its shorter duration. However, when $\lambda$ exceeds 3.8 1/s, the increase of the number of sticks compensates no more the reduction of its duration, so $\hat{\mu}_R^{\star}$ decreases. In the numerical simulations performed, the maximum value to the parameter $\lambda$ was 30 1/s. It has been considered not necessarily to verify the behavior of system for a larger range of $\lambda$ because the value 30 1/s means that on average the base has 30 changes changes of the velocity sign of the base motion per second, which is already sufficiently large. It is a limit case and can not be realized by an experiment. It should be noted that if the parameter $\lambda$ were increased, it would be necessary smaller time-step, and smaller relative and absolute tolerances. Consequently, the computational cost of integrations would increase even more, turning the parametric analysis unfeasible.

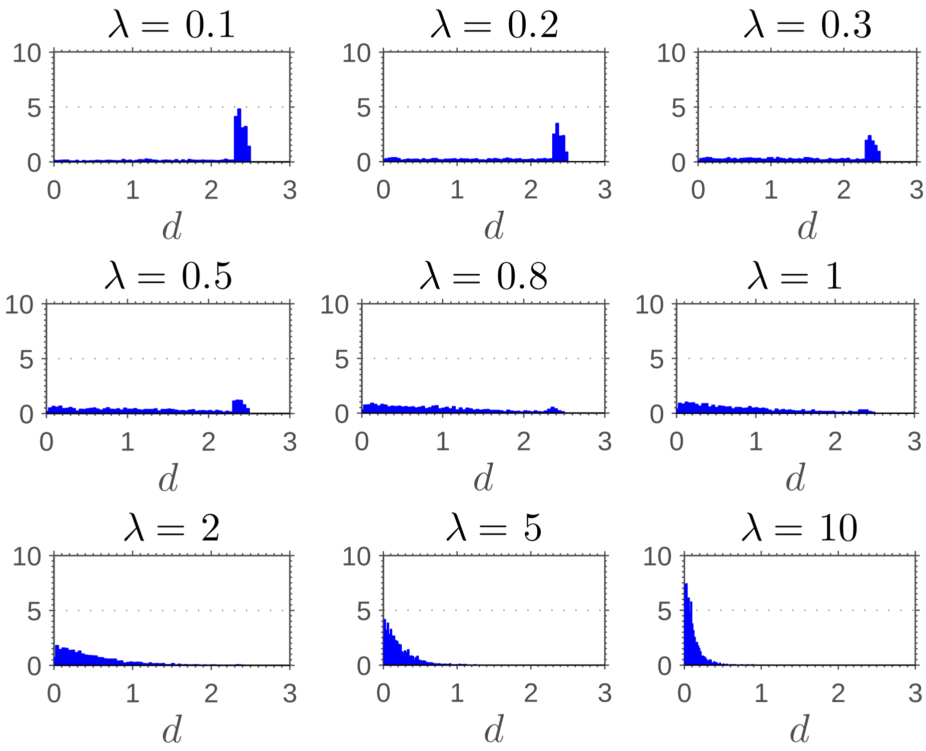



**Figure 6 – Normalized histograms constructed with $2,000$ samples of the duration of the first stick for six different values of $\lambda$.**

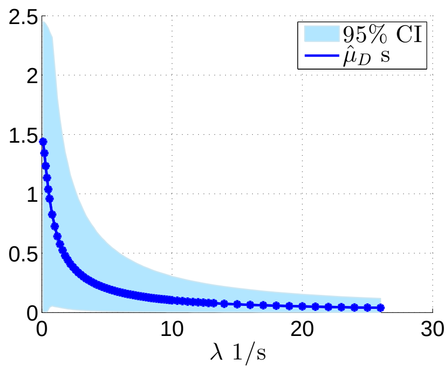
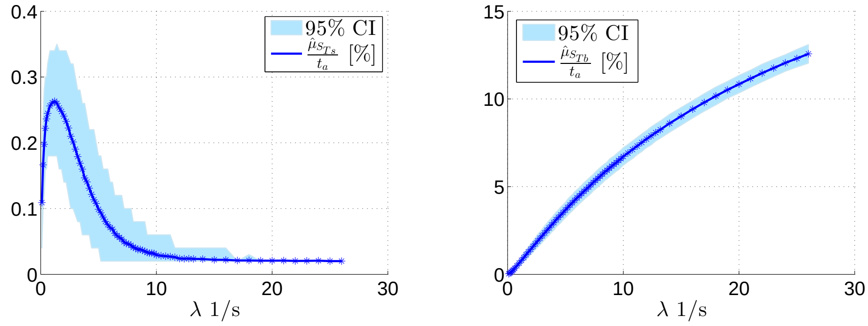



**Figure 7 – Estimated mean of the stick duration, $\hat{\mu}_D$, and 95 % confidence interval as a function of $\lambda$.**

## INFLUENCE OF THE FRICTION COEFFICIENT ON THE STATISTICAL MODEL OF THE STICK-SLIP PROCESS

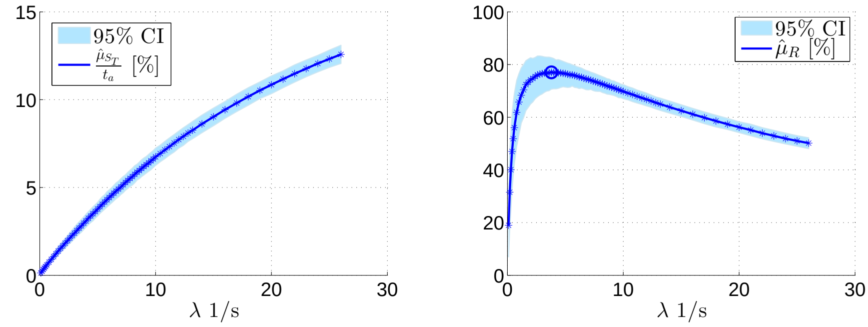
To quantify the influence of the friction coefficient on the statistical model of the stick-slip process, we analyzed the graph of the estimated mean of this random variable total time of stick $\hat{\mu}_R$ as a function of $\lambda$ for 8 values of $\mu$. The graph is shown in Fig. 10. Observing it, we verify that the maximum of the estimated mean of total time of stick, $\hat{\mu}_R^\star$, grows as $\mu$ increases. For $\mu = 0.5$, the maximum is 15.0%, and for $\mu = 7.0$, the maximum is 81.89%. Besides, we verify also that the maximum is always reached for $\lambda$ in the short interval $[3.5, 3.8]$ 1/s. From these results, we conclude that the friction coefficient has a lot of influence on $\hat{\mu}_R$. However, very little influence on the position of the maximum.

By the results shown in Fig. 10, it is possible to observe a limit case: when $\mu$ grows, $\hat{\mu}_R^\star$ approaches 100% and it is always reached for $\lambda$ in the interval $[3.5, 3.8]$ 1/s. But it should be remembered that due to the discontinuities of the base velocity, the duration of the stick-mode cannot be 100%. As explained in Section , if the block is in the stick-mode in the instant just before the base discontinuity, it must be in the slip-mode in the instant just after the discontinuity. Thus, the stick is interrupted by the discontinuities on the base velocity.




(a) (b)

**Figure 8 – (a) Estimated mean of the number of sticks that end due to the growth of the spring force up to the maximum friction force, $\hat{\mu}_{S_{Ts}}$, divided by the duration $t_a$, and 95 % confidence interval as a function of $\lambda$. (b) Estimated mean of the number of sticks that end due to the occurrence of a discontinuity on the base velocity, $\hat{\mu}_{S_{Tb}}$, divided by the duration $t_a$, and 95 % confidence interval as a function of $\lambda$.**




(a) (b)

**Figure 9 – (a) Estimated mean of the number of sticks, and 95 % confidence interval as a function of $\lambda$. (b) Estimated mean of the total stick duration, $\hat{\mu}_R$, and 95 % confidence interval as a function of $\lambda$.**

## CONCLUSIONS

In this paper, we analyzed the dynamics of a dry-friction oscillator which is stochastically excited by an imposed bang-bang base motion. The base velocity is modeled by a Poisson process with probabilistic model fully specified. The system response which is composed by a random sequence alternating stick and slip-modes is modeled as a stochastic process. This stochastic process is described by random variables that are the number of time intervals in which stick or slip occur, the instants at which they begin, and their duration (see Fig. 2). To estimate statistics and histograms for them, samples of the random sequence of stick and slip-modes were computed by the integration of the dynamic equation of the system using independent samples of the base motion. The objective was to determine how the statistics and histograms vary with the system parameters, i.e., to make a parametric analysis of the statistical model of the stick-slip process. In this stochastic parametric analysis, the influence of two system parameters were analyzed. One of them related to the probabilistic model of the base motion ($\lambda$) and the other related to friction force (the friction coefficient, $\mu$). For different combinations of these parameters, $2,000$ numerical integrations of the dynamic equation of the system were computed. In total $640,000$ integrations were performed. As the computation of each one of them is time consuming due to the non-smooth behavior of the system, the total computational cost was huge. To reduce it, we adopted the strategy of parallelization of the simulations. We used a cluster composed of sixteen computers (see Fig. 3).

The obtained results showed that the normalized histograms of the stick duration are sensitive to variations on $\lambda$, and as $\lambda$ grows, the estimated mean of the stick duration decreases. Besides of this, for small values of $\lambda$, the majority of the sticks end due to growth of the spring force up to the maximum friction force. For high values of $\lambda$, the majority of the sticks end due to occurrence of a discontinuity on the base velocity. The conclusion is that, as $\lambda$ grows, the system response presents on average a higher number of sticks, however these sticks have on average a lower duration. The relative total time of stick, $R$, showed that for $\lambda$ lower than approximately 3.8 1/s, the increase of the number of sticks causes the increase of the total time of stick, even though the sticks have a lower duration, in a way that the large number of sticks compensates its shorter duration. However, when $\lambda$ exceeds approximately 3.8 1/s, the increase of the number of sticks does not compensate the reduction of its duration anymore, so the total time of stick decreases.

From the analysis of the influence of the friction coefficient on the statistical model of the stick-slip process, we concluded that maximum of the estimated mean of total time of stick, $\hat{\mu}_R^\star$, grows as $\mu$ increases. However, we verified that the value of $\mu$ does not change the behavior of $R$ in relation to the position of its maximum.

We believe that the parametric analysis of the stick-slip process made in this paper can help in the design and project of dynamical systems with dry-friction forces. Considering, for example, systems with undesirable stick-slip vibrations,

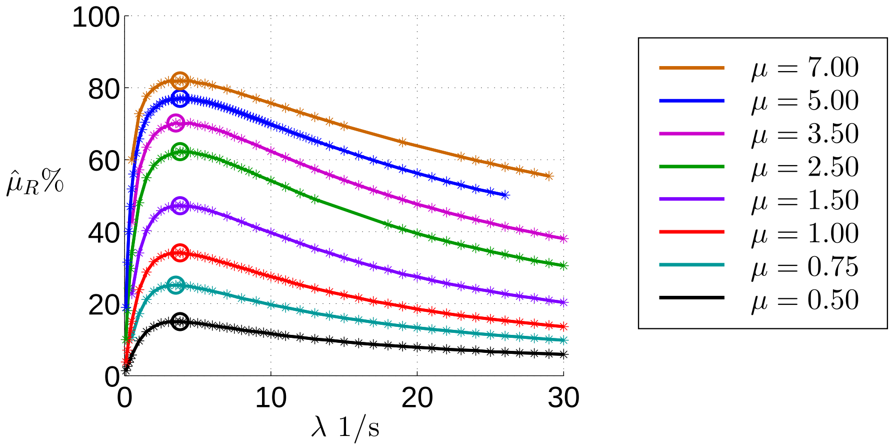



**Figure 10 – Estimated mean of the total stick duration $\hat{\mu}_R$ as a function of $\lambda$ for different values of $\mu$. The $\hat{\mu}_R^\star$ is highlighted for each $\mu$ with markers.**

disturbing the system functionality, the goal is to use the results of our analysis to design the systems in a way that the duration of the stick-mode is minimized by a careful selection of the parameters. It is clear that for each different system, a different stochastic parametric analysis should be computed. This paper is an example of how we can make it for a simple system. It should be noted that even addressing a simple dry-friction oscillator, computation of the stochastic parametric analysis is very time consuming. The non-smooth behavior of dry-friction force associated with the high number of numerical integrations required to compute statistics and histograms make the problem computationally expensive. The characterization of the random sequence of sticks or slips of the system response by a sequence of random variables, and the construction of a statistical model to them by the estimation of moments and histograms represent a new approach to the study of systems with dry-friction force.

Concerning the design of dynamical systems in which undesirable stick-slip vibrations occur, it is important to remark that a low value of total stick duration is not the only relevant variable to an optimization of the system performance. As we can see in this paper, it is possible that different systems configurations present similar total stick duration. To distinguish and choose among these configurations, we should take into account others relevant variables to the system performance. An example is the rate of the number of sticks per unit of time. In drilling, this variable is related with the stability of the system dynamic. Thus, the optimization of a system performance comprises a multi-objective optimization problem, in which the total stick duration is only one of the relevant variables.

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