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EFFECT OF PRESSURE GRADIENT ON THE GÖRTLER VORTICES FASTEST GROWING SPANWISE WAVENUMBER

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Abstract. *Centrifugal instability is one of the many routes to transition from laminar to turbulent flow in boundary layers. The main characteristic of centrifugal instability is the development of counter rotating vortices known as Görtler vortices. As other transition scenarios, centrifugal instability is influenced by flow conditions such as compressibility, heat transfer and pressure gradient. The present investigation is an extension of a previous work on the effect of pressure gradient on the development of Görtler vortices, where we investigated the effect of pressure gradient on the wavelength corresponding to the fastest-growing disturbances. The present work reviews previous results presented in the literature in the light of the relation between these fastest growing modes and pressure gradient.*

Keywords: *hydrodynamic stability, centrifugal instability, Görtler vortices, linear stability theory*

1. INTRODUCTION

The development of Görtler vortices for boundary layers over concave walls are due to centrifugal instabilities and the resulting vortices are stationary longitudinal vortices aligned with the flow direction (Floryan, 1991; Saric, 1994). As a consequence of the development of Görtler vortices, transition to turbulence, which may be an undesirable effect, results from secondary instabilities.

As for Tollmien-Schlichting waves, pressure gradients are known to affect the development of Görtler vortices and previous studies have identified that favorable pressure gradients are stabilizing while adverse pressure gradients are destabilizing (Ragab and Nayfeh, 1981; Mangalam *et al.*, 1985; Finnis and Brown, 1994; Goulpié *et al.*, 1996; Itoh, 2001; Matsson, 2008; Rogenski *et al.*, 2013, 2016a,b). A detailed review of the main references on pressure gradient effects on Görtler vortices have been presented by Fernandes (2019) and by Rogenski (2015).

The present work is concerned with the most recent results presented by Matsson (2008) and Rogenski *et al.* (2016a). Matsson (2008) studied Görtler vortices development under favorable and adverse pressure gradient. As concluded by previous investigations, they found that adverse pressure gradient results in more unstable disturbances on the linear regime, saturate earlier and at a lower amplitude compared to the Blasius results. Favorable pressure gradient result in more stable vortices in a sense that the vortices evolve farther in the streamwise direction and is also more stable to secondary instability, as discussed in (Fernandes and Mendonca, 2019). These results depend also on the spanwise wavenumber but only two wavenumbers were tested by the Matsson (2008). Taking the Blasius boundary layer solution as reference, the smaller wavenumber ($\beta = 0.22$) chosen by the authors is more unstable than their higher wavenumber ($\beta = 0.52$). It is possible to infer that the former is closer to the wavenumber associated with the maximum amplification.

The results for the nonlinear development and saturation of Görtler vortices obtained by Matsson (2008) can be compared with the results obtained by Souza and his group (Rogenski *et al.*, 2013, 2016a,b), who have been using direct numerical simulation (DNS) to study the development of Görtler vortices in boundary layers with pressure gradient.

Rogenski *et al.* (2016a) studied the effect of variable pressure gradient, considering constant pressure gradient, constant Hartree parameter and linear varying pressure gradient. Both favorable and adverse pressure gradients were considered for three different values of spanwise wavenumber. They observed that large spanwise wavenumbers are more unstable when the pressure gradient is favorable. For adverse pressure gradient the opposite is true, small spanwise wavenumbers are more unstable. These results are an indication that the fastest growing spanwise wavelength changes with pressure gradient. The study presented by Fernandes and Mendonca (2019) investigated the results presented by Rogenski *et al.* (2016a) in order to clarify the relation between pressure gradient and the fastest growing spanwise wavenumber.

Non-linear results presented by Rogenski *et al.* (2016b) corroborate results presented by Matsson (2008) up to the nonlinear breakdown stage in boundary layers with pressure gradient. Rogenski *et al.* (2016b) also found that adverse pressure gradient anticipates saturation while favorable pressure gradient delay saturation. One of the conclusions presented in their work is that the saturation point depends on the disturbance wavenumber. Due to the high cost of direct numerical simulation, only three wave length were tested, $\Lambda = 160$, 305 and 450. Saturation takes place earlier for the lower spanwise wavenumber. The results indicate that saturation and the development of secondary instability also depend both on the pressure gradient and on the corresponding spanwise wavelength of the fastest growing mode.

The present investigation complements the results presented by Fernandes and Mendonca (2019) by investigating the results presented by Goulpié *et al.* (1996), Matsson (2008) and Rogenski *et al.* (2016b). The relation between the spanwise wavenumber of the fastest growing mode and pressure gradient discussed by Fernandes and Mendonca (2019) is used to discuss the results presented by these authors. By using linear stability theory it is possible to investigate a much larger number of spanwise wavelength conditions than possible to explore using more computational expensive models. Therefore, in the present work additional conclusions and insights are offered to existing results regarding the effect of pressure gradient on the stability of Görtler vortices.

2. METHODOLOGY

The present study uses linear stability theory to identify the fastest growing mode spanwise wavenumber β. Since the growth rate of the disturbances are of the same order of the boundary layer thickness growth, parallel flow assumption is not valid for certain problem parameters. For Görtler number greater than 7 Bottaro and Luchini (1999) have shown that the local normal mode approach is valid and the results recover the results obtained with the nonparallel parabolic model. Therefore, the problem is formulated considering that the disturbances may be represented as the classic normal modes. Görtler vortices are stationary disturbances, aligned with the flow direction with spanwise periodicity wavenumber β that grow downstream with amplification rate α. According to experimental evidence, the dimensional spanwise wavenumber remains constant in the streamwise direction even under pressure gradient conditions (Finnis and Brown, 1994).

$$v^*(x,y,z) = \hat{v}(y)\exp\left(\alpha x + i\beta z\right), \tag{1}$$

where $\hat{v}$ is the normal disturbance velocity component complex amplitude, with information on disturbance amplitude and phase. Similar expressions are used for the for pressure, streamwise and spanwise velocity components.

The base flow is given by the boundary layer over a flat plate modeled by the solution of the Falkner-Skan similarity profile (Goulpié *et al.*, 1996). The free stream velocity away from the wall U_e varies in the streamwise direction x according to $U_e = x^m$.

The similarity transformation in the normal to the wall direction y is

$$\eta = \frac{y}{\delta}, \qquad \delta = \sqrt{\frac{\nu x}{U_e}}, \qquad \frac{u}{U_e} = f(\eta). \tag{2}$$

Where δ is the boundary-layer thickness parameter and ν is the kinematic viscosity. This choice of similarity transformation results in the following ordinary differential equation

$$f''' + \frac{m+1}{2} f f'' + m[1 - f^2] = 0. \tag{3}$$

The stability equations are

$$v^{IV} + \left(2UGo^2\beta^2 + \alpha U_{xy} + \beta V_x\right)\hat{u} + \alpha U_x \hat{u}' +$$
$$\left(\alpha U\beta^2 + \beta^4 + \alpha U_{yy} + \beta^2 V_y\right)\hat{v}' - \left(\alpha U + 2\beta^2 + V_y\right)\hat{v}'' - V\hat{v}''' = 0, \tag{4}$$

and

$$\hat{u}'' - \left(U\alpha + \beta^2 + U_x\right)\hat{u} - V\hat{u}' - U_y\hat{v} = 0, \tag{5}$$

where $'$ represents eigenfunction $\hat{u}$ and $\hat{v}$ derivation with respect to the normal direction and the subscript y and x represent derivation of the base flow with respect to the corresponding normal and streamwise directions. Nonparallel terms in the above equations will be neglected (Fernandes and Mendonca, 2019).

Choosing $\delta = \sqrt{\nu x_d/U}$ as the reference length scale in the spanwise and normal directions, the Görtler and Reynolds numbers are defined as

$$Go^2 = k\delta Re^2, \qquad Re = \frac{U_e\delta}{\nu}, \tag{6}$$

where k is the wall curvature.

The boundary conditions are no slip at the wall and exponential decay away from the wall, as shown bellow.

$$u = v = v' = 0 \quad \text{em} \quad y = 0, \tag{7}$$

$$u,\, v \quad \to 0 \quad \text{para} \quad y \to \infty. \tag{8}$$

The problem posed by Eqs. 4 and 5 and by the boundary conditions above, results in an eigenvalue problem where the dispersion relation is $f(Go, \beta, \alpha) = 0$.

Further details of the methodology may be found in Fernandes and Mendonca (2019).

3. RESULTS AND ANALYSIS

3.1 Effect of pressure gradient on the fastest growing mode

The study of the dependence of the fastest growing mode on the pressure gradient was presented by Fernandes and Mendonca (2019). Some of these results are reproduced here. Table 1 shows the nondimensional wavelength Λ corresponding to the fastest growing mode for different values of the acceleration parameter m. Note that, unlike the constant pressure $m = 0$ condition, the nondimensional wavelength varies downstream.

The nondimensional wavelength Λ is defined as

$$\Lambda = \frac{U_e \lambda}{\nu} \sqrt{k\lambda} = Go \left(\frac{2\pi}{\beta} \right)^{3/2} \quad , \tag{9}$$

Table 1. Fastest growing spanwise wavelength Λ versus pressure gradient acceleration parameter m. Λ_i is the upstream wavelength at $Go = 7$ and Λ_f is the downstream wavelength at $Go = 14$

favorable			adverse		
m	$\Lambda_i(Go = 7)$	$\Lambda_f(Go = 14)$	m	$\Lambda_i(Go = 7)$	$\Lambda_f(Go = 14)$
0,0	212	212	0,0	212	212
0,1	183	200	-0,01	216	214
0,2	165	197	-0,02	220	216
0,3	152	196	-0,03	225	219
0,4	141	197	-0,04	230	221
0,5	133	199	-0,05	236	225
0,6	127	202	-0,06	243	229
0,7	121	206	-0,07	251	235
0,8	116	210	-0,08	260	241
0,9	112	212	-0,09	275	253

The value of Λ corresponding to the fastest growing mode at $Go = 7$ decreases with increasing m for favorable pressure gradient. For adverse pressure gradient the fastest growing mode wavelength increases with decreasing m (or increasing absolute value). This results is shown in Fig. 1.

These results show that the wavelength of the fastest growing mode varies considerably with pressure gradient. Therefore the investigation of the effect of pressure gradient on the development of Görtler vortices has to consider how the growth rate changes with the acceleration parameter, but has to consider also, how close the given disturbance wavelength is to the fastest growing wavelength for that acceleration parameter. This discussion is presented in the next section.

3.2 Comparisons with previous investigations

Regarding the wavelength corresponding to the fastest growing mode Λ_{mx}, the information available on the literature is scarce for boundary layers with favorable and adverse pressure gradient. On the results presented by Goulpié *et al.* (1996) for $m = 0.075$, the fastest growing mode has $\Lambda = 210$. This wavelength corresponds to the fastest growing mode of the Blasius boundary layer, which is consistent with the fact that the acceleration parameter is very low. The present result for a favorable pressure gradient for $m = 0.1$ is consistent with Goulpie's result.

Matsson (2008) considered two different values for β, 0.22 and 0.52 at $Go = 2$ for different values of the acceleration parameter m. These values correspond to upstream conditions for $Go = 7$ as given in Tab. 2.

Table 2 shows that the wavelengths studied by Matsson do not correspond to the wavelength of the fastest growing modes. For the case $\beta = 0.22$ the corresponding Λ are greater than the Λ_{max} identified in the present work and for the

Figure 1. Λ corresponding to the fastest growing mode for different values of the acceleration parameter.

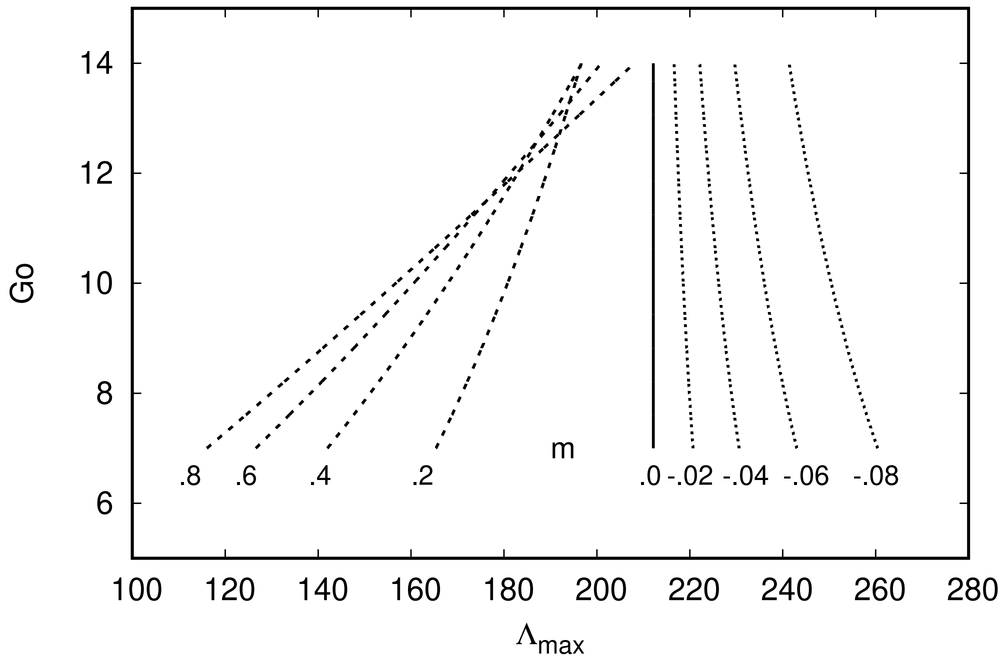

Table 2. Relation between the results from Matsson (2008) and the present results. The second and third columns present Λ corresponding to Matsson's $\beta = 0.22$ and the fourth and fifth to $\beta = 0.52$, transported to the condition $Go = 7$. The last column presents the wavelength corresponding to the fastest growing mode computed in the present work.

m	$\beta(Go = 7)$	$\Lambda(Go = 7)$	$\beta(Go = 7)$	$\Lambda(Go = 7)$	Λ_{max}
-0,075	0,553	268	1,306	74	255
0	0,507	305	1,199	84	212
0,075	0,467	345	1,105	95	201
0,15	0,433	388	1,022	107	173
0,3	0,374	481	0,885	132	152

case $\beta = 0.52$ the corresponding Λ are lower than Λ_{max}. According to Matsson's Fig. 5(b) the amplification rates for $\beta = 0.52$ are lower than the amplification for $\beta = 0.22$, but according to the idea that the results should also take into account the proximity to Λ_{max} it would not be possible to infer that increasing β would always result in a reduction of the amplification rate.

A greater number of the parameter space was considered by Rogenski *et al.* (2016a), where results are presented for constant pressure gradient, linear varying pressure gradient and pressure gradients associated to the Hartree acceleration parameter γ. The following analysis corroborates the results presented by Rogenski regarding the relation between pressure gradient and wavelength of the fastest growing disturbance.

Rogenski considered $\gamma = -0.15$, -0.075, 0, 0.125 and 0.25 for $Go = 2.5$ and wavelength Λ equal to 100, 250 and 450. The corresponding values downstream at $Go = 7$ are presented in Tab. 3, along with the values of Λ_{max} obtained in the present investigation and the corresponding values of the acceleration parameter m.

The results from Rogenski *et al.* (2016a) show that the greater the adverse pressure gradient, the greater the amplification rate. It also shows that the greater the favorable pressure gradient, the lower the amplification rate. Rogenski's cases $\gamma = -0.15$ and $\gamma = -0.075$ for $\Lambda = 227$ e 238 at $Go = 7$ have very similar amplification rates despite the difference in the acceleration parameters. The fastest growing wavelength for these cases would be $\Lambda_{max} = 251$ e $\Lambda_{max} = 228$. Since the amplification rate is a function of both the pressure gradient and the proximity of Λ to the value of Λ_{max} the similar value for the amplification rate is justified, even if the acceleration parameter differ by a factor of two.

Rogenski *et al.* (2016a) compare the amplification rates and the total amplification downstream for the cases $\Lambda(Go = 2.5) = 100$ and $\Lambda(Go = 2.5) = 450$ for $\gamma = 0.25$ and -0.15 ($m = 0.143$ and -0.07). They observed that the favorable pressure gradient case has higher amplification rate than the adverse pressure gradient case. For $\Lambda(Go = 2.5) = 100$, the two values of m (0.143 and -0.07) correspond to $\Lambda(Go = 7) = 121$ e 91. The corresponding values of Λ_{max} for

Table 3. Relation between Rogenski *et al.* (2016a) data for different values of Λ for different acceleration parameters, γ and m, and the present work value of Λ_{max}.

γ	m	$\Lambda(Go = 7)$	$\Lambda(Go = 7)$	$\Lambda(Go = 7)$	Λ_{max}
0,0	0,0	100	250	450	212
-0,15	-0,07	91	227	408	251
-0,075	-0,036	95	238	428	228
0,125	0,067	109	274	492	197
0,25	0,143	121	301	543	174

the two given m considered are 174 and 251, respectively. The higher growth rate observed for the favorable pressure gradient case with respect to the adverse pressure gradient case may be explained by the fact that the total amplification is a function of both the pressure gradient and the proximity of Λ to the fastest growing mode, which varies with the value of m. The relative difference between $\Lambda = 121$ and $\Lambda_{max} = 174$ for the favorable pressure gradient case is lower than the difference between $\Lambda = 91$ and $\Lambda = 251$ for the adverse pressure gradient case. In other words, the spanwise wavenumber of the favorable gradient case is closer to the spanwise wavenumber of the fastest growing mode, while the adverse pressure gradient case have a wavenumber that is much lower than the wavenumber of the fastest growing mode, resulting in a lower amplification rate. That may justify the greater growth rate of the favorable pressure gradient case. The chosen wavelength of the adverse pressure gradient case results in lower amplification rate than the favorable pressure gradient case.

For Rogenski's case $\Lambda(Go = 2.5) = 450$ it is possible to observe that the fastest growing mode corresponds to the adverse pressure gradient case. The values $\gamma = -0.15$ and 0.25 ($m = -0.07$ and 0.143) correspond to $\Lambda(Go = 7) = 408$ and 543, respectively. The corresponding values of Λ_{max} are 251 and 174. In this case the wavelength of the favorable pressure gradient, 543, case is much larger that the fastest growing wavelength mode, 174. For the adverse pressure gradient case the wavelength, 408, is higher than the fastest growing mode, 251, but the difference is lower and so the wavelength of the adverse pressure gradient case is closest to the fastest growing wavelength. This analysis is qualitative, but it corroborates the results found by Rogenski.

The zero pressure gradient boundary layer $m = 0$ results presented by Rogenski are in agreement with the results presented in this work. The value of $\Lambda = 250$ in their work is the one with the highest amplification rate and is close to the value found in this work for the fastest growing wavelength $\Lambda_{max} = 212$. The disturbance with $\Lambda = 100$ has smaller growth than the disturbance with $\Lambda = 150$, which is closer to 212. For the same reason, the growth of the $\Lambda = 350$ is greater than the disturbance with $\Lambda = 450$.

4. Conclusions

The present work addressed the problem of the effect of pressure gradient on the development of centrifugal instability for boundary layers over concave walls. The conclusions regarding the effect of pressure gradient on the fastest growing disturbance wavelength obtained in a previous work was used to analyse investigations available in the literature and explain some of the results presented therein. The growth rate and downstream growth of disturbances depend both on the pressure gradient and on the spanwise wavelength with respect to the spanwise wavelength of the fastest growing mode. Boundary layers with adverse pressure gradients may have lower growth rates that favorable pressure gradients depending on the choice of wavelength. For a proper identification of pressure gradient effects it is necessary to identify the fastest growing wavelength for the given pressure gradient. The classic value of $\Lambda_{max} = 210$ is only valid for Blasius boundary layers.

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