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APPLICATION OF THE GALERKIN METHOD, IN A STOCHASTIC BEAM BENDING PROBLEM, SUPPORTED BY A PASTERNAK FOUNDATION.

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Abstract. *This article presents a stochastic beam bending problem in a Pasternak foundation solved using the Galerkin method. Galerkin's method is directly applied to the abstract variational problem, obtained by the simplification of the beam displacement differential equation. The uncertainty of the problem is present in the physical parameters that characterize the rigidity of the beam itself, and the physical aspects of the foundation, on which the beam is supported. The Monte Carlo simulation is the mathematical tool used to deal with the stochastic aspect of the problem. Monte Carlo simulation generates random samples that will be properly applied to the physical parameters present in the differential equation. The aim of this article is therefore to analyze how much of the uncertainty present in the individualized parameters is related to the uncertainty of the displacement of the beam.*

Keywords: *Beam bending problem, Galerkin Method, Pasternak foundation, Monte Carlo simulation.*

1. INTRODUCTION

This article deals with a stochastic problem of beam bending, in which the uncertainty is directly related to the imprecision of the measurements of the physical parameters of the beam itself, EI and of the foundation on which the beam is supported (κ_P, κ_W). It is important to mention that for this problem in particular the beam rests on a foundation mathematically described by Pasternak's model, this relationship is evident by the κ_P parameter. The influence of the Pasternak parameter makes the problem more precise to be described mathematically by means of an approximation of the Taylor series to the first order. In zero order, the problem is described as a Winkler foundation, that is, a rougher approximation of the foundation effect.

The uncertainty of the displacement suffered by the beam is a direct consequence of the uncertainty present in the physical parameters that are part of the problem. To characterize the stochastic aspect of the problem, the Monte Carlo Simulation was used to generate a set of measurements samples for each individual physical parameter. And for each of these samples there is a characteristic displacement.

The displacement samples suffered by the beam, is obtained through the Galerkin’s Method, which is applied in the abstract variational problem. Next, the weight that the uncertainty of each individual parameter influences on the uncertainty of the beam displacement is analyzed, and for this analysis it is necessary to calculate the point-to-point standard deviation of the beam.

Galerkin's method is widely used to solve problems such as 2D fracture analysis (Zhang, et. al. 2008), non-linear problems in wave phenomena (Mahmoudi ,2005), While the use of Galerkin’s method with the Monte Carlo simulation was used by Coco, Majorana & Romano (2017) with a problem of conducting loads on a graphene substrate. Xiu & Hesthaven (2005) addressed the efficiency of the Galerkin method when applied to highly complex mathematical problem.

As previously seen, many problems can be described by different equations whose solution can be found using the Galerkin method. This method, in addition to being a very simple mathematical tool to be manipulated, does not have a significant computational cost, for these reasons the method was chosen to obtain the solution of a beam bending stochastic problem.

2. MATHEMATICAL FORMULATION

This section presents the problem of stochastic bending to be studied. The stochastic problem is defined in the probability space $(\Omega, \mathcal{F}, P)$, being "Ω " the sample space of events,"$\mathcal{F}$ " the σ-algebra de events e "P " the probability measure. Thus the problem of bending stochastic Euler Bernoulli beams supported by Pasternak foundation is defined by:

$$\begin{cases} \text{Find } \mathrm{u} \in L^2\big((\Omega,\mathcal{F},P); C^4(0,l)\big), \text{ such as,} \\ \frac{d^2}{dx^2}\left(EI\frac{d^2u}{dx^2}\right) - \frac{d}{dx}\left(\kappa_P \frac{du}{dx}\right) + \kappa_W u = q, \forall (x,\omega) \in (0,l)\times\Omega; \\ u(0,\omega) = u(l,\omega) = 0\,; \\ \left.\frac{d^2u}{dx^2}\right|_{(0,\omega)} = \left.\frac{d^2u}{dx^2}\right|_{(l,\omega)} = 0, \qquad \forall \omega \in \Omega; \end{cases} \tag{1}$$

Where, EI is the bending stiffness, κ_P and κ_W are foundation stiffness coefficients, described by Pasternak and Winkler models respectively, and u is the transversal beam stochastic displacement, and finally q is the load term.

As it is a stochastic problem, displacement function u , is defined math as $u = u(x, \omega_i, \omega_i', \omega_i'')$, being $\omega_i, \omega_i', \omega_i''$ the samples generated by the Monte Carlo method, linked to the uncertainty of the beam coefficients. In that problem, making $\kappa_P \equiv 0$ in Eq. (1), one obtains the problem of beam bending on a Winkler-type foundation, Ávila et al (2010).

The application of the Galerkin method allows to reduce the complexity of the problem by transforming Eq. (1) into an abstract variational problem (AVP), Eq.(2). It is possible to relax the order of the derivatives of the functions, which make up the terms of the approximate solution of the problem. The (AVP) is obtained by multiplying all the present terms defined by the differential equation (1), by the function v, which have the same contour conditions as the displacement function u.Then, the integral of the differential equation defined by Eq. (1) is carried out, along the entire length of the beam l, assigning the defined boundary conditions. Therefore, the problem described by Eq. (1), can be expressed according to Eq. (2):

$$a(u,v) = \int_0^l \left(EI(x,\omega_i)\frac{d^2u}{dx^2}\frac{d^2v}{dx^2} + \kappa_P(x,\omega_i')\frac{du}{dx}\frac{dv}{dx} + \kappa_W(x,\omega_i'')uv\right)dx \tag{2}$$

being $a: U \times V \to \mathbb{R}$ the bilinear form given by, and:

$$a(u,v) = f(v) = \int_0^l (qv)(x)\,dx. \tag{3}$$

and being $f: V \to \mathbb{R}$

2.1 Galerkin Method

The Galerkin method is used to find the approximate solution of the beam displacement, in a Pasternak foundation. Considering the base space of functions of the approximate solution, defined as $V_m = span\{\varphi_1, \dots, \varphi_m\}$, a subspace $V = \overline{span\{\varphi_\iota\}_{\iota\in\mathbb{N}}}$, and since the displacement is treat as a stochastic process, the parameters, coeficients present in Eq.(1), are taken as variables of the random type, describe as $\xi(\omega) = \{\xi_i(\omega)\}_{i=1}^M$, therefore the displacement as a function of the *x* position of the beam can be described as:

$$u(x,\omega) = u\big(x,\xi(\omega)\big) = u\big(x,\xi_1(\omega),\dots,\xi_M(\omega)\big), \tag{4}$$

being the vector of independent random variables,

$\xi(\omega) = \{\xi_i(\omega)\}_{i=1}^M$, is a function of the sample, that meets the following conditions:

$$\begin{cases} \mathbb{E}[\xi_i] = 0, \forall i \in \{1,\dots,M\}, \\ P(\omega \in \Omega: \xi_i(\omega) \in \Gamma_i) = 1, \forall i \in \{1,\dots,M\}, \end{cases} \tag{5}$$

where $\mathbb{E}[\cdot]$ is the expected value operator, Γ_i is the image of random variable ξ_i, that is, $\Gamma_i = \xi_i(\Omega)$, with $\Gamma_i = [a_i, b_i] \subset \mathbb{R}$, $|\Gamma_i| = |b_i - a_i| < \infty, \forall i \in \{1, \dots, N\}$.

In this form, the image of random vector $\xi: \Omega \to \Gamma$, with $\Gamma \subset \mathbb{R}^N$, and in terms of $\{\Gamma_i\}_{i=1}^{M}$, is given by $\Gamma = \prod_{i=1}^{M} \Gamma_i$. The conditions defined by Eq. (5), determine that the random variables are extremely positive, and that the function of the probability distribution is uniform.

From the AVP, represented by Eq. (2,3), and from Doob-Dynkin lemma, Rao (2010), that that determines the spatial functions can be separated from the stochastic process express by the random variables $\xi(\omega) = \{\xi_i(\omega)\}_{i=1}^{M}$. The approximate numerical solutions, order m, have the following form:

$$u_m(x, \xi_k) = \sum_{i=1}^{m} u_i(\xi_k)\varphi_i(x) = \big(U(\xi_k)\big) \cdot \Phi(x), \tag{6}$$

being $u_i's$ the parameters to be determined by minimizing the residue generated from the differential equation problem. The solution approximate from the variational problem for the structural k-th sample is defined as:

$$\begin{cases} Determine\ \ u_m(\xi_k) \in V_m\ such\ as, \\ a\big(u_m(\xi_k), \varphi_j\big) = f\big(\varphi_j\big),\ \forall \varphi_j \in V_m. \end{cases} \tag{7}$$

When replacing the numerical approximation for the k-th sample function of the stochastic process of displacement, Eq. (6), in AVP, Eq. (2,3), generating a linear system of algebraic equations represented by:

$$K(\xi_k)U = F \quad \Rightarrow \quad \mathrm{U} = \mathrm{H}(\xi_k)F, \tag{8}$$

being $\mathbf{K}(\boldsymbol{\xi}_\mathbf{k}) \in \mathbb{M}_\mathbf{m}(\mathbb{R})$ the matrix coefficients or stiffness, $H(\xi_k) = \big(K(\xi_k)\big)^{-1}$, $H(\xi_k) = \big[h_{ij}(\xi_k)\big]_{m\times m}$ e $U = [u_1(\xi_k), \dots, u_m(\xi_k)]^T$. The i-th vector entry $U(\xi_k)$ is given by:

$$u_i(\xi_k) = \sum_{j=1}^{m} h_{ij}(\xi_k)\, f_j. \tag{9}$$

From Eqs. (9) and (6), the numerical approximation of the sampled function of the stochastic process of transverse displacement, $u_m = u_m(x, \xi_k)$, it is expressed as:

$$u_m(x, \xi_k) = \sum_{i=1}^{m} \sum_{j=1}^{m} \big(h_{ij}(\xi_k) f_j\big)\, \varphi_i(x) = F \,.\, \big(H(\xi_k)\Phi(x)\big). \tag{10}$$

By Monte Carlo simulation method, estimates for the statistical moments of the stochastic process of displacement, are obtained from the set of the samples of the system expressed in Eq. (7). As suggested by Eq. (8), to determine the performance of the system is necessary to analyze the reverse operator, of the coefficient matrix $\big(H(\cdot)\big)$. It is important to mention, to determination of this operator, it involves a high computational cost, even when it is used iterative methods, such as Jacobi, Gauss-Seidel, or conjugates gradients (Kincaid and Cheny, 2002).

Once exposed mathematically the beam bending problem. In the next section, it will be discussed how the uncertainty about the physical parameters of the beam interfere in the sample space of all displacements suffered by the beam.

3. NUMERICAL RESULTS

To validate the alternative method, three examples were proposed, and for each case the uncertainty will be present: a) coefficient flexural rigidity of the beam (EI); b) Coefficient of foundation stiffness (κ_P); c) Coefficient of foundation stiffness (κ_W). The modeling of the uncertainty will be made through a parameterized random process, as Eq. (2). For all examples , the beam is simply supported at the ends, the beam length is normalized and the cross-section has a width $b = \frac{1}{100}\ m$ and height $h = \frac{1}{50}\ m$ and the beam is subjected to an uniformly distributed load $q(x) = 1\ kPa.m, \forall x \in [0,1]$, as it is shown in the Figure 2.

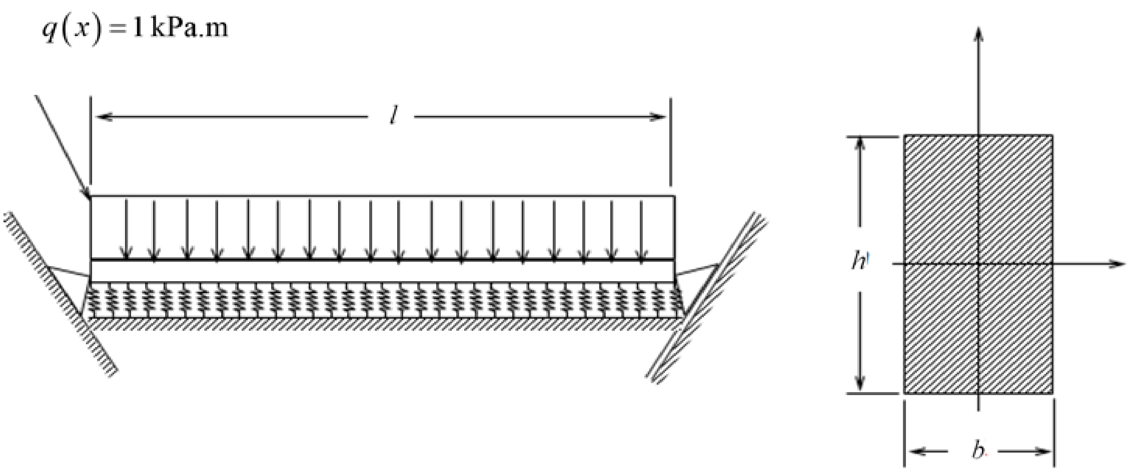

Figure 1 - a) Simply supported beam on Pasternack foundation;

b) beam cross-section.

In the numerical examples considered, the modulus of Young and the foundation stiffness parameters present the following mean-value functions:

$$\begin{cases} \mu_{EI}(x) = 1400 \text{ N.m}^2 & \wedge \quad \delta_{EI}(x) = \frac{1}{10}, \forall x \in [0,l]; \\ \mu_{\kappa_P}(x) = 1000 \text{ N} & \wedge \quad \delta_{\kappa_P}(x) = \frac{1}{10}, \forall x \in [0,l]; \\ \mu_{\kappa_W}(x) = 1000 \text{ Pa} & \wedge \quad \delta_{\kappa_W}(x) = \frac{1}{10}, \forall x \in [0,l]. \end{cases} \quad (12)$$

Being the function $\delta(x)$ the standard deviation function of the parameters.

Once exposed the physical aspects, before is necessary making an analysis on the minimum number of simulations necessary to obtain a satisfactory result, to do that is important, to define mathematically the concepts of the first statistical moment known as mean, and the second moment also known as variance.

The estimates of statistical moments of first and second order, i.e, the expected value and auto-correlation, from a set of outputs $\left(\left\{u_m\left(x,\xi(\omega_j)\right)\right\}_{j=1}^{N}, \forall x \in [0,l]\right)$ of the stochastic process of transverse displacement of a beam length "l", are defined by,

$$\begin{cases} \mu_{u_m}(x) = \frac{1}{N}\sum_{i=1}^{N} u_m\left(x,\xi(\omega_j)\right), \forall x \in [0,l]; \\ \mu_{u_m}^{(2)}(x) = \frac{1}{N-1}\sum_{j=1}^{N} u_m\left(x,\xi(\omega_j)\right) u_m\left(y,\xi(\omega_j)\right), \forall (x) \in [0,l]^2. \end{cases} \quad (11).$$

How will it be presented in the sequel the displacement of the beam's midpoint presents a greater dispersion, to all cases that will be abord. Figure 1 shows the average displacement of the central point of the beam, for the situation in which the uncertainty falls on all the physical parameters involved in the problem, as a function of the number of samples generated by the Monte Carlo Simulation.

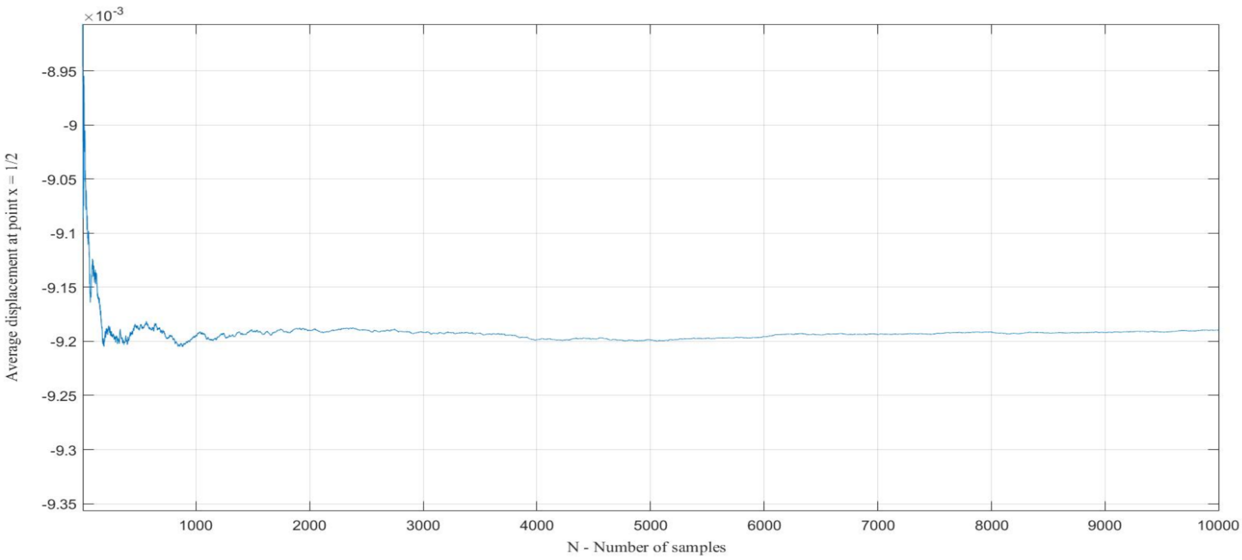

Figure 2- Calculation of the average displacement at the central point of the beam, according to the displacement samples generated through Monte Carlo Simulation.

From Figure 1 it is already possible to denote that the average value as a function of the samples does not present a great variability, for a sample space composed of more than 3.000 samples. In other words, the Monte Carlo simulation used to generate 3000 samples of random values attributed to the parameters, is more than enough to describe the bending problem more appropriately.

3.1 Example 1: random beam stiffness (*EI*).

In this example, the uncertainty is assumed on the coefficient of stiffness rigidity of the beam, $EI: [0, l] \times \Omega \rightarrow [\underline{a}, \overline{a}]$, which is modeled as a parameterized random process,

$$EI(x, \xi(\omega)) = \mu_{EI} + \left(\frac{\sqrt{3}}{3}\right) . \sigma_{EI} \sum_{k=1}^{3} \left[\xi_{2.k-1}(\omega) \cos\left(\frac{k\pi x}{4.l}\right) + \xi_{2.k}(\omega) \sin\left(\frac{k\pi x}{4.l}\right)\right], \tag{13}$$

where σ_{EI} is the standard deviation the stiffness coefficient of bending of the beam.

Figure 3 shows that, for each of the samples of the beam stiffness, parameter EI generated by the Monte Carlo simulation, the corresponding displacement function suffered by the beam. In Figure 3, it is observed that in the central point of the beam occurs the greatest amplitude of displacement. For the example represented by Figure 3, all the displacements in the middle of the beam are ranging between -0.0081 to -0.0105, being the average displacement value is -0.092 m at the beam's midpoint.

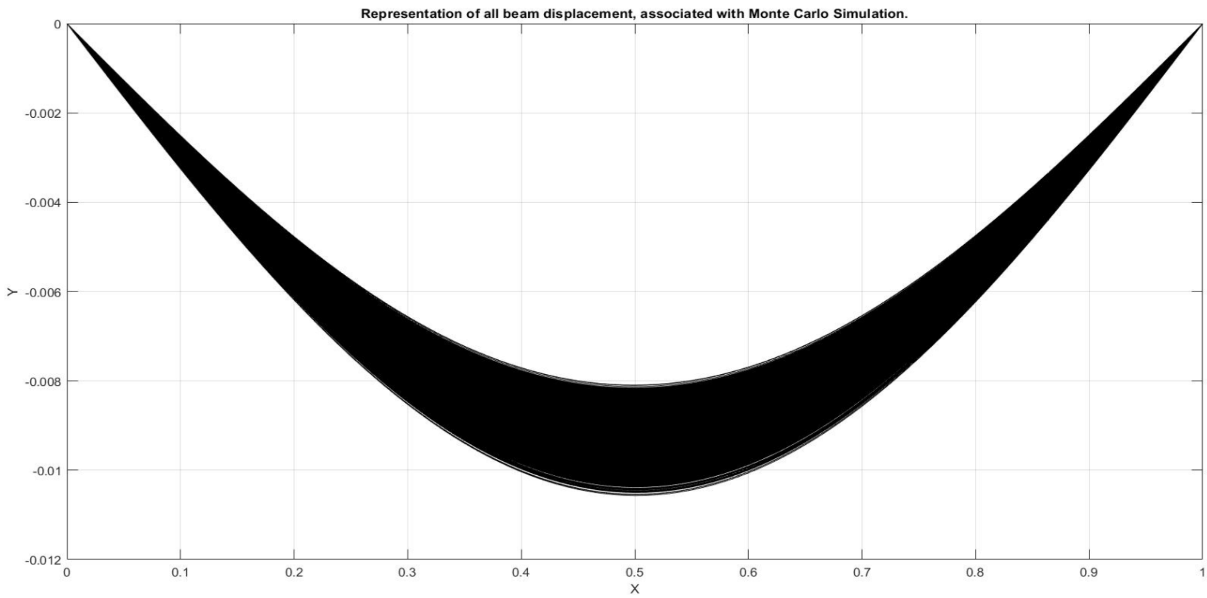

Figure 3 - All displacements generated, by countless samples of stiffness coefficient, generated by Monte Carlo simulation.

In order to better analyze how the uncertainty in the parameter influences the amplitude of the displacements of the samples generated by the Monte Carlo simulation, it is necessary to evaluate the second statistical moment, also called variance, as seen in Figure 4.

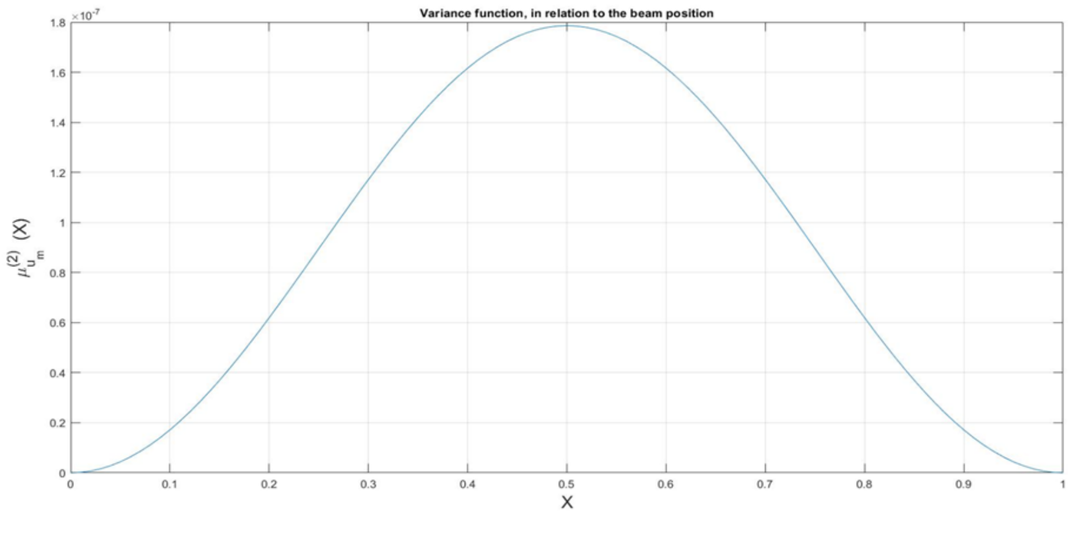

Figure 4 - Function of the sample variances corresponding to the beam stiffness, in relation to the position in the beam.

According to the graph, it is possible to observe that the maximum variance occurs at the central point of the beam, which was expected because in this particular point there is a greater dispersion, as already observed in Figure 3, in the middle of beam. The variance assumes the maximum value, and this value is 1.8 E-7.

3.2 Example 2: random foundation stiffness ($\boldsymbol{\kappa_P}$)

In this example, uncertainty is on the coefficient associated with the diffusive term Pasternak foundation. Uncertainty is modeled by a parameterized stochastic process $\kappa_P : [0, l] \times \Omega \to [\underline{b}, \bar{b}]$ given by,

$$\kappa_P\big(x, \xi(\omega)\big) = \mu_{\kappa_P} + \left(\frac{\sqrt{3}}{3}\right).\sigma_{\kappa_P} \sum_{k=1}^{3} \left[\xi_{2.k-1}(\omega)\, cos\left(\frac{k\pi x}{4.l}\right) + \xi_{2.k}(\omega)\, sin\left(\frac{k\pi x}{4.l}\right)\right], \quad (14)$$

The figure 5 show all the displacements suffered by the beam as a function of multiple pasternak´s stiffness coefficients, whose samples are generated by the Monte Carlo simulation. As can be seen in Figure 5, the displacement samples at the beam's central point are not very dispersed in relation to the average displacement value, which, as in the previous case, presents a value of -0.092m. In this case, the maximum value of the displacement and the minimum are respectively -0.0091m and -0.0092 m.

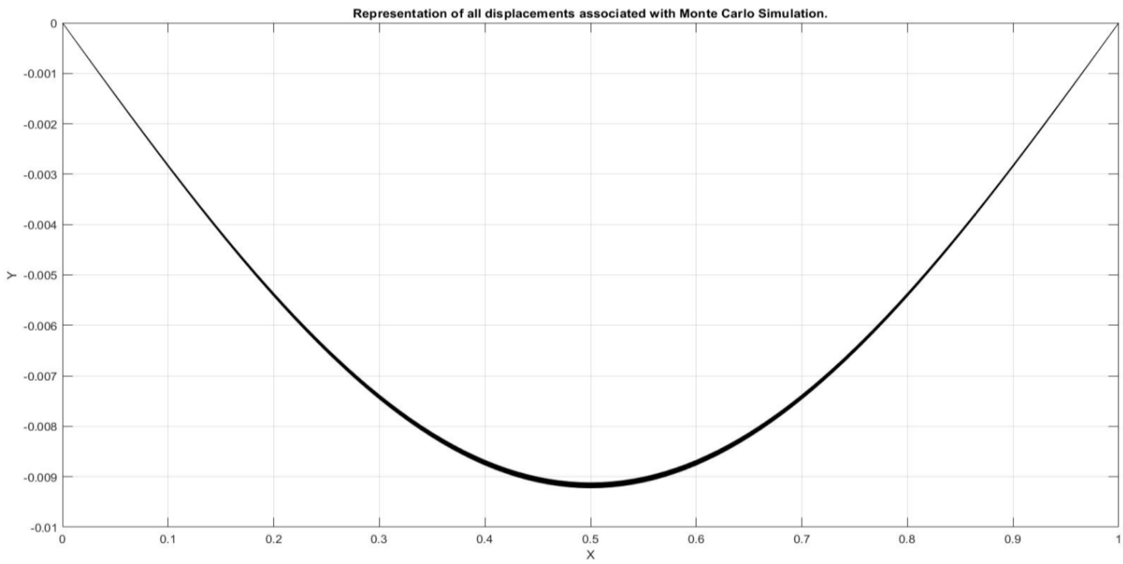

Figure 5 - All displacements generated, by countless samples of Pasternak´s stiffness coefficient, generates by MC Simulation.

The Figure 6 represents the function of variance, along the entire length of the beam. And it represents quantitatively how the degree of uncertainty of a parameter associated with a lower order term has little influence on the dispersion of displacements of all points of the beam in relation to the average displacement.

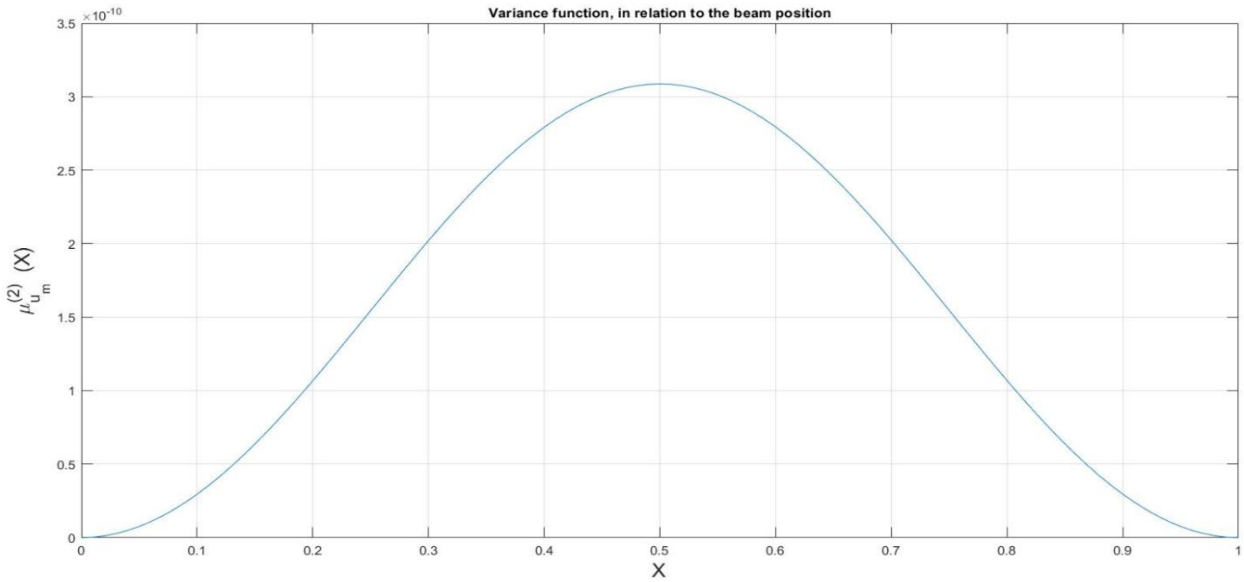

Figure 6 Function of the sample variances corresponding to the beam stiffness, in relation to the position in the beam.

According to the graph presented in Figure 8 the maximum value of the variance occurs at the central point, and has a value of 3.1E-10. It is necessary to already observe that when compared to the case in which the uncertainty falls on the rigidity of the beam, the variance function for the second case, represented by Figure 6, is significantly smaller throughout the domain.

In the next case, the uncertainty falls on the stiffness coefficient of the foundation described by the Winkler model.

3.3 Example 3: random foundation stiffness ($\boldsymbol{\kappa_W}$)

In this example, uncertainty is considered on the reaction coefficient Pasternak foundation, it is modeled by a parameterized random process $\kappa_W\colon [0, l] \times \Omega \to [\underline{c}, \overline{c}]$, is given by,

$$\kappa_W(x, \xi(\omega)) = \mu_{\kappa_W} + \left(\frac{\sqrt{3}}{3}\right) . \sigma_{\kappa_W} \sum_{k=1}^{3} \left[\xi_{2.k-1}(\omega) \cos\left(\frac{k\pi x}{4.l}\right) + \xi_{2.k}(\omega) \sin\left(\frac{k\pi x}{4.l}\right)\right], \tag{15}$$

When the uncertainty lies in the parameter referring to the Winkler type foundation model, the dispersion of the displacement samples among themselves, in the center of the beam, is of order of 10^{-4}, as can be seen in the width of the midpoint of the beam, show in Figure 7. As expected the variance is negligible in this case. For this situation, analyzing the average displacement graph is practically irrelevant. Given that the dispersion of samples around the average displacement for the entire domain of the beam, as previously expressed and as shown in Figure 6), is practically null. Just to mention the average displacement at the center point of the beam is -0.0092m, which is a coherent value considering the previous results.

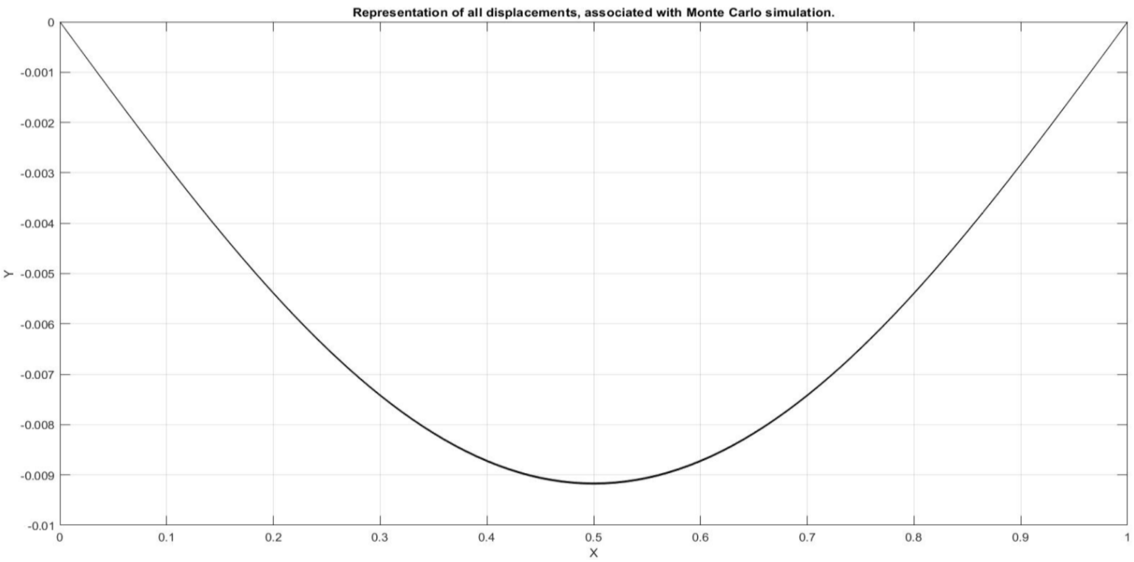

Figure 4- All displacements generated, by countless samples of Winkler´s stiffness coefficient, generated by Monte Carlo simulation.

According to the graph present in Figure 8, the variance in the whole doom is practically null, that is, the Monte Carlo Simulation for parameters associated with different operators of null order, is practically dispensable, which means a lower computational consumption. The uncertainty in the Winkler parameter does not significantly influence the stochastic process. The maximum variance that occurs in the central point of the beam was in the order of 10^{-11}

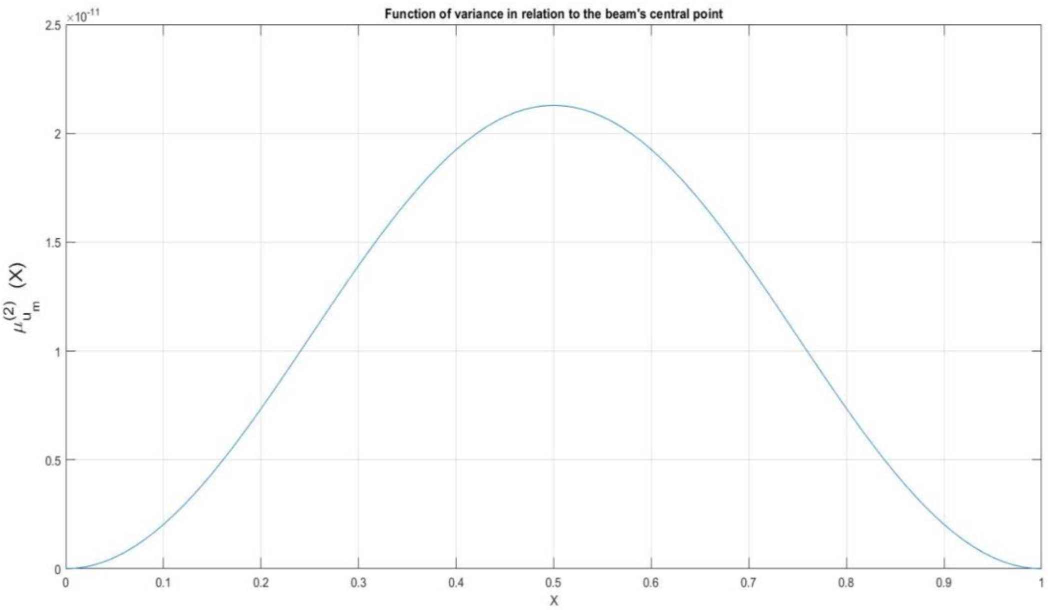

Figure 8- Function of the sample variances corresponding to the beam stiffness, in relation to the position in the beam.

4. CONCLUSION

The present problem presents the solution of a stochastic problem by a beam supported on the extremities, that rests on a foundation described by the Pasternak model. The approximate solution of the problem was proposed using the Galerkin method, which generates a stiffness matrix, on which an inverse operator is needed to calculate the solution to the problem.

First, the equation related to the displacement suffered by the beam was described mathematically, considering as restrictions imposed on the extremities. Following the requirement for the degree of differentiation of the solution of the problem, it was relaxed from the fourth order to the second order, once a displacement differential equation is reduced in the form of an abstract variational problem (PVA).

The article proposes in a second stage to analyze how the uncertainty in the stiffness parameters influences in the first and second statistical moment of the beam displacement function. And according to the results, it was possible to conclude that when the parameter uncertainty falls on the indexed coefficient the variable with the highest degree of differentiability, in this case, EI, the greater is the dispersion of the displacement samples in relation to the average displacement. And that the lower the degree of the variable's differentiability, the smaller is the dispersion, and consequently the greater the reliability of the beam displacement response.

Therefore, care regarding the determination of the parameters that are part of the equation, is different considering its weight in the final solution of the problem, and that obviously when all parameters are uncertain, which corresponds to the worst-case scenario, the greater the degree of dispersion of the final solution.

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