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ATTITUDE DETERMINATION FOR MULTIROTOR AERIAL VEHICLE USING A SUPER-TWISTING SLIDING MODE OBSERVER

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Abstract. *A plethora of attitude determination methods has been studied in the aerospace field in the past. The miniaturization and the falling cost of technology have made it viable for a multirotor aerial vehicles (MAV) field to emerge, and for its researchers to revisit some of these well-known methods. In an attempt to develop a more modern solution to this classical problem, the present work is concerned with the attitude determination for an MAV using a robust sliding-mode estimator. Without angular velocity measurements, the proposed method estimates the MAV three-dimensional attitude through a QUEST algorithm and the vector measurements provided by an accelerometer and a magnetometer. Next, a multivariable super-twisting observer is proposed as the estimation method for the attitude and its derivative, which in turn are used to estimate the MAV's angular velocity. The finite-time convergence of the estimation errors is mathematically proven via a Lyapunov approach. The method's effectiveness is demonstrated by simulation, through which the vehicle's attitude and angular velocity are accurately and robustly estimated.*

Keywords: *Attitude determination, vector measurements, sliding-mode observer, finite time convergence, robust estimation.*

1. INTRODUCTION

The attitude determination from vector measurements has been investigated for many decades in the aerospace field (Wertz, 1978; Wahba, 1965), from which, more recently, the literature on multirotor aerial vehicles (MAVs) has extracted some formulations (Santos *et al.*, 2016; Santos and Gonçalves, 2016). It consists of estimating the attitude matrix or some parameterization (Markley, 2003; Shuster and Junkins, 1993) of it using a set of at least two pairs of non-collinear vector measurements. Following the path paved by the so-called Wahba problem (Wahba, 1965), most of the literature on attitude determination have applied some kind of statistical or stochastic estimator, such as least squares (Shuster and Oh, 1981; Bar-Itzhack, 1996; Oshman *et al.*, 2001) or Kalman filtering (Bar-Itzhack and Cohen, 2005; Markley, 2003; Lefferts *et al.*, 1982; Bar-Itzhack and Idan, 1987; Bar-Itzhack and Oshman, 1985; Bar-Itzhack and Reiner, 1984). We argue that there is still room for investigating the attitude determination problem from the point of view of the robust sliding mode state estimation, from which the MAV literature can benefit from.

Regarding the attitude representation, the attitude quaternion is the preferred one. This is because the quaternion has the minimal number of parameters (four) for a global parameterization of the three-dimensional attitude without singularity (Stuelpnagel, 1964). Moreover, it presents a linear kinematics equation (Wertz, 1978), and therefore allows a good computational efficiency in attitude simulation. A seminal batch solution to the Wahba problem in terms of quaternion is the so-called QUEST algorithm (Shuster and Oh, 1981). It became very popular in the aerospace community (see Markley and Crassidis, 2014, p. 189), due to its simplicity and low computational burden. As a batch method, the QUEST does not use vector measurements taken in the past to estimate the attitude at the current time. To overcome this limitation, Bar-Itzhack (1996) has added a prediction step based on the attitude kinematics and angular velocity measurements, thus originating the so-called REQUEST algorithm. The latter method has been applied to the attitude determination of MAVs using camera vector measurements in Santos *et al.* (2016). Since the optimal Gibbs vector that represents the attitude estimate is a byproduct of the QUEST algorithm, and the experiment studied in this paper does not excite this parametrization's singularity, the system's attitude kinematics and dynamics, as well as the observer equations, are represented in Gibbs vector.

Unfortunately, because of the measurement noise, obtaining the angular velocity through differentiation of the estimated attitude vector would be unwise. Hence, using an observer is a better option. Due to their properties, such as insensitivity to bounded disturbances and finite-time stability of the estimation errors, sliding-mode observers (Utkin *et al.*, 1999; Barbot *et al.*, 2002; Edwards *et al.*, 2007; Shtessel *et al.*, 2014) have been sought after when robust estimation is needed. The super-twisting observer is a second order sliding-mode (SOSM) observer proposed by Levant (1998), with applications studied by Davila *et al.* (2005, 2006); Floquet and Barbot (2007); Moreno and Osorio (2012) and Boiko and Chehadeh (2018). The finite-time convergence of this method has been analyzed via a Lyapunov approach for SISO (Polyakov and Poznyak, 2009; Moreno and Osorio, 2008, 2012) and MIMO (Nagesh and Edwards, 2014) systems. However, in these aforementioned studies, the proposed observer was fed with perfect measurement of the states, which is unrealistic in practical experiments. By mathematically modelling the sensors and estimating the MAV attitude using the QUEST algorithm, our method demonstrates the effects that noisy and indirect attitude measurements would have on the overall estimation accuracy.

The present paper investigates the three-dimensional attitude determination from vector measurements taken from the direction of the gravity acceleration (measured by a 3-axis accelerometer) and the direction of the local magnetic field (measured by a 3-axis magnetometer). These vector measurements are the conventional one in Attitude & Heading Reference Systems (AHRS) of MAVs (Magnussen *et al.*, 2013; Martin and Salaün, 2010). Different from most of the filtering-based and least-squares methods (Santos and Gonçalves, 2016; Santos *et al.*, 2016; Bar-Itzhack, 1996; Markley, 2003; Lefferts *et al.*, 1982; Bar-Itzhack and Idan, 1987; Bar-Itzhack and Oshman, 1985; Bar-Itzhack and Reiner, 1984), the proposed method does not depend on angular velocity measurements. Instead, at each algorithm step, we use the QUEST algorithm to estimate the MAV three-dimensional attitude from the two available vector measurements and use the obtained estimate to feed a multivariable super-twisting observer (STO) formulated in terms of Gibbs vector. The STO, in turn, provides estimates of the Gibbs vector as well as its time derivative, from which, in a final step, the attitude kinematic equation is used to compute an angular velocity estimate.

The remaining text is organized as follows. Section 2 presets the adopted notation. Section 3 defines the main problem of the paper. Section 4 presets the QUEST algorithm. Section 5 formulates the multivariable STO. Section 6 evaluates the proposed method using computer simulation. Finally, Section 7 concludes the paper.

2. NOTATION

Denote the set of real and integer numbers by $\mathbb{R}$ and $\mathbb{Z}$. Moreover, denote by $k \in \mathbb{Z}_{>0}$ the index of discrete time subdivisions. To clarify the vector notation adopted here, consider two different kinds of vectors: physical (geometric) vectors and algebraic vectors. Physical vectors are denoted by italic letters with a right arrow superscript, *e.g.*, $\vec{r}$, while algebraic vectors are represented by boldface lowercase letters, *e.g.*, $\mathbf{r}$. A Cartesian coordinate system (CCS) is defined as $\mathcal{S}^a \triangleq \{A; \vec{x}^{\,a}, \vec{y}^{\,a}, \vec{z}^{\,a}\}$, where a is its identification label, A is a point representing its origin, and $\vec{x}^{\,a}$, $\vec{y}^{\,a}$, and $\vec{z}^{\,a}$ are its orthogonal unit vectors. The corresponding algebraic vector resulting from the projection of $\vec{r}$ onto an arbitrary CCS $\mathcal{S}^a$ is denoted by the same letter, but in a regular bold format, with the subscript a, *i.e.*, $\mathbf{r}_a \in \mathbb{R}^3$. Matrices are denoted by bold uppercase letters, *i.e.*, $\mathbf{A}$. Denote the special orthogonal group by $\mathrm{SO}(3) \triangleq \{\mathbf{A} \in \mathbb{R}^{3\times3} : \mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{I}_3\}$. The 2-norm of an algebraic vector is denoted by $\|.\|$. The notation $\mathbf{A} \succ \mathbf{0}$ is adopted to say that $\mathbf{A} \in \mathbb{R}^{n\times n}$ is positive definite. Consider the arbitrary physical vector $\vec{r}$ again. The relation between its representations $\mathbf{r}_a$ and $\mathbf{r}_b$ is $\mathbf{r}_a = \mathbf{D}^{a/b}(\alpha)\,\mathbf{r}_b$, where $\mathbf{D}^{a/b}(\alpha) \in \mathrm{SO}(3)$ is the attitude matrix of $\mathcal{S}^a$ with respect to $\mathcal{S}^b$ corresponding to a chosen attitude parametrization α. The inverse of $\mathbf{D}^{a/b}$ (which coincides with its transpose) is sometimes denoted by $\mathbf{D}^{b/a}$. Now, let $\vec{r}^{\,a/b}$ represent a physical quantity of $\mathcal{S}^a$ relative to $\mathcal{S}^b$, *e.g.*, $\vec{v}^{\,a/b}$ denotes the (relative) velocity of $\mathcal{S}^a$ with respect to $\mathcal{S}^b$. Let $\mathbf{e}_i$ denote an n-dimensional standard unit vector. The ith component of some algebraic vector $\mathbf{a}$ is denoted by a_i and sometimes by $\mathbf{e}_i^{\mathrm{T}}\mathbf{a}$. Now, consider two physical vectors $\vec{a}$ and $\vec{b}$ as well as the respective algebraic representations in $\mathcal{S}^a$, $\mathbf{a}_a = [a_1\ a_2\ a_3]^{\mathrm{T}}$ and $\mathbf{b}_a$, respectively. Denote the $\mathcal{S}^a$ representation of the vector product $\vec{a} \times \vec{b}$ by the matrix multiplication $[\mathbf{a}_a\times]\mathbf{b}_a$, where $[\mathbf{a}_a\times]$ is the following skew-symmetric matrix

$$[\mathbf{a}_a\times] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Denote the attitude Gibbs vector of $\mathcal{S}^a$ with respect to $\mathcal{S}^b$ by $\mathbf{g}^{a/b} \in \mathbb{R}^3$. This notation can also be abbreviated as $\mathbf{g}$ to avoid notation conflicts with transposition superscripts. The corresponding attitude matrix is given by

$$\mathbf{D}^{a/b}(\mathbf{g}) = \frac{\left(1 - \mathbf{g}^{\mathrm{T}}\mathbf{g}\right)\mathbf{I}_3 + 2\mathbf{g}\mathbf{g}^{\mathrm{T}} - 2\,[\mathbf{g}\times]}{1 + \mathbf{g}^{\mathrm{T}}\mathbf{g}}. \tag{1}$$

The attitude kinematics $\mathcal{S}^a$ with respect to $\mathcal{S}^b$ are described in terms of the attitude Gibbs vector by

$$\dot{\mathbf{g}}^{a/b} = \frac{1}{2}\boldsymbol{\Gamma}\left(\mathbf{g}^{a/b}\right)\omega_a^{a/b}, \tag{2}$$

where $\boldsymbol{\omega}_b^{a/b}$ represents the angular velocity of $\mathcal{S}^a$ with respect to $\mathcal{S}^b$ and

$$\boldsymbol{\Gamma}(\mathbf{g}) \triangleq \mathbf{g}\mathbf{g}^{\mathrm{T}} + [\mathbf{g}\times] + \mathbf{I}_3.$$

3. PROBLEM DEFINITION

Let $\mathcal{S}^b$ represent a CCS with its origin on the center of mass of a Multirotor Aerial Vehicle (MAV) body. Also, $\mathcal{S}^r$ represents an inertial CCS with its origin at a reference point fixed on the ground, with its z-axis, $\vec{z}^r$, aligned with the local vertical. The MAV attitude dynamics can be described by

$$\dot{\omega}_b^{b/r} = \mathbf{J}_b^{-1}\left[(\mathbf{J}_b\,\omega_b^{b/r})\times\right] + \mathbf{J}_b^{-1}\mathbf{u}(t) + \mathbf{d}(t), \tag{3}$$

where $\mathbf{J}_b \in \mathbb{R}^{3\times 3}$ is the inertia matrix in $\mathcal{S}^b$, $\mathbf{u}(t) \in \mathbb{R}^3$ is an input torque, and $\mathbf{d}(t) \in \mathbb{R}^3$ is a bounded disturbance vector.

This MAV is equipped with an accelerometer and a magnetometer. The accelerometer measure $\check{\mathbf{a}}_b \in \mathbb{R}^3$ can be modeled by

$$\check{\mathbf{a}}_b = \mathbf{D}^{b/r}\check{\mathbf{a}}_r + \boldsymbol{\delta}_b^{ac}, \quad \text{for} \quad \check{\mathbf{a}}_r \triangleq \dot{\mathbf{v}}_r^{b/r} + g\mathbf{e}_3, \tag{4}$$

where $\check{\mathbf{a}}_r \in \mathbb{R}^3$ is the $\mathcal{S}^r$ representation of the accelerations affecting the MAV, $\dot{\mathbf{v}}_r^{b/r} \in \mathbb{R}^3$ is the $\mathcal{S}^r$ representation of the relative acceleration of $\mathcal{S}^b$ with respect to $\mathcal{S}^r$, $g\mathbf{e}_3 \in \mathbb{R}^3$ is the gravity acceleration vector, and $\boldsymbol{\delta}_b^{ac} \in \mathbb{R}^3$ is a bounded measurement error. To simplify, assume that $\dot{\mathbf{v}}_r^{b/r} = 0$ throughout the experiment. Consequently, $\check{\mathbf{a}}_r$ keeps constant throughout the experiment.

The magnetometer measure $\check{\mathbf{m}}_b \in \mathbb{R}^3$ can be modeled by

$$\check{\mathbf{m}}_b = \mathbf{D}^{b/r}\check{\mathbf{m}}_r + \boldsymbol{\delta}_b^{mg}, \tag{5}$$

where $\check{\mathbf{m}}_r \in \mathbb{R}^3$ is the $\mathcal{S}^r$ representation of the local magnetic field and $\boldsymbol{\delta}_b^{mg} \in \mathbb{R}^3$ is a bounded measurement error. To simplify, assume that $\check{\mathbf{m}}_r$ keeps constant throughout the experiment.

The derivative of equation (2), followed by the substitution of (3) results in

$$\ddot{\mathbf{g}} = \frac{1}{2}\left[\dot{\Gamma}\left(\mathbf{g}\right)\omega_b^{b/r} + \Gamma\left(\mathbf{g}\right)\left(\mathbf{J}_b^{-1}\left[(\mathbf{J}_b\omega_b^{b/r})\times\right] + \mathbf{J}_b^{-1}\mathbf{u}(t) + \mathbf{d}(t)\right)\right], \tag{6}$$

where $\dot{\Gamma} = \mathbf{g}\dot{\mathbf{g}}^{\mathrm{T}} + \dot{\mathbf{g}}\mathbf{g}^{\mathrm{T}} + [\dot{\mathbf{g}}\times]$. By defining the states as $\mathbf{x}_1 \triangleq \mathbf{g}$ and $\mathbf{x}_2 \triangleq \dot{\mathbf{g}}$, the system state equations can be written as

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2, \tag{7}$$

$$\dot{\mathbf{x}}_2 = \frac{1}{2}\left[\dot{\Gamma}\left(\mathbf{g}\right)\omega_b^{b/r} + \Gamma\left(\mathbf{g}\right)\left(\mathbf{J}_b^{-1}\left[(\mathbf{J}_b\omega_b^{b/r})\times\right] + \mathbf{J}_b^{-1}\mathbf{u}(t) + \mathbf{d}(t)\right)\right], \tag{8}$$

$$\mathbf{y} = \begin{bmatrix} \check{\mathbf{a}}_b^{\mathrm{T}} & \check{\mathbf{m}}_b^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}. \tag{9}$$

Let $\tilde{\mathbf{x}}_1$ and $\tilde{\mathbf{x}}_2$ represent the estimation error of states $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. The goal of this paper is to estimate the MAV attitude and angular velocity using the system model in (7)-(8) and the vector measurements provided by the accelerometer and magnetometer, as well as to present the finite-time stability of the estimation errors at the point $(\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) = (\mathbf{0}, \mathbf{0})$ when using a multivariable STO.

4. THE QUEST ALGORITHM

The Wahba problem for computing the attitude of a body represented in quaternion $\hat{\mathbf{q}}(k)$ at instant k can be stated as the minimization of

$$J(\mathbf{q}(k)) = \frac{1}{2}\sum_{i=1}^{n} a_i \|\check{\boldsymbol{\mu}}_b^i(k) - \mathbf{D}(\mathbf{q}(k))\check{\boldsymbol{\mu}}_r^i(k)\|^2, \tag{10}$$

subject to $\|\mathbf{q}(k)\| = 1$, where $\left(\check{\boldsymbol{\mu}}_b^i(k), \check{\boldsymbol{\mu}}_r^i(k)\right)$ is the pair of vector measurements, $\mathbf{D}(\mathbf{q}(k))$ is the attitude matrix corresponding to the quaternion $\mathbf{q}(k)$, a_i is a positive weight associated with the ith measurement pair, and n is the number of vector measurements available at instant k.

From the work of Shuster and Oh (1981), the minimization problem of equation (10) can be replaced by the maximization of

$$G(\mathbf{q}(k)) = \mathbf{q}(k)^{\mathrm{T}}\mathbf{K}(k)\mathbf{q}(k), \tag{11}$$

where

$$\mathbf{K}(k) \triangleq \begin{bmatrix} \mathbf{S}(k) - \sigma(k)\mathbf{I}_3 & \mathbf{z}(k) \\ \mathbf{z}(k)^{\mathrm{T}} & \sigma(k) \end{bmatrix} \in \mathbb{R}^{4\times 4}, \tag{12}$$

$$\mathbf{S}(k) \triangleq \mathbf{B}(k) + \mathbf{B}(k)^{\mathrm{T}} \in \mathbb{R}^{3\times 3}, \tag{13}$$

$$\sigma(k) \triangleq \frac{1}{m(k)} \sum_{i=1}^{n} a_i \check{\boldsymbol{\mu}}_b^i(k)^{\mathrm{T}} \check{\boldsymbol{\mu}}_r^i(k) \in \mathbb{R}, \tag{14}$$

$$\mathbf{B}(k) \triangleq \frac{1}{m(k)} \sum_{i=1}^{n} a_i \check{\boldsymbol{\mu}}_b^i(k) \check{\boldsymbol{\mu}}_r^i(k)^{\mathrm{T}} \in \mathbb{R}^{3\times 3}, \tag{15}$$

$$\mathbf{z}(k) \triangleq \frac{1}{m(k)} \sum_{i=1}^{n} a_i \left[\check{\boldsymbol{\mu}}_b^i(k)\times\right] \check{\boldsymbol{\mu}}_r^i(k) \in \mathbb{R}^{3}, \tag{16}$$

and

$$m(k) \triangleq \sum_{i=1}^{n} a_i. \tag{17}$$

The solution $\hat{\mathbf{q}}_Q(k)$ to the maximization of $G(\mathbf{q}(k))$ in (11) is given by following eigenvalue/eigenvector equation:

$$\mathbf{K}(k)\hat{\mathbf{q}}_Q(k) = \lambda \hat{\mathbf{q}}_Q(k), \tag{18}$$

where λ is the maximum eigenvalue of $\mathbf{K}(k)$. In other words, the solution $\hat{\mathbf{q}}_Q(k)$ is the eigenvector corresponding to the maximum eigenvalue of $\mathbf{K}(k)$. Shuster and Oh (1981) present an efficient algorithm for solving the above eigenvalue/eigenvector problem; this is the well-known QUEST algorithm. The same paper also shows that λ is close to 1 (for noise-free measurements, it is exactly 1). In short, the cited work shows that the optimal Gibbs vector (see Wertz, 1978, for information about different attitude parameterizations) is given by

$$\hat{\mathbf{g}}_Q(k) = \left[(\lambda + \sigma(k))\,\mathbf{I}_3 - \mathbf{S}(k)\right]^{-1} \mathbf{z}(k), \tag{19}$$

and the corresponding quaternion is given by

$$\hat{\mathbf{q}}_Q(k) = \frac{1}{\sqrt{(1 + \hat{\mathbf{g}}_Q(k)^{\mathrm{T}}\hat{\mathbf{g}}_Q(k))}} \begin{bmatrix} \hat{\mathbf{g}}_Q(k) \\ 1 \end{bmatrix}. \tag{20}$$

5. MULTIVARIABLE SUPER-TWISTING SLIDING MODE OBSERVER

A second-order sliding algorithm for single input systems, based on the one proposed by Levant (1998), can be described by

$$\dot{\sigma}_1 = \kappa_1 |\sigma_1| \,\mathrm{sign}\,(\sigma_1) + \sigma_2, \tag{21}$$
$$\dot{\sigma}_2 = -\kappa_2 \,\mathrm{sign}\,(\sigma_1) + \delta, \tag{22}$$

where $\sigma_1 \in \mathbb{R}$ and $\sigma_2 \in \mathbb{R}$ are its state variables, $\kappa_1, \kappa_2 > 0$ are design scalar parameters, and $\delta \in \mathbb{R}$ a unknown, but bounded, disturbance signal. This particular algorithm receives the denomination of Super-Twisting. In order to extend its application to multi-input systems, it is necessary to substitute the sign function for a suitable multivariable version. Here, the unit-vector approach will be considered, altering equations (21)-(22) to their mutivariable version given by

$$\dot{\boldsymbol{\sigma}}_1 = \kappa_1 \frac{\boldsymbol{\sigma}_1}{\|\boldsymbol{\sigma}_1\|^{1/2}} + \boldsymbol{\sigma}_2, \tag{23}$$
$$\dot{\boldsymbol{\sigma}}_2 = -\kappa_2 \frac{\boldsymbol{\sigma}_1}{\|\boldsymbol{\sigma}_1\|} + \boldsymbol{\delta}, \tag{24}$$

where $\boldsymbol{\sigma}_1 \in \mathbb{R}^n$, $\boldsymbol{\sigma}_2 \in \mathbb{R}^n$, and $\boldsymbol{\delta} \in \mathbb{R}^n$ are now states and disturbance vectors, respectively. As the differential equations in (10) to (13) have discontinuous right hand sides, their solutions must be understood in the Filippov sense (Filippov, 1989). Also, assume the disturbance satisfies the restriction $\|\boldsymbol{\delta}\| \leq \rho$, and that the upper bound ρ is known. It can be shown that, for the suitable choice of the gains κ_1 and κ_2, $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2) = (\mathbf{0}, \mathbf{0})$ is finite-time stable.

For the system (23)-(24), let us consider a Lyapunov-function candidate based on the one proposed by Nagesh and Edwards (2014)

$$V\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)=2\kappa_2\|\boldsymbol{\sigma}_1\|+\frac{1}{2}\boldsymbol{\sigma}_2^{\mathrm{T}}\boldsymbol{\sigma}_2+\boldsymbol{\zeta}^{\mathrm{T}}\boldsymbol{\zeta}, \tag{25}$$

where $\boldsymbol{\zeta}\triangleq\kappa_1\frac{\boldsymbol{\sigma}_1}{\|\boldsymbol{\sigma}_1\|^{1/2}}-\boldsymbol{\sigma}_2$. It can be verified that $V\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)$ is radially unbounded and positive definite. Differentiating (25) and then substituting for (23)-(24) results in

$$\dot{V}\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)=\left(2\kappa_2+\frac{\kappa_1^2}{2}\right)\frac{\boldsymbol{\sigma}_1\dot{\boldsymbol{\sigma}}_1}{\|\boldsymbol{\sigma}_1\|}+2\boldsymbol{\sigma}_2^{\mathrm{T}}\dot{\boldsymbol{\sigma}}_2-\kappa_1\left(-\frac{1}{2}\frac{\left(\boldsymbol{\sigma}_2^{\mathrm{T}}\dot{\boldsymbol{\sigma}}_1\right)\left(\boldsymbol{\sigma}_1^{\mathrm{T}}\dot{\boldsymbol{\sigma}}_1\right)}{\|\boldsymbol{\sigma}_1\|^{5/2}}+\frac{\left(\boldsymbol{\sigma}_1^{\mathrm{T}}\dot{\boldsymbol{\sigma}}_2\right)+\left(\boldsymbol{\sigma}_2^{\mathrm{T}}\dot{\boldsymbol{\sigma}}_1\right)}{\|\boldsymbol{\sigma}_1\|^{1/2}}\right), \tag{26}$$

$$\dot{V}\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)=-\left(\kappa_1\kappa_2+\frac{\kappa_1^3}{2}\right)\|\boldsymbol{\sigma}_1\|^{1/2}+2\boldsymbol{\sigma}_2^{\mathrm{T}}\boldsymbol{\delta}+\frac{\kappa_1}{2}\frac{|\boldsymbol{\sigma}_2^{\mathrm{T}}\boldsymbol{\sigma}_1|^2}{\|\boldsymbol{\sigma}_1\|^{5/2}}-\kappa_1\frac{\boldsymbol{\sigma}_1^{\mathrm{T}}\boldsymbol{\delta}}{\|\boldsymbol{\sigma}_1\|^{1/2}}+\kappa_1^2\frac{\boldsymbol{\sigma}_2^{\mathrm{T}}\boldsymbol{\sigma}_1}{\|\boldsymbol{\sigma}_1\|}-\kappa_1\frac{\boldsymbol{\sigma}_2^{\mathrm{T}}\boldsymbol{\sigma}_2}{\|\boldsymbol{\sigma}_1\|^{1/2}}. \tag{27}$$

Increasing the right hand side of equation (27) yields the inequality

$$\dot{V}\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)\leq-\left(\kappa_1\kappa_2+\frac{\kappa_1^3}{2}\right)\|\boldsymbol{\sigma}_1\|^{1/2}+2|\boldsymbol{\sigma}_2^{\mathrm{T}}\boldsymbol{\delta}|+\frac{\kappa_1}{2}\frac{|\boldsymbol{\sigma}_2^{\mathrm{T}}\boldsymbol{\sigma}_1|^2}{\|\boldsymbol{\sigma}_1\|^{5/2}}+\kappa_1\frac{|\boldsymbol{\sigma}_1^{\mathrm{T}}\boldsymbol{\delta}|}{\|\boldsymbol{\sigma}_1\|^{1/2}}+\kappa_1^2\frac{|\boldsymbol{\sigma}_2^{\mathrm{T}}\boldsymbol{\sigma}_1|}{\|\boldsymbol{\sigma}_1\|}-\kappa_1\frac{\|\boldsymbol{\sigma}_2\|^2}{\|\boldsymbol{\sigma}_1\|^{1/2}}. \tag{28}$$

By using the Cauchy-Schwarz inequality and substituting the disturbance $\boldsymbol{\delta}$ by its upper bound, equation (28) right hand side can be increased to

$$\dot{V}\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)\leq-\left(\kappa_1\kappa_2+\frac{\kappa_1^3}{2}\right)\|\boldsymbol{\sigma}_1\|^{1/2}+2\|\boldsymbol{\sigma}_2\|\rho-\frac{\kappa_1}{2}\frac{\|\boldsymbol{\sigma}_2\|^2}{\|\boldsymbol{\sigma}_1\|^{1/2}}+\kappa_1\|\boldsymbol{\sigma}_1\|^{1/2}\rho+\kappa_1^2\|\boldsymbol{\sigma}_2\|. \tag{29}$$

Let $\mathbf{x}\in\mathbb{R}^2$ be a vector defined as

$$\mathbf{x}\triangleq\left[\begin{array}{cc}\|\boldsymbol{\sigma}_1\|^{1/2} & \|\boldsymbol{\sigma}_2\|\end{array}\right]^{\mathrm{T}}. \tag{30}$$

With this new vector, (29) can be rearranged as

$$\dot{V}\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)\leq\frac{-1}{\|\boldsymbol{\sigma}_1\|^{1/2}}\mathbf{x}^{\mathrm{T}}\boldsymbol{\Omega}\mathbf{x},\quad\text{for}\quad\boldsymbol{\Omega}\triangleq\left[\begin{array}{cc}\frac{1}{2}\kappa_1^3+\kappa_1\kappa_2-\kappa_1\rho & \frac{1}{2}\kappa_1^2-\rho\\ \frac{1}{2}\kappa_1^2-\rho & \frac{1}{2}\kappa_1\end{array}\right], \tag{31}$$

where $\boldsymbol{\Omega}$ is symmetric and positive definite for $\kappa_1>\sqrt{2\rho}$ and $\kappa_2>3\rho+\frac{2\rho^2}{\kappa_1^2}$. Using Raleigh's inequality, (31) can be rewritten as

$$\dot{V}\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)\leq\frac{-1}{\|\boldsymbol{\sigma}_1\|^{1/2}}\lambda_{min}(\boldsymbol{\Omega})\|\mathbf{x}\|^2. \tag{32}$$

Now, let $\mathbf{X}\in\mathbb{R}^6$ be a vector defined as

$$\mathbf{X}\triangleq\left[\begin{array}{cc}\frac{\boldsymbol{\sigma}_1}{\|\boldsymbol{\sigma}_1\|^{1/2}}^{\mathrm{T}} & \boldsymbol{\sigma}_2^{\mathrm{T}}\end{array}\right]^{\mathrm{T}}. \tag{33}$$

It is possible to note that $\|\mathbf{X}\|=\|\mathbf{x}\|,\forall\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)$. As pointed by Nagesh and Edwards (2014) and Moreno and Osorio (2008), for this new vector, equation (25) can be rearranged as

$$V\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)=\mathbf{X}^{\mathrm{T}}\mathbf{P}\mathbf{X},\quad\text{for}\quad\mathbf{P}\triangleq\left[\begin{array}{cc}2\kappa_2+\kappa_1^2 & -\kappa_1\\ -\kappa_1 & 1.5\end{array}\right], \tag{34}$$

with $\mathbf{P}$ symmetric and positive definite for the aforementioned values of the gains κ_1 and κ_2. From Rayleigh's inequality, $V\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)\leq\lambda_{max}(\mathbf{P})\|\mathbf{X}\|^2$ and $V^{1/2}>\sqrt{\lambda_{min}(\mathbf{P})}\|\boldsymbol{\sigma}_1\|^{1/2}$. Therefore, equation (32) can be written as

$$\begin{aligned}\dot{V}\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)&\leq\frac{-1}{\|\boldsymbol{\sigma}_1\|^{1/2}}\lambda_{min}(\boldsymbol{\Omega})\|\mathbf{X}\|^2\leq\frac{-1}{\|\boldsymbol{\sigma}_1\|^{1/2}}\frac{\lambda_{min}(\boldsymbol{\Omega})}{\lambda_{max}(\mathbf{P})}V,\\ \dot{V}\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)&\leq-\eta V^{1/2},\quad\text{for}\quad\eta=\frac{\lambda_{min}(\boldsymbol{\Omega})\sqrt{\lambda_{min}(\mathbf{P})}}{\lambda_{max}(\mathbf{P})},\end{aligned} \tag{35}$$

which, since $\boldsymbol{\Omega}\succ\mathbf{0}$ and $\mathbf{P}\succ\mathbf{0}$, satisfies the finite-time stability theorem of Bhat and Bernstein (2000) and proves that $\left(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2\right)=(\mathbf{0},\mathbf{0})$ is finite-time stable. Therefore, it is feasible to assign the super-twisting algorithm dynamics to the estimation error. Define the estimation error of the states in (7)-(8) as

$$\tilde{\mathbf{x}}_1\triangleq\mathbf{x}_1-\hat{\mathbf{x}}_1, \tag{36}$$

$$\tilde{\mathbf{x}}_2\triangleq\mathbf{x}_2-\hat{\mathbf{x}}_2, \tag{37}$$

where $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are the states estimates. Then, the super-twisting algorithm in (23)-(24) can be rewritten as

$$\dot{\tilde{\mathbf{x}}}_1 = -\kappa_1 \frac{\tilde{\mathbf{x}}_1}{\|\tilde{\mathbf{x}}_1\|^{1/2}} + \tilde{\mathbf{x}}_2, \tag{38}$$

$$\dot{\tilde{\mathbf{x}}}_2 = -\kappa_2 \frac{\tilde{\mathbf{x}}_1}{\|\tilde{\mathbf{x}}_1\|} + \boldsymbol{\delta}, \tag{39}$$

with $\boldsymbol{\delta} \triangleq (f(\mathbf{x})-f(\hat{\mathbf{x}}))+(b(\mathbf{x})-b(\hat{\mathbf{x}}))\mathbf{u}(t)+\boldsymbol{\Gamma}(\mathbf{g})\mathbf{d}(t)$. From equations (7)-(8) and (38)-(39), the following multivariable super-twisting sliding mode observer is obtained

$$\dot{\hat{\mathbf{x}}}_1 = \hat{\mathbf{x}}_2 + \kappa_1 \frac{\tilde{\mathbf{x}}_1}{\|\tilde{\mathbf{x}}_1\|^{1/2}}, \tag{40}$$

$$\dot{\hat{\mathbf{x}}}_2 = \frac{1}{2}\left[\dot{\Gamma}\left(\hat{\mathbf{x}}_1\right)\omega_b^{b/r} + \Gamma\left(\hat{\mathbf{x}}_1\right)\left(\mathbf{J}_b^{-1}\left[(\mathbf{J}_b\omega_b^{b/r})\times\right] + \mathbf{J}_b^{-1}\mathbf{u}(t)\right)\right] + \kappa_2 \frac{\tilde{\mathbf{x}}_1}{\|\tilde{\mathbf{x}}_1\|}, \tag{41}$$

where $\dot{\Gamma}\left(\hat{\mathbf{x}}_1\right) = \hat{\mathbf{x}}_1\hat{\mathbf{x}}_2^{\mathrm{T}} + \hat{\mathbf{x}}_2\hat{\mathbf{x}}_1^{\mathrm{T}} + [\hat{\mathbf{x}}_2\times]$.

6. METHOD EVALUATION

Consider the following angular velocity signal

$$\boldsymbol{\omega}_b^{b/r}(t) = \frac{2\pi}{20}\begin{bmatrix} \sin(0.5\pi t) \\ \sin(0.5\pi t + 0.5\pi) \\ \sin(0.5\pi t + \pi) \end{bmatrix}, \tag{42}$$

as a forcing function acting on equations (8) and (41). Also, consider that the system (7)-(8) is perturbed by a disturbance vector

$$\mathbf{d}(t) = 0.05\begin{bmatrix} \sin(50\pi t) \\ \sin(50\pi t + 0.5\pi) \\ \sin(50\pi t + \pi) \end{bmatrix}. \tag{43}$$

At every instant k, the QUEST algorithm will use the two pairs of vector measurements obtained by simulating equations (4) and (5), according to the following definitions

$$\check{\boldsymbol{\mu}}_b^1(k) \triangleq \check{\mathbf{a}}_b, \quad \check{\boldsymbol{\mu}}_r^1(k) \triangleq \check{\mathbf{a}}_r, \quad \check{\boldsymbol{\mu}}_b^2(k) \triangleq \check{\mathbf{m}}_b, \quad \check{\boldsymbol{\mu}}_r^2(k) \triangleq \check{\mathbf{m}}_r. \tag{44}$$

Since the system does not provide a perfect measurement of $\mathbf{x}_1$, the state estimation error described in (36), and used in the observer model in equations (40)-(41), will be rewritten as

$$\tilde{\mathbf{x}}_1 = \hat{\mathbf{g}}_Q(k) - \hat{\mathbf{x}}_1, \tag{45}$$

where $\hat{\mathbf{g}}_Q(k)$ is the optimal Gibbs vector obtained from computing the accelerometer and magnetometer vector measurements through the QUEST algorithm (see equation (19)). It is possible to see that the existence of measurement error directly affects how close $\hat{\mathbf{g}}_Q(k)$ is to the real attitude vector $\mathbf{g}$. Consequently $\tilde{\mathbf{x}}_1$ and the observed states will also be affected by the aforementioned error.

After obtaining the estimates $\hat{\mathbf{g}}$ and $\dot{\hat{\mathbf{g}}}$, the attitude kinematic equation (2) is used to compute an angular velocity estimate. Since the matrix $\boldsymbol{\Gamma}(\mathbf{g})$ in equation (2) is non-singular for every $\mathbf{g}$ in this simulation, the MAV angular velocity can be calculated by

$$\omega_b^{b/r} = 2\boldsymbol{\Gamma}^{-1}(\mathbf{g})\dot{\mathbf{g}}, \tag{46}$$

and, inspired by the certainty equivalence principle (de Water and Willems, 1981), the angular velocity estimate $\hat{\omega}_b^{b/r}$ can be calculated by (46), by assuming the estimated parameters $\hat{\mathbf{g}}$ and $\dot{\hat{\mathbf{g}}}$ as the real ones. The angular velocity estimation error is given by $\tilde{\omega}_b^{b/r} \triangleq \omega_b^{b/r} - \hat{\omega}_b^{b/r}$.

Table 1: Simulation Parameters

Symbol	Description	Value
t_f	Simulation Time	5s
Ts	Integration Step	0.0001s
J	Inertia Matrix	$\mathrm{diag}\,(0.05, 0.05, 0.02)$ kgm^2
$\mathbf{g}_0$	Initial State	$[0.11\ \ 0.09\ \ 0.11]^{\mathrm{T}}$
$\hat{\mathbf{g}}_0$	Initial State Estimate	$[-0.17\ -0.10\ \ 0.3]^{\mathrm{T}}$
$\dot{\hat{\mathbf{g}}}_0$	Initial State Estimate	$[-0.15\ \ 0.07\ \ 0.11]^{\mathrm{T}}$
g	Gravity Acceleration	9.81 m/s^2
$\check{\mathbf{m}}_r$	Local Magnetic Field	$[13.7\ \ -4.6\ \ -10.9]^{\mathrm{T}}$ μT

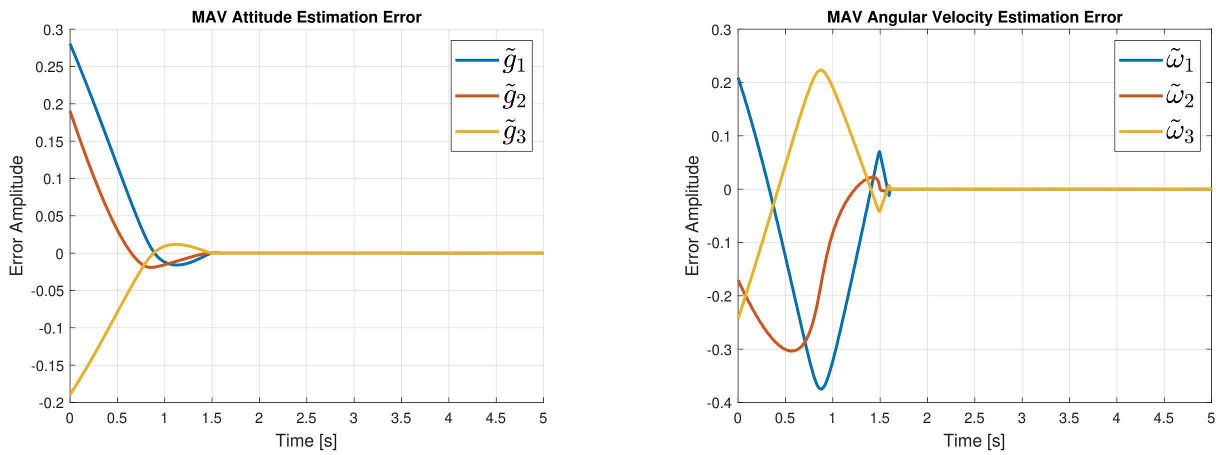
Initially, consider a perfect measurement of $\breve{\mathbf{a}}_b$ and $\breve{\mathbf{m}}_b$, *i.e.* $\boldsymbol{\delta}_b^{ac} = \boldsymbol{\delta}_b^{mg} = 0$. For the parameters in Table 1 and gains $\kappa_1 = 1$ and $\kappa_2 = 0.5$, which satisfy the conditions stated in Section 5, figures 1a and 1b illustrate the behaviour of the estimation error of $\mathbf{g}$ and $\boldsymbol{\omega}$. It is possible to see that the sliding mode observer is able to correctly estimate the system's states, even under the influence of an external disturbance, driving the estimation error to zero at approximately 1.6s.

Finally, consider the following bounded noise signals

$$\boldsymbol{\delta}_b^{ac}(t) = 1.5 \times 10^{-5} \begin{bmatrix} \sin(125\pi t) + 0.333 \sin(625\pi t) \\ 0 \\ \sin(250\pi t + \pi) + 0.333 \sin(1250\pi t + \pi) \end{bmatrix}, \tag{47}$$

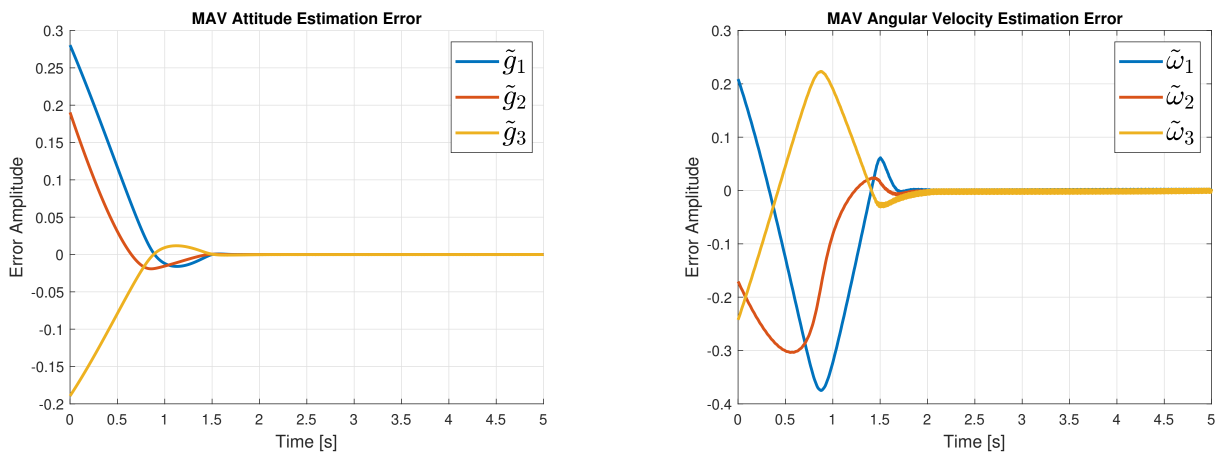
$$\boldsymbol{\delta}_b^{mg}(t) = 2 \times 10^{-2} \begin{bmatrix} \sin(125\pi t) + 0.333 \sin(625\pi t) \\ 0 \\ \sin(250\pi t + \pi) + 0.333 \sin(1250\pi t + \pi) \end{bmatrix}. \tag{48}$$

For the same simulation parameters and gains, figures 2a and 2b illustrate the behaviour of the estimation error of $\mathbf{g}$ and $\boldsymbol{\omega}$ when the system is affected by sensor measurement noise. The figures demonstrate how the inclusion of measurement error has an enduring effect on the states estimation.



(a) (b)

Figure 1: Attitude and angular velocity estimation errors over time.



(a) (b)

Figure 2: Attitude and Angular Velocity estimation errors over time, under the influence of measurement noise.

7. CONCLUSION

In this paper, a multivariable super-twisting sliding-mode observer was employed to estimate the attitude of an MAV, represented in Gibbs vector, and its angular velocity. The stability of the algorithm was theoretically proven and, with the help of a numerical simulation, its effectiveness was demonstrated. The observer is shown to be robust to the influence of a bounded disturbance, seen as a model uncertainty. However, the same property is not seen when the system is under the influence of a bounded disturbance applied as a measurement error, in which case the second state estimation error does not converge to zero. For future works, we intend to investigate our method's effectiveness using vector measurements from real sensors embedded on an MAV, as well as compare the accuracy of the estimate obtained from different attitude determination algorithms.

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9. REFERENCES

Bar-Itzhack, I.Y., 1996. "Request - a recursive quest algorithm for sequential attitude determination". *Astrodynamics Conference*. doi:10.2514/6.1996-3617.

Bar-Itzhack, I.Y. and Cohen, Y., 2005. "Geometry based euler-vector and quaternion recursive-estimators". *AIAA Guidance, Navigation, and Control Conference and Exhibit*. doi:10.2514/6.2005-6398.

Bar-Itzhack, I.Y. and Idan, M., 1987. "Recursive attitude determination from vector observations euler angle estimation". *Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 2, p. 152–157. doi:10.2514/3.22911.

Bar-Itzhack, I.Y. and Oshman, Y., 1985. "Attitude determination from vector observations: Quaternion estimation". *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-21, No. 1, p. 128–136. doi:10.1109/taes.1985.310546.

Bar-Itzhack, I.Y. and Reiner, P., 1984. "Recursive attitude determination from vector observations: Direction cosine matrix identification". *Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 1, p. 51–56. doi:10.2514/3.56362.

Barbot, J., Djemai, M. and Boukhobza, T., 2002. "Sliding mode observers". *Sliding mode control in engineering*, Vol. 11, p. 33.

Bhat, P.S. and Bernstein, D.S., 2000. "Finite-time stability of continuous autonomous systems". *SIAM Journal on Control and Optimization*, Vol. 38, No. 3, p. 751–766. doi:10.1137/s0363012997321358.

Boiko, I. and Chehadeh, M., 2018. "Sliding mode differentiator/observer for quadcopter velocity estimation through sensor fusion". *International Journal of Control*, Vol. 91, No. 9, p. 2113–2120. doi:10.1080/00207179.2017.1421775.

Davila, J., Fridman, L. and Levant, A., 2005. "Second-order sliding-mode observer for mechanical systems". *IEEE Transactions on Automatic Control*, Vol. 50, No. 11, p. 1785–1789. doi:10.1109/tac.2005.858636.

Davila, J., Fridman, L. and Poznyak, A., 2006. "Observation and identification of mechanical systems via second order sliding modes". *International Workshop on Variable Structure Systems, 2006. VSS06*. doi:10.1109/vss.2006.1644523.

de Water, H.V. and Willems, J., 1981. "The certainty equivalence property in stochastic control theory". *IEEE Transactions on Automatic Control*, Vol. 26, No. 5, p. 1080–1087. doi:10.1109/tac.1981.1102781.

Edwards, C., Spurgeon, S.K., Tan, C.P. and Patel, N., 2007. "Sliding-mode observers". *Mathematical Methods for Robust and Nonlinear Control Lecture Notes in Control and Information Sciences*, p. 221–242. doi: 10.1007/978-1-84800-025-4_8.

Filippov, A.F., 1989. *Differential equations with discontinuous righthand sides*. Kluwer Academic Pubr.

Floquet, T. and Barbot, J.P., 2007. "Super twisting algorithm-based step-by-step sliding mode observers for nonlinear systems with unknown inputs". *International Journal of Systems Science*, Vol. 38, No. 10, p. 803–815. doi:10.1080/00207720701409330.

Lefferts, E., Markley, F. and Shuster, M., 1982. "Kalman filtering for spacecraft attitude estimation". *20th Aerospace Sciences Meeting*. doi:10.2514/6.1982-70.

Levant, A., 1998. "Robust exact differentiation via sliding mode technique". *Automatica*, Vol. 34, No. 3, p. 379–384. doi:10.1016/s0005-1098(97)00209-4.

Magnussen, O., Ottestad, M. and Hovland, G., 2013. "Experimental validation of a quaternion-based attitude estimation with direct input to a quadcopter control system". *2013 International Conference on Unmanned Aircraft Systems (ICUAS)*. doi:10.1109/icuas.2013.6564723.

Markley, F.L., 2003. "Attitude error representations for kalman filtering". *Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 2, p. 311–317. doi:10.2514/2.5048.

Markley, F.L. and Crassidis, J.L., 2014. *Fundamentals of spacecraft attitude determination and control*. Springer.

Martin, P. and Salaün, E., 2010. "Design and implementation of a low-cost observer-based attitude and heading reference system". *Control Engineering Practice*, Vol. 18, No. 7, p. 712–722. doi:10.1016/j.conengprac.2010.01.012.

Moreno, J.A. and Osorio, M., 2008. "A lyapunov approach to second-order sliding mode controllers and observers". *2008 47th IEEE Conference on Decision and Control*. doi:10.1109/cdc.2008.4739356.

Moreno, J.A. and Osorio, M., 2012. "Strict lyapunov functions for the super-twisting algorithm". *IEEE Transactions on Automatic Control*, Vol. 57, No. 4, p. 1035–1040. doi:10.1109/tac.2012.2186179.

Nagesh, I. and Edwards, C., 2014. "A multivariable super-twisting sliding mode approach". *Automatica*, Vol. 50, No. 3, p. 984–988. doi:10.1016/j.automatica.2013.12.032.

Oshman, Y., Bar-Itzhack, I.Y. and Choukroun, D., 2001. "Optimal request algorithm for attitude determination". *AIAA Guidance, Navigation, and Control Conference and Exhibit*. doi:10.2514/6.2001-4153.

Polyakov, A. and Poznyak, A., 2009. "Reaching time estimation for "super-twisting" second order sliding mode controller via lyapunov function designing". *IEEE Transactions on Automatic Control*, Vol. 54, No. 8, pp. 1951–1955. doi: 10.1109/TAC.2009.2023781.

Santos, D.A., Ballet, R., Moura, E.A. and Góes, L.C.S., 2016. "Visual-inertial REQUEST for attitude determination of multirotor aerial vehicles". *Anais do IX Congresso Nacional de Engenharia Mecânica*. doi:10.20906/cps/con-2016-0648.

Santos, D.A. and Gonçalves, P.F.S.M., 2016. "Attitude determination of multirotor aerial vehicles using camera vector measurements". *Journal of Intelligent & Robotic Systems*, Vol. 86, No. 1, p. 139–149. doi:10.1007/s10846-016-0418-0.

Shtessel, Y., Edwards, C., Fridman, L. and Levant, A., 2014. *Sliding mode control and observation*. Birkhäuser.

Shuster, M. and Junkins, J.L., 1993. *A survey of attitude representations*. American Astronautical Society.

Shuster, M.D. and Oh, S.D., 1981. "Three-axis attitude determination from vector observations". *Journal of Guidance and Control*, Vol. 4, No. 1, p. 70–77. doi:10.2514/3.19717.

Stuelpnagel, J., 1964. "On the parametrization of the three-dimensional rotation group". *SIAM Review*, Vol. 6, No. 4, p. 422–430. doi:10.1137/1006093.

Utkin, V., Guldner, J. and Jingxin, S., 1999. *Sliding mode control in electromechanical systems*. Taylor & Francis.

Wahba, G., 1965. "A least squares estimate of satellite attitude". *SIAM Review*, Vol. 7, No. 3, p. 409–409. doi:10.1137/1007077.

Wertz, J.P., 1978. *Spacecraft attitude determination and control*. Kluwer.

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