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ON THE INFLUENCE OF NONLINEAR STIFFNESS ON THE DESIGN OF VIBRATION ABSORBER AND NEUTRALIZERS

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Abstract. *Vibration neutralizers are devices used to reduce vibration on a primary structure and are usually tuned to the frequency of an external harmonic force. The vibration absorber consists of a similar device, but the target tuning frequency is a resonance of the primary structure. Their result is the creation of a notch filter in frequency. A single degree vibration neutralizer has stiffness and mass and damping. The objective frequency can be adjusted by changing the stiffness or mass elements. Regarding the nonlinear behaviour of such devices, there are two cases to be considered. One occurs when there is an interest to improve the performance of neutralizer exploring nonlinearity. The second case occurs when the neutralizer can exhibit nonlinear behaviour if submitted to large amplitude oscillations but not intentionally designed. This paper focuses on the analysis of such cases when they operate in a nonlinear regime. Nonlinearity in systems is possible with greater deformations of the spring, called hardening or softening stiffness. The cases of hardening and softening stiffness can be modelled using a cubic polynomial to represent the elastic restoring force. The analysis is based on a system with at least two degrees of freedom where the neutralizer is attached. Frequency response functions are obtained considering an approximation of nonlinear forces obtained by the harmonic balance method, based on the fundamental frequency. Differently from the traditional harmonic balance method, here the equations of motion are computed assuming a complex exponential excitation force, so are the displacements variables expanded into real and imaginary components and solved using a continuation method. In some cases, the nonlinear vibration neutralizer can improve the system's performance compared to the linear defined, but only if correctly designed. The analytical results obtained by the harmonic balance method are also verified by numerical integration of the equations of motion.*

Keywords: *vibration isolation, nonlinear dynamic, absorber.*

1. INTRODUCTION

Vibration neutralizers are devices typically composed of three elements, a spring, a mass and a damper used to reduce vibration in a structure and are usually tuned to the frequency of an external harmonic force. The vibration absorber is a similar device, but it is designed so that the resonant frequency matches a certain frequency of interest, usually the frequency of an external harmonic force.

According to (Qian and Zuo, 2021), nonlinear vibration absorbers have been extensively investigated for passive vibration control, motion isolation and synchronous energy collection. This study focuses on the exact nonlinear dynamics of a supported beam simply carrying a nonlinear energy absorber with a spring damper for primary resonance vibration reduction. In (Brennan and Gatti, 2012), the authors investigate the use of cubic nonlinearity in the vibration neutralizer to improve its effectiveness. Assume that the frequency of the harmonic excitation is well above the resonant frequency of the machine to which the neutralizer is connected and that the machine acts as a simple mass. Other research, studies the beneficial effects of the nonlinearity of geometric stiffness with case studied in the context of a common structure (Gatti *et al.*, 2019). There are studies where the article presents a study of the dynamic behavior of a specific two-degree-of-freedom that represents the system, in which the nonlinear system does not affect the vibration of the forced linear system (Gatti *et al.*, 2010). The paper of the (Tang *et al.*, 2016) describes the characterization of such an absorber using a novel experimental procedure, where the estimation method is based on a free vibration test, which is appropriate for a lightly damped device.

Unlike the traditional harmonic balance method as most research in this area, as seen in (Brennan *et al.*, 2008), which uses simple approximate dimensionless expressions and the corresponding displacement amplitudes for the up and down jump frequencies of a Duffing oscillator using a complete set of expressions determined using the harmonic equilibrium approach, in this article the equations of motion are calculated assuming a complex exponential excitation force, then the displacement variables are expanded into real and imaginary components and solved using a continuation method.

Some additional noteworthy publications that discuss various aspects of nonlinearity applied to mechanical systems are available in the following studies, such as (Aiello and Gatti, 2017), which presents some observations on the frequency response of a primary linear oscillator when an auxiliary nonlinear oscillator is coupled to it, acting as a vibration neutralizer, or (Howard, 2012) which uses a sliding Goertzel algorithm for adaptive passive neutralizers to extract vibration signals at the frequency of interest.

The use of components that replace the linear spring in order to make the dynamic vibration absorber/neutralizer have a better performance is something much studied in the literature for which the structure (single degree of freedom) and/or the device coupled to it are considered as having cubic nonlinear stiffness. Bringing the possibility of using a nonlinear device coupled to a multi-degree-of-freedom structure and differently the traditional harmonic equilibrium method as most researches involving this type of analysis, here the equations of motion are calculated assuming a complex exponential excitation force, and solved using a numerical continuation method.

2. MATHEMATICAL MODEL

The proposed analysis discussed in this paper takes into the influence of a vibration neutralizer on two types of structure. The first is a single degree of freedom where the neutralizer is attached (Case I). The second case is a two degree of freedom where the neutralizer is also attached (Case II). These systems are shown in Figure 1.

In both configurations, a harmonic force $f = F\,e^{i\omega t}$ is applied at the first mass of the primary structure, m, k and c are values of mass, stiffness and viscous damping. The x_i is the displacement of the masses of the primary structure. The vibration neutralizers was characterized by a mass m_n, a viscous damping coefficient c_n and nonlinear stiffness coefficient k_n. The nonlinear stiffness coefficient produces a cubic restoring force. Two cases are considered; hardening and softening types. The cubic restoring force can be defined as

$$f_n = k_1 z \pm k_3 z^3 \tag{1}$$

where z is the relative displacement in which the nonlinear spring is subject to.

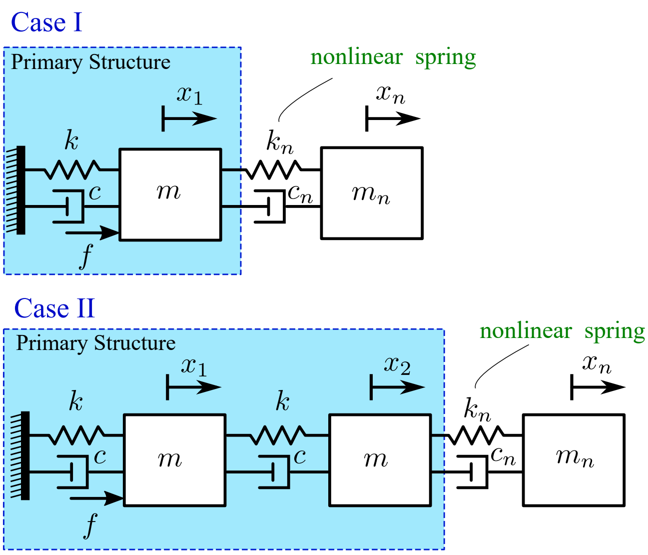

Figure 1. Mechanical systems with a nonlinear dynamic vibration neutralizer (nDVN) attached. Case I consist of a single degree of freedom with nDVN, and Case II consists of a multiple degree of freedom with a nDVN.

2.1 Mathematical modelling

The equations of motion for the system shown in Case I are defined in terms of the displacement x_1 and x_n, such that, the force equilibrium allows writing the equations in nondimensional form as

$$x_1'' + 2\zeta x_1' + x_1 + 2\zeta\beta(x_1' - x_n') + \gamma_1(x_1 - x_n) + \gamma_3(x_1 - x_n)^3 = \delta_{st} e^{i\Omega\tau} \tag{2}$$

$$\mu x_n'' - 2\zeta\beta(x_1' - x_n') - \gamma_1(x_1 - x_n) - \gamma_3(x_1 - x_n)^3 = 0 \tag{3}$$

where $x' = \omega_0 \frac{dx}{d\tau}$ is the derivative with respect to the nondimensional time τ, and $\omega_0 = \sqrt{k/m}$. The other parameters are $\mu = m_n/m$, $\gamma_1 = k_1/k$, $\gamma_3 = k_3/k$, $\zeta = c/2m\omega_0$, $\beta = c_n/c$, $\delta_{st} = F/k$ and $\Omega = \omega/\omega_0$.

For Case II, where the structure has two degrees of freedom plus the neutralizers degree freedom, allows writing the equations

$$x_1'' + 2\zeta x_1' + 2\zeta\beta(x_1' - x_2') + (2x_1 - x_2) = \delta_{st} e^{i\Omega\tau} \tag{4}$$

$$x_2'' - 2\zeta\beta(x_1' - x_2') - (x_1 - x_2) - 2\zeta\beta(x_2' - x_n') + \gamma_1(x_2 - x_n) + \gamma_3(x_2 - x_n)^3 = 0 \tag{5}$$

$$\mu x_n'' - 2\zeta\beta(x_2' - x_n') - \gamma_1(x_2 - x_n) - \gamma_3(x_2 - x_n)^3 = 0 \tag{6}$$

Assuming harmonic motion for the masses of the form

$$x_j = X_j\, e^{i\omega t}, j = 1, 2, n, \tag{7}$$

The equations for Case I can be written as

$$-\Omega^2 X_1 + i2\Omega\zeta X_1 + X_1 + i2\zeta\beta\Omega\,(X_1 - X_n) + \gamma_1\,(X_1 - X_n) + F_{NL} = \delta_{st}, \tag{8}$$

$$-\mu\Omega^2 X_n - i2\zeta\beta\Omega\,(X_1 - X_n) - \gamma_1\,(X_1 - X_n) - F_{NL} = 0, \tag{9}$$

Following the same exponential expansion solution it is possible to obtain the equations for Case II

$$-\Omega^2 X_1 + i2\Omega\zeta X_1 + i2\zeta\Omega\,(X_1 - X_2) + (2X_1 - X_2) = \delta_{st}, \tag{10}$$

$$-\Omega^2 X_2 - i2\Omega\zeta\,(X_1 - X_2) - (X_1 - X_2) + i2\zeta\beta\Omega\,(X_2 - X_n) + \gamma_1\,(X_2 - X_n) + F_{NL} = 0, \tag{11}$$

$$-\mu\Omega^2 X_n - i2\zeta\beta\Omega\,(X_2 - X_n) - \gamma_1\,(X_2 - X_n) - F_{NL} = 0, \tag{12}$$

The term F_{NL} shown in the set of equations for Case I and II is related to the transformation of the time to the frequency domain using the first order harmonic balance method. This term is defined in Appendix A in real and imaginary parts. The equations in the frequency domain are also given in the appendix in real and imaginary equations.

The set of nonlinear equations in the frequency domain can be solved using, for instance, a root-finding method, such as the Levenberg-Marquardt algorithm, but this would require many initial guesses for each frequency of interest. In this case, a continuation procedure has shown to be a more interesting approach to calculate the frequency response curves. For this, Python programming language with the module PyDSTool was used (Clewley *et al.* (2007)).

3. RESULTS AND DISCUSSIONS

In this section, an investigation on the effects of the parameters nonlinear of the neutralizer is presented. In all simulation conditions the neutralizer linear resonance was tuned to $\Omega = 1$. This means that without the nonlinear stiffness term $\gamma_3 = 0$, the primary system would present an antiresonance at $\Omega = 1$. For this, the neutralizer mass was assumed to be 25% of the system mass m (shown in Fig. 1, such that $\mu = m_n/m = 0.25$. As a result, $\gamma_1 = 0.25$ ($k_n/k = 0.25$). Damping on the primary system was also considered a fixed value in all numerical simulations, using $\zeta = 0.01$ and $\beta = c_n/c = 1$. The influence of the nonlinear spring can be observed by either changing the excitation force amplitude or changing the nonlinear parameter γ_3. Neutralizers with hardening and softening springs were considered by adjusting $\gamma_3 > 0$ for hardening and $\gamma_3 < 0$ for softening.

3.1 CASE I - Results and discussion

In Case I, the primary system consists of a single degree of freedom spring mass damper, where only a resonance peak is expected before the DVN is attached. The response of the primary structure subject to a harmonic force is shown in Figure 2(a) with a black dashed line (showing a resonance peak at $\Omega = 1$). The blue and red lines are for the system with a linear dynamic vibration neutralizer (DVN) and a nonlinear dynamic vibration neutralizer (nDVN), respectively. As in the linear system, in the nonlinear system, it is possible to observe the appearance of antiresonance. For the nDVN, a hardening configuration was assumed initially with $\gamma_3 = 0,005$. It is possible to observe that second resonance bends towards higher frequencies as an effect of the nonlinearity. It is possible to observe a frequency response loop occurring in the absolute displacement of the structure in Figure 2(a). This loops only appears in the absolute displacement and is not observed in the relative response Figure 2(b), the visualization of the loop occurs simply because the graph is presented in 2D and the curves do not cross. The loops only occur for high nonlinearity values, the unstable part of the curve appears with a greater amplitude than the stable parts, the upper part of the loop being the unstable part of the frequency response, a fact proven by the location of the limit points (where the stability of the system changes) of the curve. This effect is also discussed in Brennan and Gatti (2012). Most important in this figure is the shift in the tuning frequency $\Omega = 1$, which is influenced by the nonlinearity.

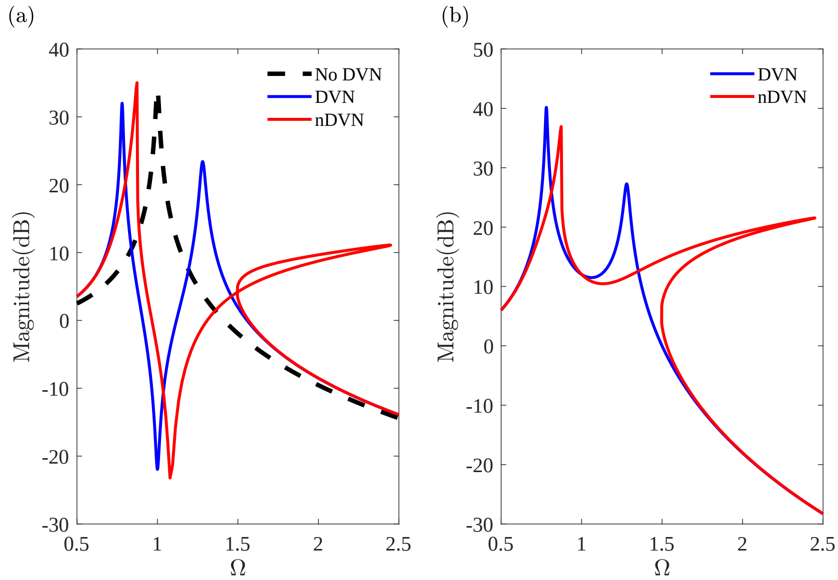

Figure 2. Frequency response for Case I - hardening spring. (a) Absolute frequency response (X_1). (b) Relative frequency response ($|Z| = |X_n - X_1|$). (dB ref. 1 N/m.)

In order to understand the influence of the nonlinear parameter on the tuning frequency, the parameter γ_3 was varied to some range of values and illustrated in Figure 3. it is possible to verify that as γ_3 increases, the peaks to bend towards higher frequencies, showing regions where multiple amplitude of response can occur. It is also important to note the change of about 40% of the linear tuned frequency of the vibration neutralizer (from $\Omega = 1$ shifting to $\Omega \approx 1.4$) for larger values of γ_3. This effect clearly shows that performance at the tuned frequency can be significantly affect by the non-linearity.

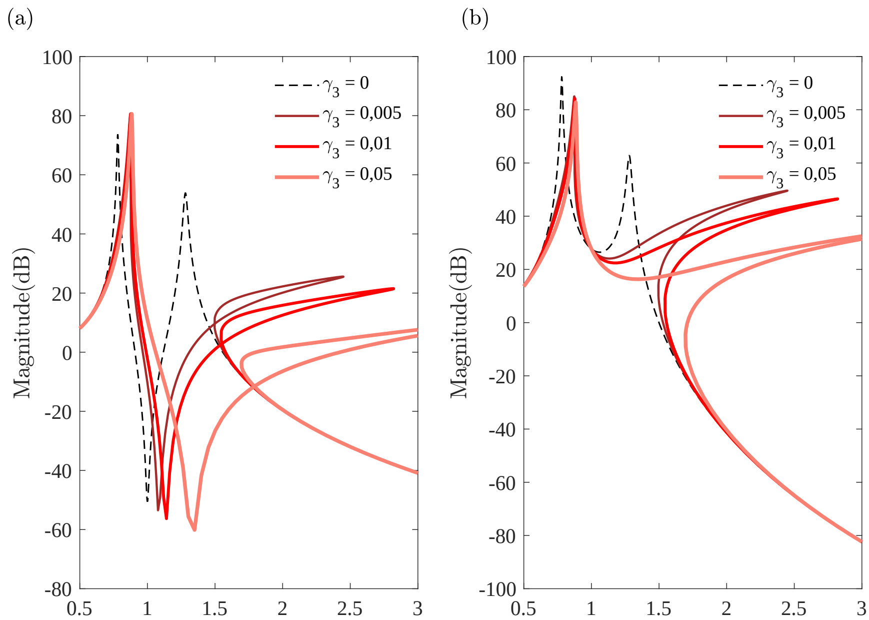

Figure 3. Frequency response for Case I for different values of γ_3 - hardening spring. (a) Absolute frequency response (X_1). (b) Relative frequency response ($|Z| = |X_n - X_1|$). (dB ref. 1 N/m.)

The softening effect of the nDVN was also investigated for Case I. This was done by adjusting the nonlinear parameter such that $\gamma_3 < 0$. The results for different values of nonlinearity are shown in Figure 4.

It is possible to verify that as the parameter decreases, the resonance peaks bends towards lower frequencies, as well as a small shift of the antiresonance to the left, as the value of the parameter nonlinear is extrapolated in a way negative, it is possible to observe the disappearance of the antiresonance, because due to the decrease of parameter the loss of stiffness is greater, that is, the spring effect decreases, the spring softening causes a region of instability, observed in absolute displacements as shown in Figure 4(a), causing the curves to tend to the left but not only that, they start to rise (as if there was a kind of barrier that prevents the curvature and makes them rise), a possible explanation would be that with a higher loss of rigidity, it would be as if the neutralizer were coupled to the structure only by the damper.

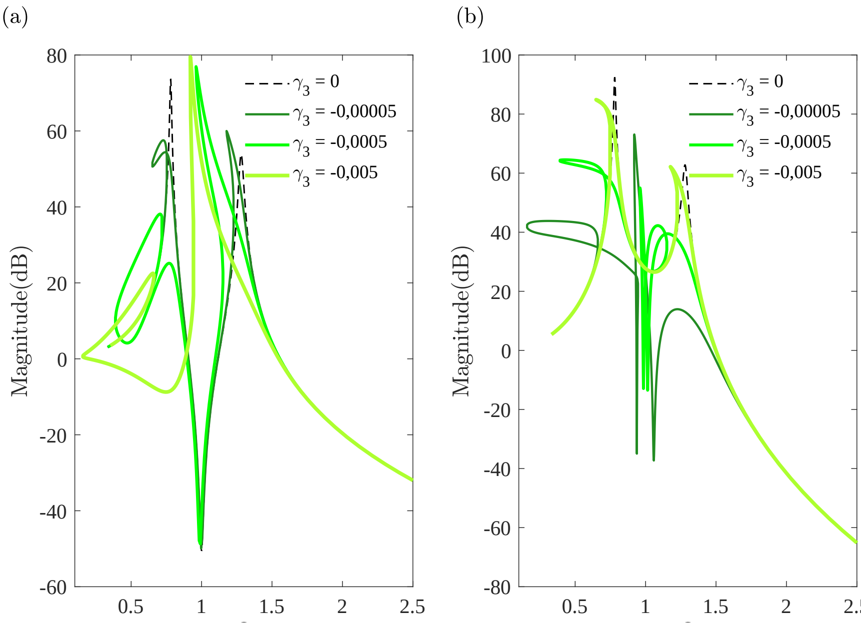

Figure 4. Frequency response for Case I for different values of γ_3 - softening spring. (a) Absolute frequency response (X_1). (b) Relative frequency response ($|Z| = |X_n - X_1|$). (dB ref. 1 N/m.).

In the linear systems, the increase in the amplitude of the force applied to the structure causes the increase in the amplitude of the transfer function and the displacement of the curve upwards. It is possible to note that although the nonlinear parameter does not change, but only the amplitude of the force that the curve becomes more accentuated in the hardening characteristic when there is an increase in force and less accentuated or curved when there is a tendency of the curve to bend towards higher frequencies as shown in Figure 5(a) for the absolute displacements and Figure 5(b) for the relative displacement.

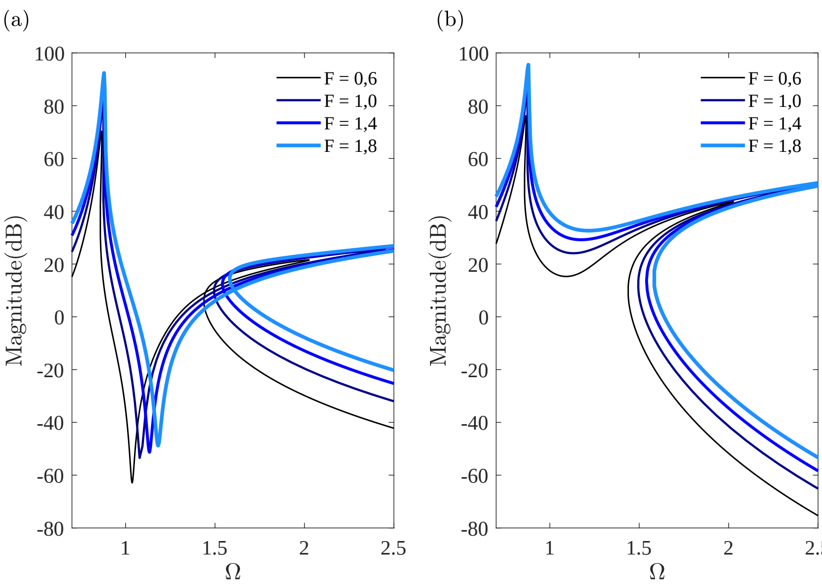

Figure 5. Frequency response for Case I for different values of F - hardening spring. (a) Absolute frequency response (X_1). (b) Relative frequency response ($|Z| = |X_n - X_1|$). (dB ref. 1 N/m.)

3.2 CASE II - Results and discussion

The system in Case II consists of a multi-degree-of-freedom structure, where at least two resonance peaks and one antiresonance are expected to be observed before the attachment of the nDVN. The analyses in this section are similar to what was done for Case I, where the linear resonance of the DVN was tuned to $\Omega = 1$. In this case, the only parameter that is different from the previous example was $\beta = 0.25$, to reduce the influence of the vibration neutralizer damping on the primary structure.

The frequency response curves are shown in Fig. 6(a), where the black dashed line is for the system without the presence of a dynamic vibration neutralizer (No DVN), the blue and red lines are for the system with a linear dynamic vibration neutralizer (DVN) and a nonlinear dynamic vibration neutralizer (nDVN) respectively. As in the linear system, in the nonlinear system it is possible to observe the appearance of two antiresonances but shifted to the right of the antiresonance of the linear system as shown in Figure 6. It is possible to observe a loop occurring in the absolute displacement of the structure and in Figure 6(a).

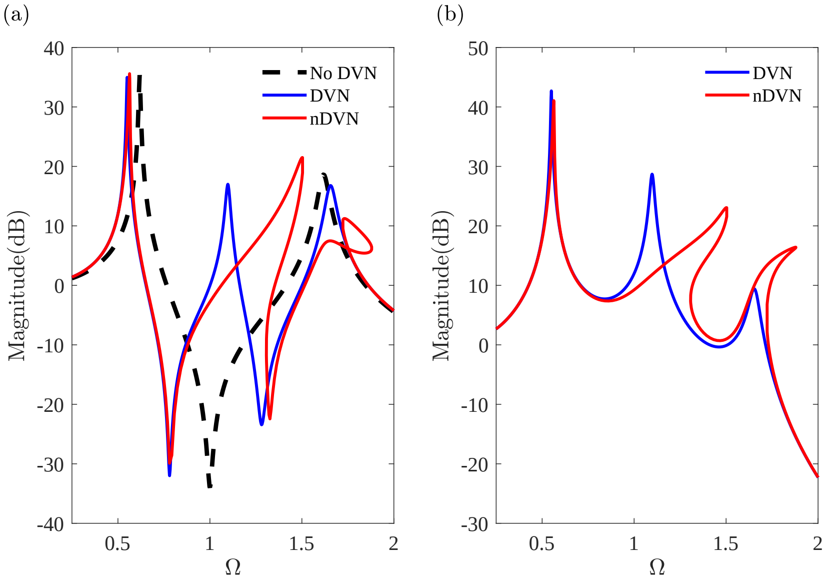

Figure 6. Frequency response for Case II - hardening spring. (a) Absolute frequency response (X_1). (b) Relative frequency response ($|Z|= |X_n - X_2|$). (dB ref. 1 N/m.)

The influence of the nonlinear parameter within the system can be observed by changing γ_3 as shown in Fig. 7

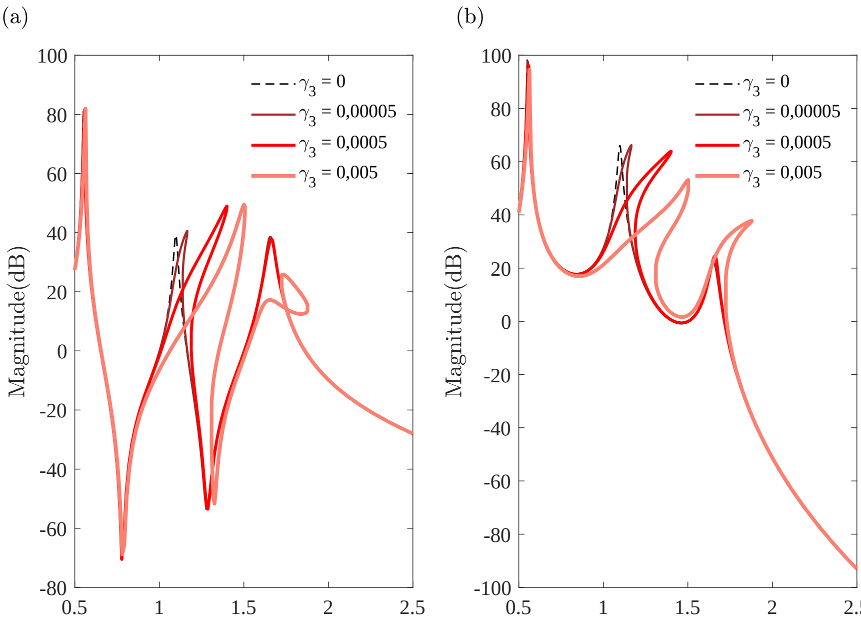

Figure 7. Frequency response for Case II for different values of γ_3 - hardening spring. (a) Absolute frequency response (X_1). (b) Relative frequency response ($|Z|= |X_n - X_2|$). (dB ref. 1 N/m.)

As shown in Figure 7, it is possible to notice that when the γ_3 increases, most of the resonance peaks shift towards higher frequencies (similar to what was observed in Case I), as well as antiresonance shift to the right by a small amount. It can be observed frequency response loops occurring in the absolute displacement of the structure on all curves as the nonlinear parameter is increased as shown in Figure 7(a). This effect is not observed in the relative displacement in Figure 7(b).

It is possible to vary the nonlinear parameter negatively exactly as in Case I, so that all responses present softening-type deformations in which there is loss of stiffness. As shown in Figure 8, it is possible to verify that as the parameter decreases, the frequency response peaks bend towards lower frequencies, as well as a small shift of the antiresonance to the left, as the value of the nonlinear parameter is extrapolated negatively, it is possible to observe the disappearance of the antiresonance, because with the decrease of the parameter the loss of stiffness is greater, that is, the spring effect decreases exactly as in the two degrees of freedom system, the spring softening causes a region of greater instability when compared to the two-degree structure, a fact observed in the absolute displacement as shown in Figure 8(a), causing the curves to tend to the left but not only that, they start to rise (as if there was a kind of barrier that prevents bending and makes them go up) and the curves referring to the second amplitude peak shift so far to the left that they reach the first amplitude peak, a possible explanation would be that with a greater loss of stiffness as well as an increase in the degree of freedom of the structure, it would be as if the neutralizer were only coupled to the structure by the damper.

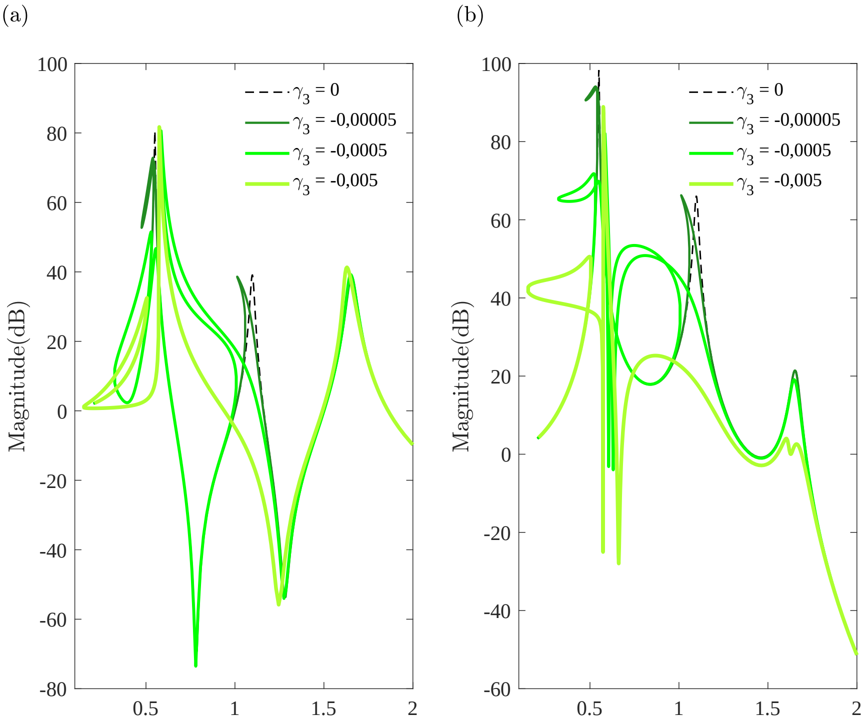

Figure 8. Frequency response for Case II for different values of γ_3 - softening spring. (a) Absolute frequency response (X_1). (b) Relative frequency response ($|Z|= |X_n - X_2|$). (dB ref. 1 N/m.).

It is possible to note that although the nonlinear parameter does not change, but only the amplitude of the force applied to the structure that the curve becomes more accentuated to the right as in the hardening characteristic as shown in Figure 9 for the absolute displacements it is possible to notice the formation of frequency response loops in the curves as shown in the Figure 9(a) and Figure 9(b) for the relative displacement.

4. CONCLUSIONS

This paper has discussed the influence of stiffness nonlinearity on the performance of a vibration neutralizer attached to a primary structure. Two examples were used to study the influence of nonlinearity. Case I, consisting of a single degree of freedom representing the primary structure. In this condition, as the nonlinearity is increased, there is a significant change in the antiresonance caused by the nonlinear vibration neutralizer. This effect needs to be considered in the design of such devices, either for the hardening or softening cases. For Case II, a system with at least two resonance and one resonance was considered, a similar study was carried out with a nonlinear vibration neutralizer. Some additional features are observed in this system, such as the effect of resonance peak bending is restricted by the other system resonances. A smaller influence on the antiresonance was observed in this case. The method used to obtain the frequency responses was done considering an alternative approach, using the continuation method inside the PyDSTool (Python Package), which allow the analysis of nonlinear response for a system with any number of degrees of freedom.

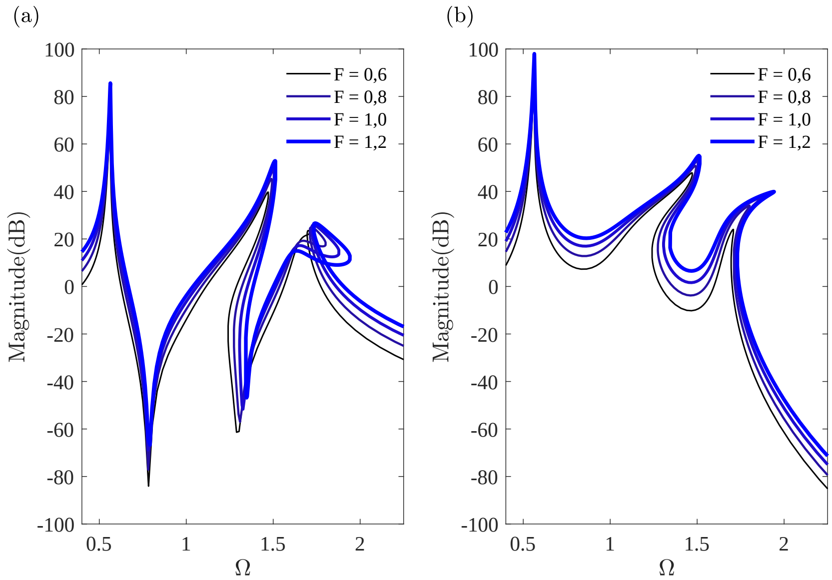

Figure 9. Frequency response for Case II for different values of F - hardening spring. (a) Absolute frequency response (X_1). (b) Relative frequency response ($|Z|= |X_n - X_2|$). (dB ref. 1 N/m.)

5. ACKNOWLEDGEMENTS

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7. RESPONSIBILITY NOTICE

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A Appendix - Expansion of equations in real and imaginary parts

In this appendix, the equations of motion in the frequency domain are expanded in real and imaginary parts using as a solution Eq. (13).

$$x_j = X_j\, e^{i\omega t}, j = 1, 2, n, \tag{13}$$

The linear stiffness k_1 and nonlinear stiffness k_3 presented in the frequency domain equations are substituted as the cubic restoring force as described above, and can be expanded in real and imaginary as per the solution below Eq. (14)

$$F_{NL} = F_{NL,R} - iF_{NL,I}, \tag{14}$$

where the subscripts R and I are the real and imaginary parts, respectively. When expanding the cubic restoring force into real and imaginary variables, the relative displacement must be taken into account, which is different for each of the cases described.

$$F_{NL,R} = \frac{3}{4}\gamma_3 \left((Z_I)^2\ (Z_R) + (Z_R)^3 \right), \tag{15}$$

$$F_{NL,I} = \frac{3}{4}\gamma_3 \left((Z_I)\ (Z_R)^2 + (Z_I)^3 \right), \tag{16}$$

Where Z is the relative displacement, for Case I, $Z_R = (X_{1,R} - X_{n,R})$ and $Z_I = (X_{1,I} - X_{n,I})$, for Case II, the relative displacement is $Z_R = (X_{2,R} - X_{n,R})$ and $Z_I = (X_{2,I} - X_{n,I})$

The first equation for Case I, Eq. (8) expanded real and imaginary variables.

$$\begin{aligned} -\Omega^2\ (X_{1,R} + i\, X_{1,I}) + \gamma_1\ (-X_{n,R} - i\, X_{n,I} + X_{1,R} + i\, X_{1,I}) + i2\,\zeta\,\beta\,\Omega\ (-X_{n,R} - i\, X_{n,I} + X_{1,R} + i\, X_{1,I}) \\ +\ i2\,\zeta\,\Omega\ (X_{1,R} + i\, X_{1,I}) + X_{1,R} + i\, X_{1,I} + \frac{3}{4}\gamma_3 \left((X_{1,R} - X_{n,R})^3 + (X_{1,I} - X_{n,I})^2\ (X_{1,R} - X_{n,R}) \right) \\ + \frac{3}{4}\gamma_3 \left((X_{1,I} - X_{n,I})^3 + (X_{1,R} - X_{n,R})^2\ (X_{1,I} - X_{n,I}) \right) = \delta_{st}, \end{aligned} \tag{17}$$

The second equation for Case I, Eq. (9) expanded real and imaginary variables.

$$\begin{aligned} -\Omega^2\,\mu\ (X_{n,R} + i\, X_{n,I}) - \gamma_1\ (-X_{n,R} - i\, X_{n,I} + X_{1,R} + i\, X_{1,I}) \\ -\ i2\,\zeta\,\beta\,\Omega\ (-X_{n,R} - i\, X_{n,I} + X_{1,R} + i\, X_{1,I}) - \frac{3}{4}\gamma_3 \left((X_{1,R} - X_{n,R})^3 + (X_{1,I} - X_{n,I})^2\ (X_{1,R} - X_{n,R}) \right) \\ - \frac{3}{4}\gamma_3 \left((X_{1,I} - X_{n,I})\ (X_{1,R} - X_{n,R})^2 + (X_{1,I} - X_{n,I})^3 \right) = 0, \end{aligned} \tag{18}$$

The equations Eq. (17) and Eq. (18) were expanded in order to separate the real part from the imaginary part, so now have 4 equations for Case I.

The equations Eq. (19) and Eq. (20) are about equation Eq. (17), separated in the form of equations with real and imaginary variables respectively.

$$\begin{aligned} \gamma_1\ (X_{1,R} - X_{n,R}) + 2\,\zeta\,\Omega\,\beta\ (X_{2,I} - X_{1,I}) - \Omega^2\, X_{1,R} + X_{1,R} - 2\,\zeta\,\Omega\, X_{1,I} \\ + \frac{3}{4}\gamma_3 \left((X_{1,I} - X_{n,I})^2\ (X_{1,R} - X_{n,R}) + (X_{1,R} - X_{n,R})^3 \right) = \delta_{st}, \end{aligned} \tag{19}$$

$$\begin{aligned} 2\,\zeta\,\Omega\,\beta\ (X_{1,R} - X_{n,R}) + \gamma_1\ (X_{1,I} - X_{n,I}) + 2\,\zeta\,\Omega\, X_{1,R} - \Omega^2\, X_{1,I} + X_{1,I} \\ + \frac{3}{4}\gamma_3 \left((X_{1,I} - X_{n,I})\ (X_{1,R} - X_{n,R})^2 + (X_{1,I} - X_{n,I})^3 \right) = 0, \end{aligned} \tag{20}$$

The equations Eq. (21) and Eq. (22) are about equation Eq. (18), separated in the form of equations with real and imaginary variables respectively.

$$\begin{aligned} -\Omega^2\,\mu\, X_{n,R} - \gamma_1\ (X_{1,R} - X_{n,R}) - 2\,\zeta\,\Omega\,\beta\ (X_{n,I} - X_{1,I}) \\ - \frac{3}{4}\gamma_3 \left((X_{1,I} - X_{n,I})^2\ (X_{1,R} - X_{n,R}) + (X_{1,R} - X_{n,R})^3 \right) = 0, \end{aligned} \tag{21}$$

$$\begin{aligned}&-2\,\zeta\,\Omega\,\beta\,\left(X_{1,R}-X_{n,R}\right)-\Omega^2\,\mu\,X_{n,I}-\gamma\_1\,\left(X_{1,I}-X-n,I\right)\\&-\frac{3}{4}\,\gamma_3\,\left(\left(X_{1,I}-X_{n,I}\right)\,\left(X_{1,R}-X_{n,R}\right)^2+\left(X_{1,I}-X_{n,I}\right)^3\right)=0,\end{aligned} \tag{22}$$

The first equation for Case II Eq. (10) expanded real and imaginary variables.

$$\begin{aligned}&-X_{2,R}+2\,i\,\zeta\,\Omega\,(-X_{2,R}-i\,X_{2,I}+X_{1,R}+i\,X_{1,I})-i\,X_{2,I}\\&-\Omega^2\,\left(X_{1,R}+i\,X_{1,I}\right)+2\,i\,\zeta\,\Omega\,\left(X_{1,R}+i\,X_{1,I}\right)+2\,\left(X_{1,R}+i\,X_{1,I}\right)=\delta_{st},\end{aligned} \tag{23}$$

The second equation for Case II Eq. (11) expanded real and imaginary variables.

$$\begin{aligned}&\gamma_1\,\left(X_{2,R}+i\,X_{2,I}-X_{n,R}-i\,X_{n,I}\right)+\,i2\,\beta\,\zeta\,\Omega\,\left(X_{2,R}+i\,X_{2,I}-X_{n,R}-i\,X_{n,I}\right)\\&-\Omega^2\,\left(X_{2,R}+i\,X_{2,I}\right)+X_{2,R}-\,i2\,\zeta\,\Omega\,\left(-X_{2,R}-i\,X_{2,I}+X_{1,R}+i\,X_{1,I}\right)+i\,X_{2,I}\\&-X_{1,R}-i\,X_{1,I}+\frac{3}{4}\,\gamma_3\,\left(\left(X_{2,R}-X_{n,R}\right)^3+\left(X_{2,I}-X_{n,I}\right)^2\,\left(X_{2,R}-X_{n,R}\right)\right)\\&+\frac{3}{4}\,\gamma_3\,\left(\left(X_{2,I}-X_{n,I}\right)\,\left(X_{2,R}-X_{n,R}\right)^2+\left(X_{2,I}-X_{n,I}\right)^3\right)=0,\end{aligned} \tag{24}$$

The third equation for Case II Eq. (12) expanded real and imaginary variables.

$$\begin{aligned}&-\gamma_1\,\left(X_{2,R}+i\,X_{2,I}-X_{n,R}-i\,X_{n,I}\right)-\,i2\,\beta\,\zeta\,\Omega\,\left(X_{2,R}+i\,X_{2,I}-X_{n,R}-i\,X_{n,I}\right)\\&-\Omega^2\,\mu\,\left(X_{n,R}+i\,X_{n,I}\right)-\frac{3}{4}\,\gamma_3\,\left(\left(X_{2,R}-X_{n,R}\right)^3+\left(X_{2,I}-X_{n,I}\right)^2\,\left(X_{2,R}-X_{n,R}\right)\right)\\&-\frac{3}{4}\,\gamma_3\,\left(\left(X_{2,I}-X_{n,I}\right)\,\left(X_{2,R}-X_{n,R}\right)^2+\left(X_{2,I}-X_{n,I}\right)^3\right)=0,\end{aligned} \tag{25}$$

The quations Eq. (23), Eq. (24) and Eq. (25) were expanded in order to separate the real part from the imaginary part, so now have 6 equations for Case II.

The equations Eq. (26) and Eq. (27) are about equation Eq. (23), separated in the form of equations with real and imaginary variables respectively.

$$-X_{2,R}+2\,\zeta\,\Omega\,\left(X_{2,I}-X_{1,I}\right)-\Omega^2\,X_{1,R}+2\,X_{1,R}-2\,\zeta\,\Omega\,X_{1,I}=\delta_{st}, \tag{26}$$

$$2\,\zeta\,\Omega\,\left(X_{1,R}-X_{2,R}\right)-X_{2,I}+2\,\zeta\,\Omega\,X_{1,R}-\Omega^2\,X_{1,I}+2\,X_{1,I}=0, \tag{27}$$

The equations Eq. (28) and Eq. (29) are about equation Eq. (24), separated in the form of equations with real and imaginary variables respectively.

$$\begin{aligned}&\gamma_1\,\left(X_{2,R}-X_{n,R}\right)-\Omega^2\,X_{2,R}+X_{2,R}-2\,\zeta\,\Omega\,\left(X_{2,I}-X_{1,I}\right)+2\,\zeta\,\Omega\,\beta\,\left(X_{n,I}-X_{2,I}\right)\\&-X_{1,R}+\frac{3}{4}\,\gamma_3\,\left(\left(X_{2,R}-X_{n,R}\right)^3+\left(X_{2,I}-X_{n,I}\right)^2\,\left(X_{2,R}-X_{n,R}\right)\right)=0,\end{aligned} \tag{28}$$

$$\begin{aligned}&2\,\zeta\,\Omega\,\beta\,\left(X_{2,R}-X_{n,R}\right)-2\,\zeta\,\Omega\,\left(X_{1,R}-X_{2,R}\right)+\gamma_1\,\left(X_{2,I}-X_{n,I}\right)-\Omega^2\,X_{2,I}\\&+X_{2,I}-X_{1,I}+\frac{3}{4}\,\gamma_3\,\left(\left(X_{2,I}-X_{n,I}\right)\,\left(X_{2,R}-X_{n,R}\right)^2+\left(X_{2,I}-X_{n,I}\right)^3\right)=0,\end{aligned} \tag{29}$$

The equations Eq. (30) and Eq. (31) are about equation Eq. (25), separated in the form of equations with real and imaginary variables respectively.

$$\begin{aligned}&-\gamma_1\,\left(X_{2,R}-X_{n,R}\right)-2\,\zeta\,\Omega\,\beta\,\left(X_{n,I}-X_{2,I}\right)\\&-\frac{3}{4}\,\gamma_3\,\left(\left(X_{2,R}-X_{n,R}\right)^3+\left(X_{2,I}-X_{n,I}\right)^2\,\left(X_{2,R}-X_{n,R}\right)\right)-\Omega^2\,\mu\,X_{n,R}=0,\end{aligned} \tag{30}$$

$$\begin{aligned}&-2\,\zeta\,\Omega\,\beta\,\left(X_{2,R}-X_{n,R}\right)-\gamma_1\,\left(X_{2,I}-X_{n,I}\right)\\&-\frac{3}{4}\,\gamma_3\,\left(\left(X_{2,I}-X_{n,I}\right)\,\left(X_{2,R}-X_{n,R}\right)^2+\left(X_{2,I}-X_{n,I}\right)^3\right)-\Omega^2\,\mu\,X_{n,I}=0,\end{aligned} \tag{31}$$