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Optimal Time-fixed Earth-Moon Trajectories

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Abstract. *This work analyses time-fixed Earth-Moon trajectories to transfer a space vehicle from a circular Low Earth Orbit to a circular Low Moon Orbit by application of two impulses. The majority of previous works deals with minimum fuel consumption trajectories without any constraint on the time of flight. However, the time of flight for crewed missions is of paramount importance. Based on this argument, this work analyses and formulates problems to solve trajectories by prescribing this parameter. At first, the classic lunar patched-conic approximation and the circular restricted three-body model are considered to study the possibilities of time fixed Earth-Moon trajectories. It is observed that if one prescribes a feasible total time of flight, there is only one Earth-Moon direct ascent trajectory solution which is obtained by a formulation of a two-point boundary value problem. Due to the existence of only one solution, the total fuel consumption, which is represented by the sum of the two velocity increments, can not be optimized. However, if the eccentricity of the Moon's orbit around the Earth is considered, an extra parameter, related to the position of the Moon on its eccentric orbit, is added to the models. In this case, an one-degree optimization problem is formulated to determine the minimum fuel consumption trajectory for a prescribed time of flight. Each optimal time-fixed solution corresponds to an optimal value of the Moon's position on its eccentric orbit at the initial time. A curve of time of flight versus optimal Moon's position is built, in which a minimum fuel consumption point is observed.*

Keywords: *Earth-Moon trajectories, Optimal trajectories, time-fixed trajectories.*

1. Introduction

The determination of trajectories on space missions can involve several types of constraint such as: geometry specification of the main gravitational bodies that determine the launch windows (Yim *et al.*, 2015); flyby altitude to precisely deflect the trajectory; operational and engineering design limitations (Ferrier and Epenoy, 2001); finite fuel consumption (Ocampo, 2005); minimum time of flight (Pino and Circi, 2017). Each one of these constraints is characterized by one or more parameters for the mathematical development in a preliminary mission analysis. Depending on the mission, a constraint can be more important than the others, so that a characteristic performance index of the mission is defined for the optimization problem. Many works have studied the optimal trajectories by minimizing the total fuel consumption of the space vehicle. Sweetser (1991) has concluded that for an Earth-Moon trajectory departing from an altitude of 167 km around the Earth and arriving in an altitude of 100 km around the Moon, the least required total fuel consumption is about 3721 m/s in a circular restricted three-body problem. Topputo (2013) has obtained thousands of solution in the context of the four-body problem and it has built a front of Pareto to show the best balance between cost and transfer time. de Almeida Junior *et al.* (2021) have developed a new approach to solve the Earth-Moon transfer problem by using the Theory of Functional Connections. In this work, minimum fuel consumption are obtained in the context of a model that considers multiple bodies and perturbations.

For crewed mission, however, the time of flight becomes an important parameter due to the biological constraint of the crew. In the severe conditions in the space, it is important to keep the crew on this environment for the shortest time or, at least, a commitment relationship between time of flight and fuel consumption must be established. In the Apollo 11 mission, for instance, the direct ascent maneuver (translunar phase) lasted 3.2504 days (Houston, 1969), which is smaller than the time of flight for minimum fuel consumption (close to 4.6 days (Gagg Filho and da Silva Fernandes, 2017)). The front of Pareto determined by Topputo (2013) has already provided important information relating time of flight and fuel consumption. Nevertheless, the majority of the works deals with time-free optimal trajectories. On the other hand, the work by Miele and Mancuso (2001) have dealt with time-fixed optimal trajectories in the context of the circular restricted three-body problem. In this work, some values of the time of flight are prescribed and the penalty on the fuel consumption is calculated with respect to the time-free optimal solution.

The present work follows the same approach of Miele and Mancuso (2001) but in the context of elliptic models.

Therefore, the time of flight is an important parameter in the mathematical formulation that determines direct ascent trajectories for an Earth-Moon mission. In this way, this parameter is prescribed for two-point boundary value problems (TPBVP) and for an optimization problem that minimizes the fuel consumption.

2. Objectives

This work analyses the possibility of time-fixed trajectories and determines optimal bi-impulsive time-fixed trajectories that minimize the fuel consumption for an Earth-Moon mission. This transfer problem is enunciated as:

It is desired to transfer a space vehicle from a circular low Earth orbit (LEO) to a circular low Moon orbit (LMO) by application of two impulsive velocity increments. The first accelerating velocity increment inserts the space vehicle into a transfer trajectory; and, the second decelerating velocity increment decelerates the space vehicle and it circularizes the motion of the space vehicle in the prescribed LMO. The fuel consumption is represented by the sum of the velocity increments Marec (1979).

3. Formulation

The Earth-Moon transfer problem is formulated based on three models. The first model is the classic lunar patched-conic approximation described by Bate *et al.* (1971), which is based on the two-body problem dynamics. The second one is the circular restricted three-body problem, in which the eccentricity of the primaries's is neglected. The last one is the elliptic restricted three-body problem, in which the eccentricity of the Earth's and Moon's orbit around the barycenter of the Earth-Moon system is included. In order to determine a time-fixed Earth-Moon transfer trajectory, one must firstly analyzes the possibility of such trajectories. In this way, the first two-models are used for this goal; then, the third model is utilized to determine the time-fixed trajectories.

3.1 Lunar patched-conic approximation

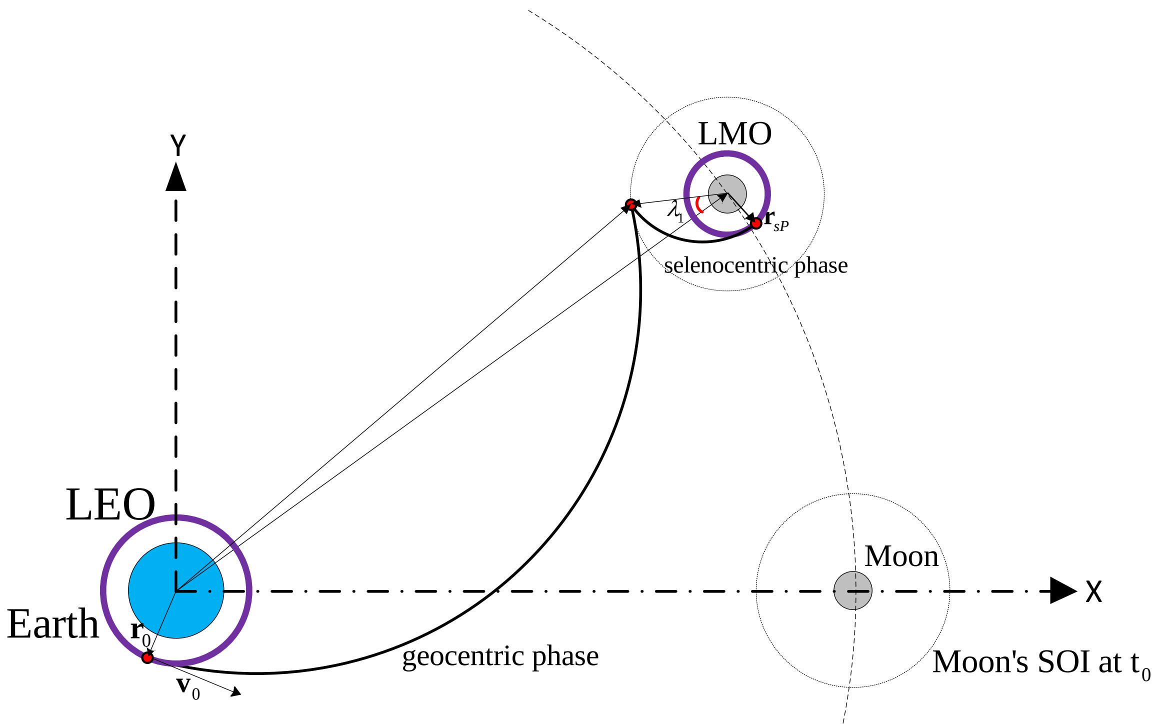


Figure 1: Lunar patched-conic approximation.

The classic lunar patched-conic approximation determines the Earth-Moon transfer trajectory by dividing it into two conics: the first one is an elliptic geocentric phase in which the space vehicle is subjected only to the gravitational field of the Earth; and the second one is a hyperbolic selenocentric phase in which the space vehicle is subjected only to the gravitational field of the Moon. The concept of Sphere of Influence (SOI) is applied to define the phases. In this way, a first accelerating velocity increment Δv_{LEO} inserts a space vehicle from the LEO into the geocentric phase. When the space vehicle reaches the boundary of the Moon's SOI, the selenocentric phase starts. A second decelerating velocity increment is applied at the periselenium to circularize the motion of the space vehicle into the LMO. The transfer trajectory based on this model is determined by solving the following two-point boundary value problem (Gagg Filho and da Silva Fernandes, 2016):

Problem 1 *Given a value of* λ_1*, determine the initial velocity* v_0 *of the space vehicle subjected to the final constraint:*

$$g_1 : \qquad r_f - r_{sp} = 0 \tag{1}$$

where v_0 is the initial velocity of the space vehicle at the geocentric phase immediately after the application of the first velocity increment at the time $t = t_0$; λ_1 is the angle that defines the geometry when the space vehicle reaches the boundary of the Moon's SOI (see Fig. 1); r_{sp} is the periselenium distance, and r_f is the prescribed radial distance of the LMO.

The Newton-Raphson algorithm (Stoer and Bulirsch, 1973) is used to solve this problem. Once the unknown v_0 is determined, others parameters of the transfer trajectory are calculated using the well-known expressions of the two-body problem (Bate *et al.*, 1971). These *a posteriori* calculations include: the total time of flight T, the velocity increments Δv_{LEO} and Δv_{LMO}, and the initial phase angle of the space vehicle $\theta_{EP}(0)$. For more details see Gagg Filho and da Silva Fernandes (2016).

3.2 Planar circular restricted Three-Body problem

The planar circular restricted three-body problem (PCR3BP) does not divide the trajectory into phases, instead it considers the gravitational field of the primaries (Earth and Moon) during the entire transfer trajectory. The mass of the space vehicle is supposed infinitesimal (restricted problem) as compared to the primary masses. Also, it is supposed that the primaries describe a circular motion around the barycenter B of the Earth-Moon system. An inertial reference frame centered at the barycenter B is adopted with the x- axis pointing towards the Moon at $t = t_0$ (Fig. 2). The motion of the space vehicle is described by the following system of differential equations.

$$\ddot{x}_P = -\frac{\mu_E}{r_{EP}^3}(x_P - x_E) - \frac{\mu_M}{r_{MP}^3}(x_P - x_M) \tag{2}$$

$$\ddot{y}_P = -\frac{\mu_E}{r_{EP}^3}(y_P - y_E) - \frac{\mu_M}{r_{MP}^3}(y_P - y_M) \tag{3}$$

where (x_P, y_P), (x_E, y_E), and (x_M, y_M) are the coordinates of the position vector of the space vehicle P, the Earth E, and the Moon M, respectively; r_{EP} and r_{MP} are, respectively, the Earth-space vehicle and the Moon-space vehicle distances evaluated at any time; and, μ_E and μ_M are the Earth's and Moon's gravitational parameters, respectively.

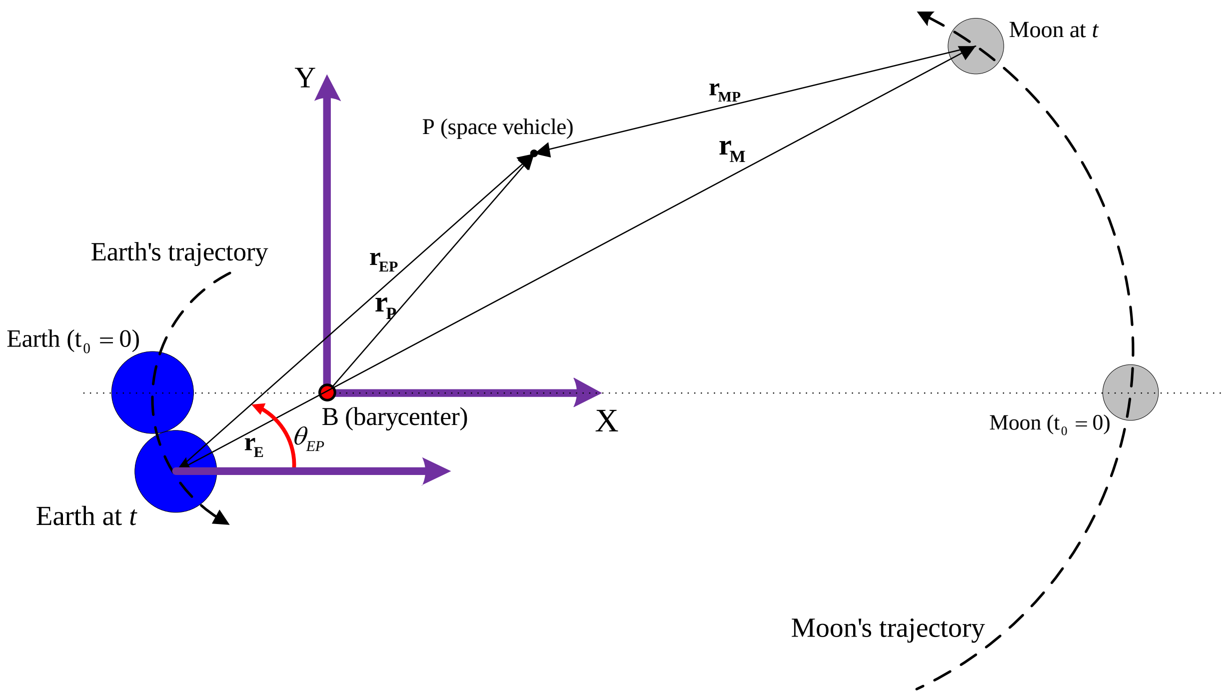


Figure 2: Planar circular restricted three-body problem.

The transfer trajectory based on this model is determined by solving the following two-point boundary value problem.

Problem 2 *Given a value of the initial phase angle $\theta_{EP}(0)$ of the space vehicle, determine the set of unknowns $(\Delta v_{LEO}$, Δv_{LMO} , $T)$ subjected to the final constraints:*

$$g_2: \qquad (x_P(T) - x_M(T))^2 + (y_P(T) - y_M(T))^2 = {r_f}^2 \tag{4}$$

$$g_3: \qquad (\dot{x}_P(T) - \dot{x}_M(T))^2 + (\dot{y}_P(T) - \dot{y}_M(T))^2 = \left[\sqrt{\frac{\mu_M}{r_f}} + \Delta v_{LMO}\right]^2 \tag{5}$$

$$g_4: \qquad (x_P(T) - x_M(T))(\dot{y}_P(T) - \dot{y}_M(T)) - (y_P(T) - y_M(T))(\dot{x}_P(T) - \dot{x}_M(T)) = \mp r_f\left[\sqrt{\frac{\mu_M}{r_f}} + \Delta v_{LMO}\right] \tag{6}$$

where g_2, g_3, and, g_4 are the final constraints and they correspond, respectively, to the position condition, velocity condition, and orthogonality condition at the arrival time $t = T$. For more details see da Silva Fernandes and Marinho (2012), and, Gagg Filho and da Silva Fernandes (2017).

3.3 Planar elliptic restricted Three-Body problem

The planar elliptic restricted three-body problem (PER3BP) is an extension of the PCR3BP by including the eccentricity of the primary (Earth and Moon) orbits around the barycenter. The system of differential equations of the space vehicle motion is the same as in the previous model. In order to evaluate the position of the primaries at any time on their eccentric orbits, the Kepler's equation must be solved. Moreover, the initial position of the Moon on its eccentric orbit, given by its true anomaly $f_M(0)$ (see Fig. 3), must also be prescribed in a first moment. Due to these additional parameters, a two-point boundary value problem with prescribed time of flight is stated as:

Problem 3 *Given a value of the time of flight T, and, prescribing the initial position of the Moon $f_M(0)$, determine the set of unknowns $(\Delta v_{LEO}, \theta_{EP}(0))$ subjected to the final constraints:*

$$g_2: \qquad (x_P(T) - x_M(T))^2 + (y_P(T) - y_M(T))^2 = r_f{}^2 \tag{7}$$

$$g_4: \qquad (x_P(T) - x_M(T))(\dot{y}_P(T) - \dot{y}_M(T)) - (y_P(T) - y_M(T))(\dot{x}_P(T) - \dot{x}_M(T)) = \mp r_f\left[\sqrt{\frac{\mu_M}{r_f}} + \Delta v_{LMO}\right] \tag{8}$$

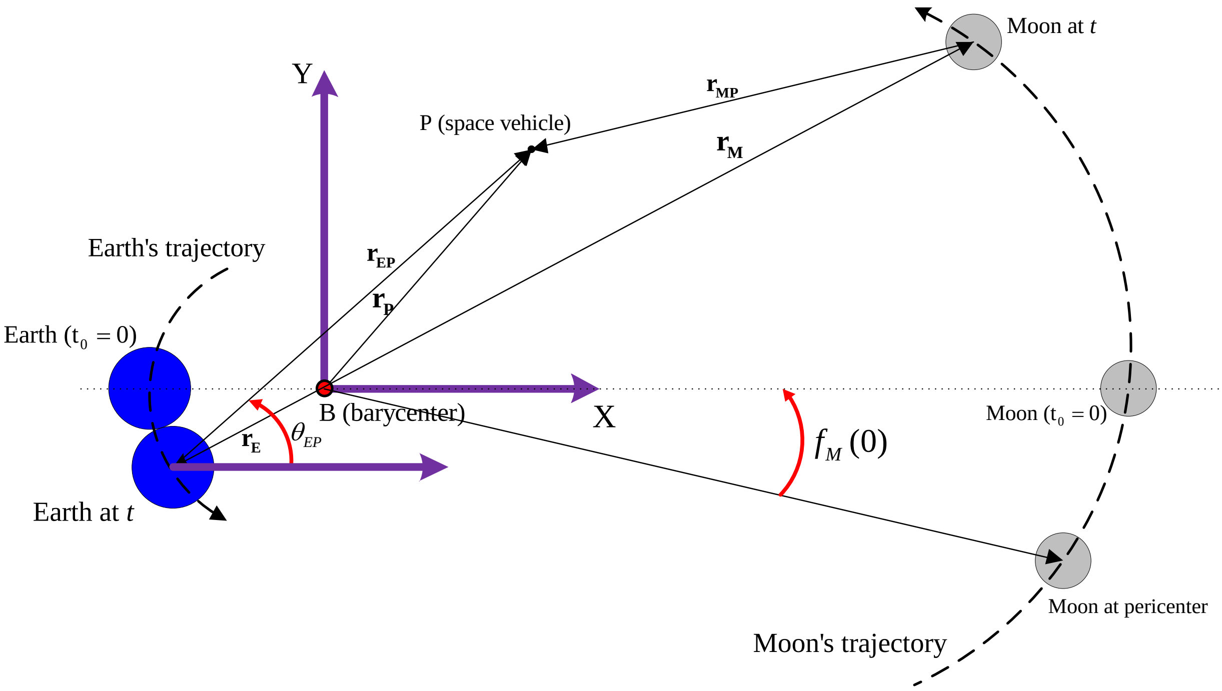


Figure 3: Planar elliptic restricted three-body problem.

Note that the velocity constraint, g_3, is removed as well as the unknown Δv_{LMO}. By doing this, the unknown Δv_{LMO} is naturally determined by Eq. 5. This procedure converts the constraint g_3 in an algebraic equation, which is solved for Δv_{LMO} as following

$$\Delta v_{LMO} = \sqrt{(\dot{x}_P(T) - \dot{x}_M(T))^2 + (\dot{y}_P(T) - \dot{y}_M(T))^2} - \sqrt{\frac{\mu_M}{r_f}} \tag{9}$$

This alternative procedure could also be performed on Problem 2.

One can set the initial position of the Moon $f_M(0)$ as an unknown to calculate an optimal time-fixed fuel consumption trajectory according to the following problem:

Problem 4 *Given a value of the time of flight T, determine the set of unknowns $(\Delta v_{LEO}, \theta_{EP}(0), f_M(0))$ that minimizes the total fuel consumption Δv_{Total}*

$$F: \qquad \Delta v_{Total} = \Delta v_{LEO} + \Delta v_{LMO} \tag{10}$$

subjected to the final constraints:

$$g_2: \qquad (x_P(T) - x_M(T))^2 + (y_P(T) - y_M(T))^2 = (r_f)^2 \tag{11}$$

$$g_4: \qquad (x_P(T) - x_M(T))(\dot{y}_P(T) - \dot{y}_M(T)) - (y_P(T) - y_M(T))(\dot{x}_P(T) - \dot{x}_M(T)) = \mp r_f\left[\sqrt{\frac{\mu_M}{r_f}} + \Delta v_{LMO}\right] \tag{12}$$

where Δv_{LMO} is determined by Eq. 9.

4. Results

In order to determine time-fixed Earth-Moon transfer trajectories, one must firstly analyze the possibility of such trajectories. Problem 1 and Problem 2 are used for this goal. The LEO's and LMO's altitude are, respectively, 167 km and 100 km. Only counterclockwise departures and clockwise arrivals are considered. Trajectories with counterclockwise arrival are not studied in order to present a brief paper, but the same procedure can be applied for this arrival sense. The eccentricity of the primaries on both models is 0.05490 (Roncoli, 2005), and, the semi-major axis of the Moon's orbit is 384400 km. Despite the time variations of the eccentricity and semi-major axis in real model, only the mean values of both variables are used. Therefore, the evection and variation phenomena in the eccentricy are neglected as well as the periodic behavior of the semi-major, which is acceptable for a preliminary analysis. The radius of the Moon's sphere of influence is 66300 km, and the Earth's and Moon's radius are, respectively, 6378.2 km and 1738 km. Also, the Earth's and Moon's gravitational parameter are, respectively, $\mu_E = 398600$ km^3/s^2 and $\mu_M = 4902.83$ km^3/s^2.

Accordingly to the statement of Problem 1, one must prescribe λ_1 to solve the TPBVP. In this way, Fig. 4 plots the total time flight for several prescribed values of λ_1, and, Fig. 5 shows the fuel consumption Δv_{Total} considering the same prescribed values of λ_1. It can be seen that for each value of λ_1 there is only one direct ascent trajectory with a specific time of flight. A specific time of flight gives a unique value of fuel consumption, as described in Fig. 5. A similar discussion occurs by using the PCR3BP (Problem 2). In this model, the initial phase angle $\theta_{EP}(0)$ of the space vehicle is prescribed instead of λ_1. The solutions are shown in Fig. 6 and 7. It can be seen that for each prescribed value of $\theta_{EP}(0)$ there is only one direct ascent trajectory with a specific time of flight (Fig. 6), which gives an unique value of fuel consumption (see Fig 7). Therefore, the results of Problems 1 and 2 shows that an optimization of the fuel consumption can not be performed if the time of flight is prescribed together with the phase angles (λ_1 or $\theta_{EP}(0)$).

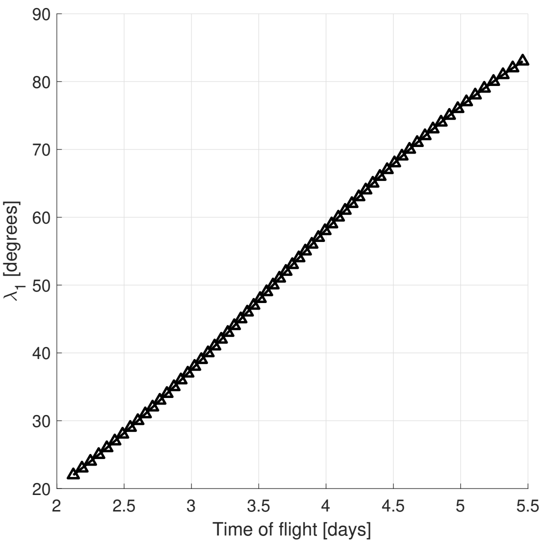


Figure 4: Patched-conic. Problem 1. Time of flight $\times$ λ.

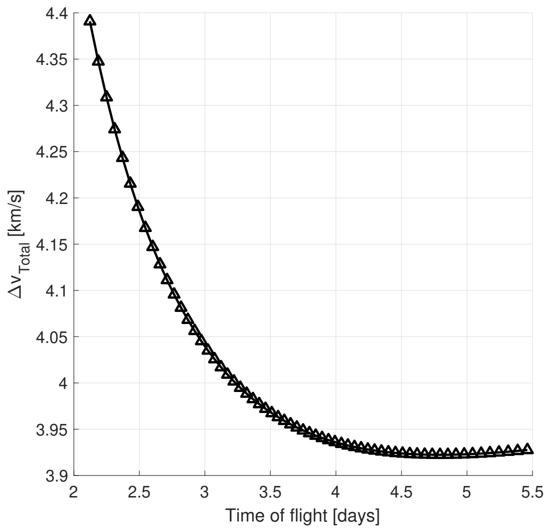


Figure 5: Patched-conic. Problem 1. Time of flight $\times$ Δv_{Total}.

However, if the eccentricity of the primaries is included in the dynamical model, as described in the PER3BP model, another parameter appears and it is related to the position of the Moon in its eccentric orbit around the Earth-Moon system's barycenter. Problem 3 uses the true anomaly of the Moon at the initial time $f_M(0)$ as this additional parameter. Due to this parameter, a time-fixed direct ascent trajectory is not unique. Indeed, by setting a time of flight equal to 5 days on Problem 3, there are infinite Earth-Moon trajectories. Each one of these solutions corresponds to a value of $f_M(0)$ as it can be seen on Fig. 8. This set of solutions generates a curve with a point of minimum close to $f_M(0) = 100°$ providing a fuel consumption between 3.945 km/s and 3.950 km/s. Moreover, due to the additional parameter, two solutions of 5 days with the same initial phase angle $\theta_{EP}(0)$ appear (Fig. 9). For instance, for $\theta_{EP}(0) = 248°$ (Fig. 9), there is the solution with small fuel consumption ($\Delta v_{Total} < 3.95$ km/s) and there is the solution with large fuel consumption ($\Delta v_{Total} > 3.96$ km/s). Each one of the these solutions corresponds to different prescribed values of $f_M(0)$. In this way, an optimization problem (Problem 4) with prescribed time of flight is solved that sets the additional parameter $f_M(0)$ as an unknown to be optimized. Figures 10 and 11 plot the optimal results of Problem 4. Note that each point of these figures is an optimal solution of Problem 4 with different prescribed values of the time of flight. For instance, given a time of flight of 5 days, the optimal value of $f_M(0)$ is 110° (Fig. 11) that gives a minimum fuel consumption of 3.946970 km/s (Fig. 10). Figure 12 compares this optimal trajectory with a non-optimal one provided by Fig. 8, near the maximum fuel

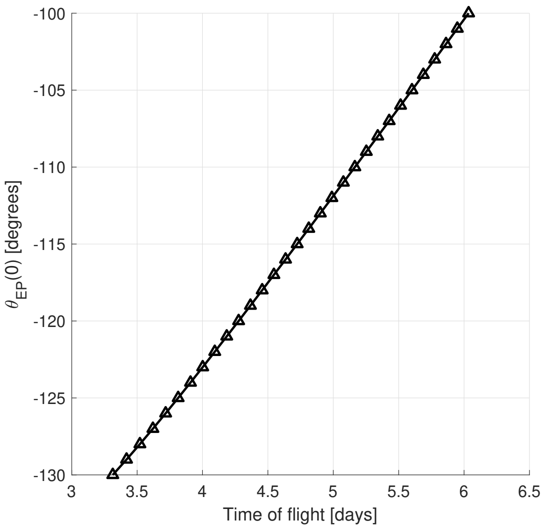


Figure 6: PCR3BP. Problem 2. Time of flight × $\theta_{EP}(0)$.

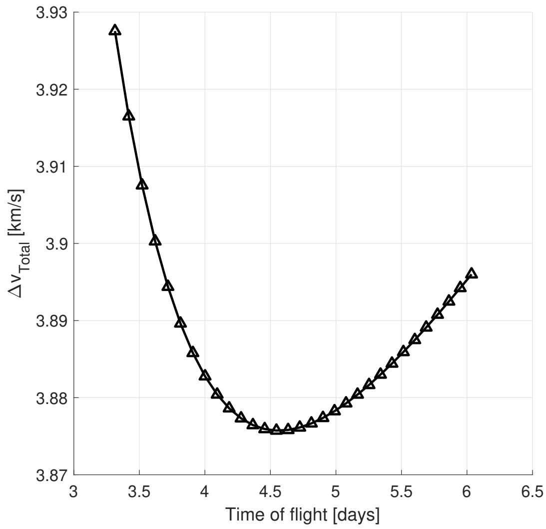


Figure 7: PCR3BP. Problem 2. Time of flight × Δv_{Total}.

consumption point. Note that the angle between these trajectories is nearly $180°$, which means that while the arrival of the minimum fuel consumption trajectory occurs with the Moon close to its apocenter on its eccentric orbit, the maximum fuel consumption trajectory occurs with the Moon close to its pericenter. This difference between the Earth-Moon distance at the arrival of both trajectory is better visualized if both initial position of Moon are set in the $x - axis$ (see Fig. 12b). Note in Fig. 12b the farther distance of the Moon (close to the apocenter) in the minimum fuel consumption trajectory.

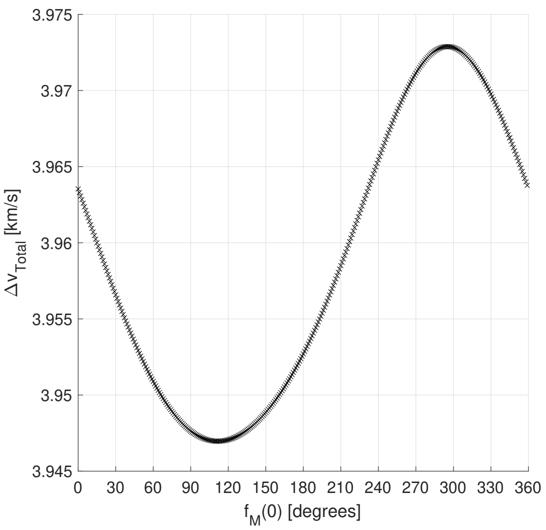


Figure 8: PER3BP. Problem 3. $f_M(0) \times \Delta v_{Total}$. Time of flight = 5 days.

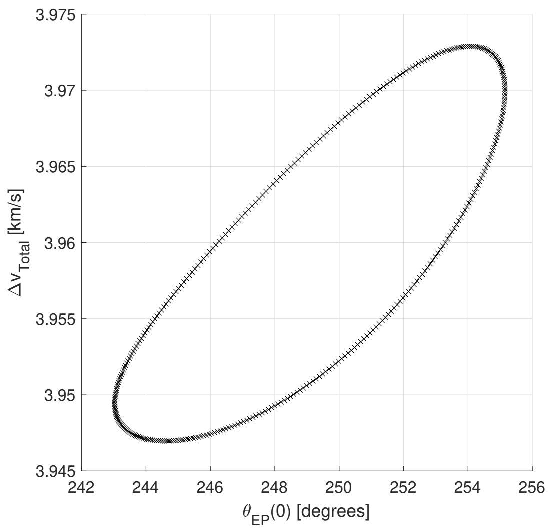


Figure 9: PER3BP. Problem 3. $\theta_{EP}(0) \times \Delta v_{Total}$. Time of flight = 5 days.

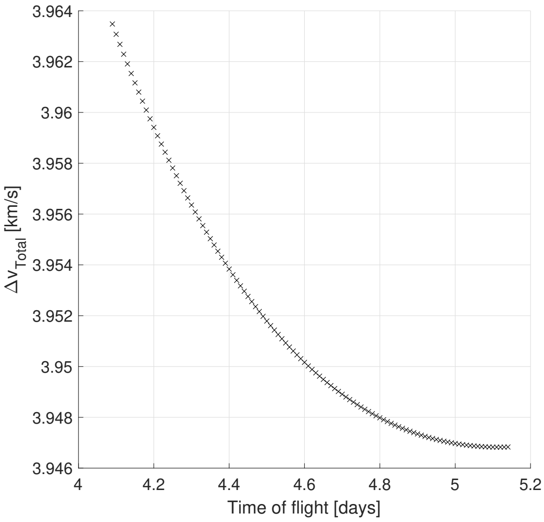


Figure 10: PER3BP. Problem 4. Time of flight × Δv_{Total}.

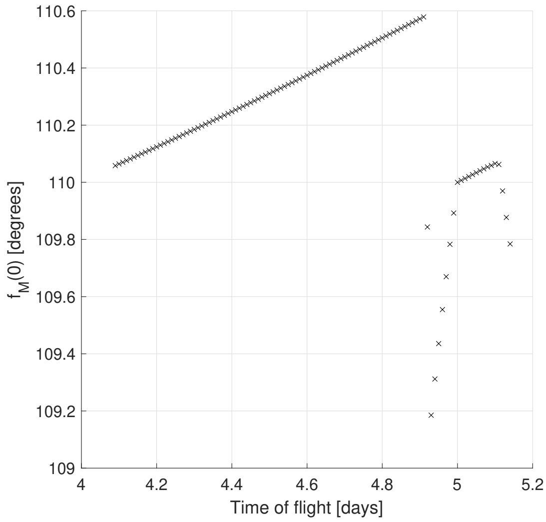
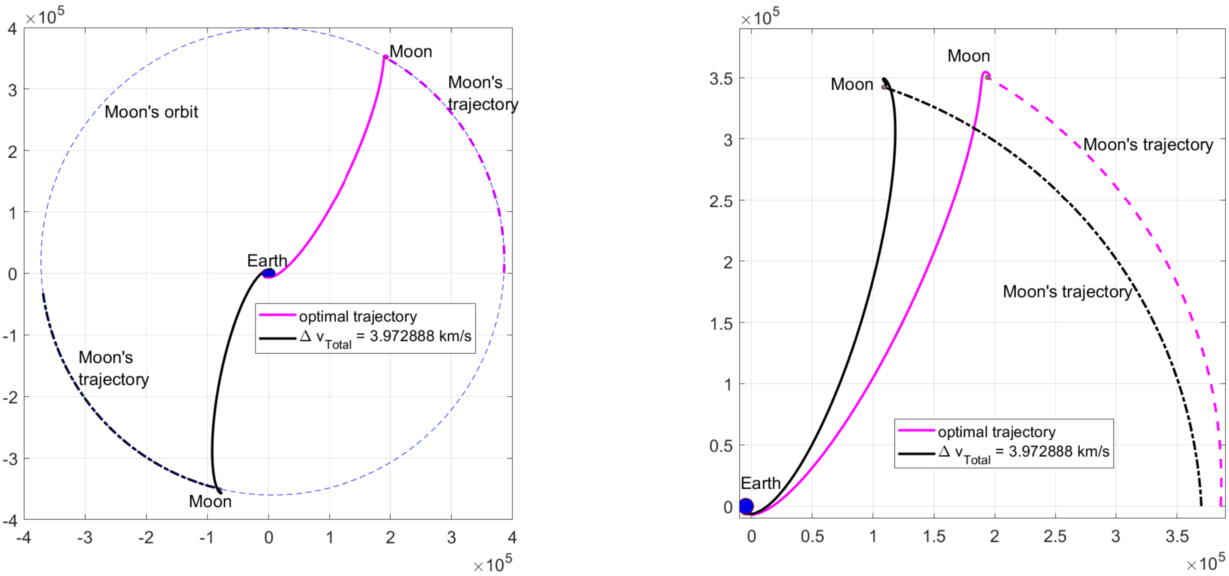


Figure 11: PER3BP. Problem 4. Time of flight × $f_M(0)$.



(a) (b)

Figure 12: PER3BP. Optimal and non-optimal trajectory. Time of flight = 5 days.

5. Conclusion

This work analyses time-fixed direct ascent Earth-Moon trajectories. Three-models are considered to perform this study: the patched-conic approximation, the planar circular restricted-three body problem and the planar elliptic restricted three-body problem. The results of the lunar patched-conic approximation and the results of the PCR3BP shows that there is only one solution trajectory for each prescribed time of flight if the initial phase angle of the space vehicle $\theta_{EP}(0)$ or the phase angle λ_1 is given. However, if more complexity is added to the model, which is the case of the PER3BP, several solution trajectories can be obtained for the same prescribed time of flight. The additional parameter that add this complexity is chosen to be the true anomaly $f_M(0)$ of the Moon on its eccentric orbit at the initial time t_0. In this way, several solutions with the same time of flight (5 days) are determined by solving a TPBVP considering several values of $f_M(0)$. Each one of these solutions presents a different fuel consumption. Indeed, there are solution trajectories with different fuel consumption, but with the same time of flight and the same initial phase angle of the space vehicle. The minimum fuel consumption trajectory of these solution is obtained by solving an one-degree of freedom optimization problem with a prescribed time of flight resulting in a saving of fuel consumption of about 25.91 m/s. A comparison between the minimum Δv_{Total} trajectory and the near maximum Δv_{Total} trajectory reveals that the Moon's position at the arrival of the minimum Δv_{Total} trajectory occurs near its apocenter, while this occurrence of the near maximum Δv_{Total} trajectory is close to the pericenter.

6. Acknowledgements

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