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DISCRETE AND CONTINUOUS SPECTRAL METHODS APPLIED TO BOUNDARY LAYER STABILITY

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Abstract. *Matrix forming is arguably the most used technique to solve linear stability problems. It generates and solves an algebraic generalized eigenvalue problem constructed from the linear differential eigenvalue problem that models the time asymptotic behavior of small amplitude disturbances. This construction is quite often done using spectral methods, but finite difference methods are used as well. In a previous study of the Orr-Sommerfeld equation modeling the linear, local and modal stability of the plane Poiseuille flow, it was shown that discrete and continuous spectral methods have equivalent performances in terms of absolute error versus CPU time. This flow, however, has an analytical steady-state. This means that the required integral coefficients in a continuous spectral method can be calculated analytically. In the present work, the plane boundary layer forming over a flat plate is investigated instead. It is significantly more difficult for continuous spectral methods because its numerical steady-state requires the integral coefficients to be calculated numerically as well. Comparisons between continuous and discrete Chebyshev transforms was preformed in terms of absolute error versus CPU time and the results show an advantage for the discrete spectral methodology .*

Keywords: *Eigenvalue Problems, Stability Analysis, Spectral Methods.*

1. INTRODUCTION

The Linear Stability Analysis (LSA) is, nowadays, an important field in fluid mechanics, especially when it is need to study transition of laminar to turbulent flows, for example, the fuel/oxidant interaction in injections systems (Manco *et al.*, 2015) as well as the boundary layer transition for aviation engineering problems and so on. An advantage in realize a LSA is the reduction of the computation cost when compared with Direct Numerical Simulation (DNS). In a DNS the different scales of vorticy formation force the construction of a very refined mesh and introduce rigidity in the algebraic systems, increasing the computational cost. Otherwise the LSA models the evolution of linear perturbations introduced at flows.

Stability equations are obtained through decomposition of governing equations between steady-state plus unsteady part. The steady-state is called base flow, otherwise the unsteady is called disturbance flow. When only linear disturbances are considered and the governing equations are the Navier-Stokes Equations, this decompostion gives the Linearized Navier-Stokes Equation (LNSE), that models the linear disturbances behavior introduced in a Newtonian fluid flow (Paredes, 2014).

In a strong level of approximation, the base flow is homogeneous in two spatial directions and the disturbance flow are given as Fourier mode in this directions. That is analogous to apply the Fourier transform, resulting in eigenvalue problem and corresponding to a local stability analysis. The matrix forming approach is arguably the most use technique to solve this problems, in this one the differential eigenvalue problem is transformed in a algebraic eigenvalue problem, often been using spectral methods Juniper *et al.* (2014).

The spectral collocation methods are historically used to discretize the linear local stability equations, and the discretization scheme plays a very important role for the matrix storing. Paredes *et al.* (2013) realized a comparative study about the discretizations schemes, showing that the finite difference method is a competitive approach when multidimensional eigenvalue problems are considered. According to Theofilis (2011) the spectral methods lose advantage for multidimensional problems with low resolution, and for a high resolution a large amount of memory is required to matrix storing.

As well as collocation approach, the Spectral Galerkin method may be used to do the differential to algebraic transformation into the eigenvalue problem. In this one, the equations are not discretized, and a truncated series solution about the trial function is proposed and the residue is minimized through of scalar product with the test function. Galerkin approach was used by Orszag (1971) to study the linear and local stability of the Plane Poiseuille flow, giving accurate solutions for the most unstable eigenvalue.

In a previous study (Silva *et al.*, 2020) of the Orr-Sommerfeld equation modeling the linear, local and modal stability of the plane Poiseuille flow, it was shown that discrete and continuous spectral methods have equivalent performances in terms of absolute error versus CPU time. This flow, however, has an analytical steady-state. This means that the required integral coefficients in a continuous spectral method can be calculated analytically. In the present work, the plane boundary layer forming over a flat plate is investigated instead through the Orr-Sommerfeld equation using continuous and discrete approaches both based on Chebyshev Polynomials.

2. METHODOLOGY

This section present the mathematical methodology used to established the Orr-Sommerdfeld Equation that models the evolution of small amplitude perturbation introduced at flow when the flow are considered local and parallel. Besides that, the spectral methodology used to calculate the numerical results are present.

2.1 Governing Equations

The incompressible equations in Cartesian coordinates with constant properties and for Newtonian fluid were used to derive the Orr-Sommerfeld equation. The dimensionless Navier Stokes equations are given by:

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{Re}\nabla^2 \mathbf{u}, \tag{2}$$

where $\mathbf{u} = (u, v, w)$ is the velocity vector, p the pressure and Re the Reynolds number. For Cartesian coordinates,

$$\mathbf{u} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \quad \text{and} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{3}$$

The flow have been decomposed into the steady-state base flow ($\bar{\mathbf{q}}$) and unsteady disturbance ($\tilde{\mathbf{q}}$) parts as given by

$$\mathbf{q}(\mathbf{x},\, t) = \bar{\mathbf{q}}(\mathbf{x}) + \epsilon\tilde{\mathbf{q}}(\mathbf{x},\, t), \quad \epsilon \ll 1, \tag{4}$$

where $\mathbf{q} = (u,\, v,\, w,\, p)^T$, $\mathbf{x} = (x,\, y\, z)$ and ϵ is a amplitude perturbation parameter. When apply the Eq. (4) into Eqs. (1) and (2), and collecting only order $\mathcal{O}(\epsilon)$ (for a linear analysis) terms, results in the Linearized Navier-Stokes Equations (LNSE)

$$\nabla \cdot \tilde{\mathbf{u}} = 0 \tag{5}$$

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \bar{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} = \nabla p + \frac{1}{Re}\nabla^2 \tilde{\mathbf{u}}. \tag{6}$$

For the modal and local stability analysis the unsteady disturbance are considered having a single no homogeneous direction in the base flow, than the separation of variables allows the Fourier decomposition into the homogeneous directions, given

$$\mathbf{q}(x,\, y,\, z,\, t) = \hat{\mathbf{q}}(y)\exp\left[i(\alpha x + \beta z - \omega t)\right], \tag{7}$$

where $i = \sqrt{-1}$. The velocity as well as pressure disturbances $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{p}$ are described as functions of α, β and ω. The eigenfunctions $\hat{u}$, $\hat{v}$, $\hat{w}$ and $\hat{p}$ are considered as a local linear disturbance amplitudes, that is, depends just on the y direction.

Applying this decomposition, Eq. (7) into Eqs. (5) to (6) yields:

$$\mathcal{L}(\hat{u}) + \frac{\partial \bar{u}}{\partial y}\hat{v} + i\alpha\hat{p} = 0, \tag{8}$$

$$\mathcal{L}(\hat{v}) + \frac{\partial \hat{p}}{\partial y} = 0, \tag{9}$$

$$\frac{\partial \bar{w}}{\partial y}\hat{v} + \mathcal{L}(\hat{w}) - i\beta\hat{p} = 0, \tag{10}$$

$$i\alpha\hat{u} + \frac{\partial \hat{v}}{\partial y} + i\beta\hat{w} = 0, \tag{11}$$

where $\mathcal{L}$ is the operator $\mathcal{L}(.) = i\alpha\bar{u}(.) + i\beta\bar{w}(.) - i\omega(.) - \frac{1}{Re}\left(\frac{\partial^2(.)}{\partial y^2} - \beta^2(.) - \alpha^2(.)\right)$. If the spanwise base flow ($\bar{w}$) is zero, then the system given by Eqs. (8 to 11) may be written as a single fourth-order differential equation by elimination the pressure with the differentiation of Eqs. (8) and (10) and using the Eq. (11) resulting in the Orr-Sommerfeld Equation (OSE)

$$\frac{i}{Re}\frac{d^4\hat{v}}{dy^4} + \left(\alpha u_b - 2k^2\frac{i}{Re}\right)\frac{d^2\hat{v}}{dy^2} + \left(-\alpha k^2 u_b - \alpha\frac{d^2u_b}{dy^2} + \frac{i}{Re}k^4\right)\hat{v} + k^2\omega\hat{v} - \omega\frac{d^2\hat{v}}{dy^2} = 0, \qquad (12)$$

where $k^2 = (\alpha^2 + \beta^2)$, subject to homogeneous Dirichlet and Neumann boundary conditions . With the matrix formation, may construct a generalized eigenvalue problem as $\mathbf{A}\hat{v} = \omega\mathbf{B}\hat{v}$, where the operators $\mathbf{A}$ and $\mathbf{B}$ are given by:

$$\mathbf{A} = \left(\frac{i}{Re}\frac{d^4(.)}{dy^4} + \left(\alpha u_b - 2k^2\frac{i}{Re}\right)\frac{d^2(.)}{dy^2} + \left(-\alpha k^2 u_b - \alpha\frac{d^2u_b}{dy^2} + \frac{i}{Re}k^4\right)(.)\right) \quad \text{and} \quad \mathbf{B} = \left(\frac{d^2(.)}{dy^2} - k^2(.)\right). \qquad (13)$$

2.2 Spectral Methods

The spectral methods are global methods, i.e. the calculation of any point of domain depends not only the neighboring information, but the all domain points information. If compare with finite difference methods, that approximate the function by local polynomial interpolation and the scheme order is the order of the polynomial used to interpolate the function, the spectral methods has infinity-order precision since use all domain information.

They are a part of the general weighted residuals methods (WRM). The WRM consider that the differential equation solution may be approximated with a truncated series about a base function (trial), and the residuals, i.e. the difference between the exact and truncated solution, is forced to be zero through the test functions.

Before forward, we need to define the scalar product as

$$\langle u, v\rangle_w = \int_{-1}^{1} u(x)\, v(x)\, w(x)\, dx, \qquad (14)$$

where $w(x)$ is a weight function.

The trial and test functions must be mutually orthogonal, i.e $\langle \phi_k, \psi_l\rangle_w = 0, \quad \forall\, k \neq l$, and the set of which orthogonal trial and test function use depends of the application and that function are directly connected with the accuracy of these methods (Canuto *et al.*, 2012). If the problem has periodic boundary conditions, the base function as periodic, and the method will results at the Fourier spectral methods, otherwise, Chebyshev and Legendre polynomials are the most used base functions (Boyd, 2001; Hesthaven *et al.*, 2007; Canuto *et al.*, 2012).

An approximate truncated solution for a differential problem is given by

$$u_N(x) = \sum_{k=0}^{N} \hat{u}_k\phi_k(x) \qquad \text{with} \quad \hat{u}_k = \frac{\langle u, \phi_k\rangle_w}{\langle \phi_k,\, \phi_k\rangle_w}. \qquad (15)$$

Finally the Residual $\mathcal{R}_N = u - u_N$ is minimized when

$$\langle \mathcal{R}_N, \psi_k\rangle_w = 0. \qquad (16)$$

The Sturm-Liouville problem is a important point in spectral methods, because their eigenfunctions are mutually orthogonal. The problem is given by

$$-(pu')' + qu = \lambda w\, u, \qquad (17)$$

with suitable boundary conditions. In the present work the Chebyshev Polynomials of first kind ($T_k(x)$) are used to provide the approximations (trial functions), for the Chebyshev polynomials the Sturm-Liouville problem has $p(x) = (1 - x^2)^{1/2}$, $q = 0$, $w(x) = (1 - x^2)^{-1/2}$, and $\lambda = k^2$, then a given solution

$$T_k(x) = \cos k \arccos x, \qquad (18)$$

and the expansion may be established as

$$u(x) = \sum_{k=0}^{\infty} \hat{u}_k T_k(x) \qquad \hat{u}_k = \frac{2}{\pi c_k}\int_{-1}^{1} u(x)T_k(x)w(x)dx \qquad \text{where} \quad c_k = \begin{cases} 2 & \text{if } k = 0 \\ 1 & \text{if } k \geq 1 \end{cases}. \qquad (19)$$

2.2.1 Discrete Approach: Chebyshev Collocation Method

The Collocation approach is characterized by the test function at Eq. (16) $\psi_j = \delta(x - x_j)$. In this way, the collocation point are not arbitrary choice, they depends on the quadrature formulas for integration. The often used quadrature points are the Gauss-Lobatto points Canuto *et al.* (2012), and this one are given as

$$x_j = cos\frac{\pi j}{N}, \tag{20}$$

with the weight

$$w_j = \begin{cases} \frac{\pi}{2N} & \text{if } j = 0 \\ \frac{\pi}{N} & \text{if } 1 \leq j \leq N-1 \end{cases}. \tag{21}$$

In other point of view, the Chebyshev-Gauss-Lobatto points may be seen as the Lagrange Polynomial interpolation. In this way, the truncated series becomes

$$u_N(x) = \sum_{k=0}^{N} h_k u(x_j), \tag{22}$$

where

$$h_k(x) = \frac{(-1)k + 1(1 - x^2)T'_N(x)}{\overline{c}_k N^2 (x - x_k)}, \tag{23}$$

is the Lagrange Polynomial. As the series do not have the spectral unknown coefficients, it may be differentiated, resulting

$$(\partial_n u)(x_j) = \sum_{k=0}^{N} h'_k(x_j) u(x_k), \tag{24}$$

where $h'_j(x_i)$ is a operator named Chebyshev pseudo-spectral matrix given by

$$h'_j(x_i) = (D)_{ij} = \begin{cases} \frac{\overline{c}_i}{\overline{c}_j}\frac{(-1)^{i+j}}{x_i - x_j} & i \neq j \\ \frac{-x_j}{2(1-x_j^2)} & 1 \leq i = j \leq N-1 \\ \frac{2N^2+1}{6} & i = j = 0 \\ -\frac{2N^2+1}{6} & i = j = N \end{cases}, \tag{25}$$

and the derivative of u is calculated as $u' = Du$.

With the Chebyshev differentiation matrix Eq. (25) the operators at the Eq. (13) are constructed and the algebraic generalized eigenvalue problem $\mathbf{A}\hat{v} = \omega \mathbf{B}\hat{v}$ is solved. The number of points N in this case is relative to the number of Chebyshev polynomials used in the approximation as Equation (15).

2.2.2 Continuous Approach: Chebyshev Galerkin Method

As well as the collocation approach, the Galerkin ones can be used to solve the OSE. In this one, the trial and test functions are the same functions, and here we note the importance of the Sturm-Liouville problem, since their eigenfunctions are mutually orthogonal.

In this way, the procedure is put the Eq.(15) into the OSE Eq.(12). This procedure is the same when apply the Integral Transform Technique and results in

$$\sum_{n=0}^{N} \langle \phi_m, \mathbf{A}\phi_n \rangle \hat{u}_n = \omega \sum_{n=0}^{N} \langle \phi_m, \mathbf{B}\phi_n \rangle \hat{u}_n. \tag{26}$$

The scalar product in the above equation are similar to the integral coefficients of the integral transformation approach. Solving the integrals we can construct the algebraic generalized eigenvalue problem and finally solve it. In this work the Chebyshev polynomial are used as orthogonal basis this resulting in

$$\sum_{n=0}^{N} \langle T_m, \mathbf{A}T_n \rangle \hat{u}_n = \omega \sum_{n=0}^{N} \langle T_m, \mathbf{B}T_n \rangle \hat{u}_n. \tag{27}$$

In the Boundary layer problem, the base flow is an numerical solution, as will be discussed in the next subsection, because of this, the integrals coefficients (scalar product) must be calculated numerically. Since the Chebyshev polynomial are used to realize the approximation, the natural choice of integration methods are Chebyshev-Gauss quadrature established by

$$\sum_{j=0}^{N_x} p(x_j) w_j = \int_{-1}^{1} p(x) w(x) dx, \quad \text{for all}\, p \in \mathbb{P}_{2N_x-1} \quad . \tag{28}$$

Apply this integration into the Eq. (27), results

$$\sum_{n=0}^{N} \left(\sum_{j=0}^{N_x} T_m(x_j)\, \mathbf{A} T_n(x_j)\, w_j \right) \hat{u}_n = \omega \sum_{n=0}^{N} \left(\sum_{j=0}^{N_x} T_m(x_j)\, \mathbf{B} T_n(x_j)\, w_j \right) \hat{u}_n. \tag{29}$$

The quadrature points and weights used here are the same in the discrete approach, characterized by Equations (20) and (21), therefore it is a Chebyshev-Gauss-Lobatto integration of Galerkin Equations.

Since the Chebyshev eigenfunctions do not respect the homogeneous Dirichlet and Neumann boundary conditions, it is necessary complete the algebraic eigenvalue problem with the boundary values. It is done substituting the last four lines with the boundary conditions as

$$\sum_{k=0}^{N} \hat{u}_k T_k(\pm 1) = 0 \tag{30}$$

and

$$\sum_{k=0}^{N} \hat{u}_k T_k'(\pm 1) = 0 \quad . \tag{31}$$

2.3 Base Flow

The steady-state of plane boundary layer is a similarity solution from the Prandtl shear layer theory, the equations that model a thin thickness layer, subject shear stress, is known as Blasius equation

$$f'''(\eta) + \frac{1}{2} f''(\eta) f(\eta) = 0, \tag{32}$$

where

$$f'(\eta) = \frac{u}{U_\infty} \tag{33}$$

is the dimensionless velocity, and

$$\eta = y \sqrt{\frac{U_\infty}{\nu\, x}} \tag{34}$$

is the similarity variable. with the boundary conditions $f(0) = f'(0) = 0$ and $f'(\infty) = 1$.

The Blasius equations is a non-linear ordinary differential equation, and to solve it was employed a *Wolfram Mathematica* routine, based on Shooting Method.

2.4 Mapping

Both methodologies discussed into previews sections are based on Chebyshev polynomials, that are orthogonal in $(-1, 1)$ interval. However, the plane Boundary Layer boundaries are $[0, \infty)$, thus its need a mapping to convert the physical and computational spaces. It is made through coordinates transformation in the Orr-Sommerfeld Equations as

$$y = y_a \frac{1 + \xi}{y_b - \xi} \quad , \tag{35}$$

where

$$y_a = \frac{y_d\, y_{max}}{y_{max} - 2 y_d} \tag{36}$$

and

$$y_b = 1 + 2\frac{y_a}{y_{max}}. \tag{37}$$

In this case, y_d is a mesh refinement parameter and y_{max} is the numerical truncation for ∞ (far field). The transformation jacobians are need to replace the derivatives terms in the new coordinate system. Juniper *et al.* (2014) present all jacobians needs to established the Orr-Sommerfeld operator.

3. RESULTS

Initially it is necessary verify if the numerical results are accurate. It is made comparing to the results obtained with the previous published results, the base flow results were obtained with the shooting method, and compared with Schlichting and Gersten (2016), and the comparison may be seen in the Figure 1 left.the eigenvalue spectrum results obtained with Chebyshev Collocation method, i.e. discrete approach, were compared with Mack (1976). The comparison may be seen in the Figure 1 right. All numerical results are obtained with the same mapping parameters, i.e. $y_{max} = 50$ and $y_d = 20$.

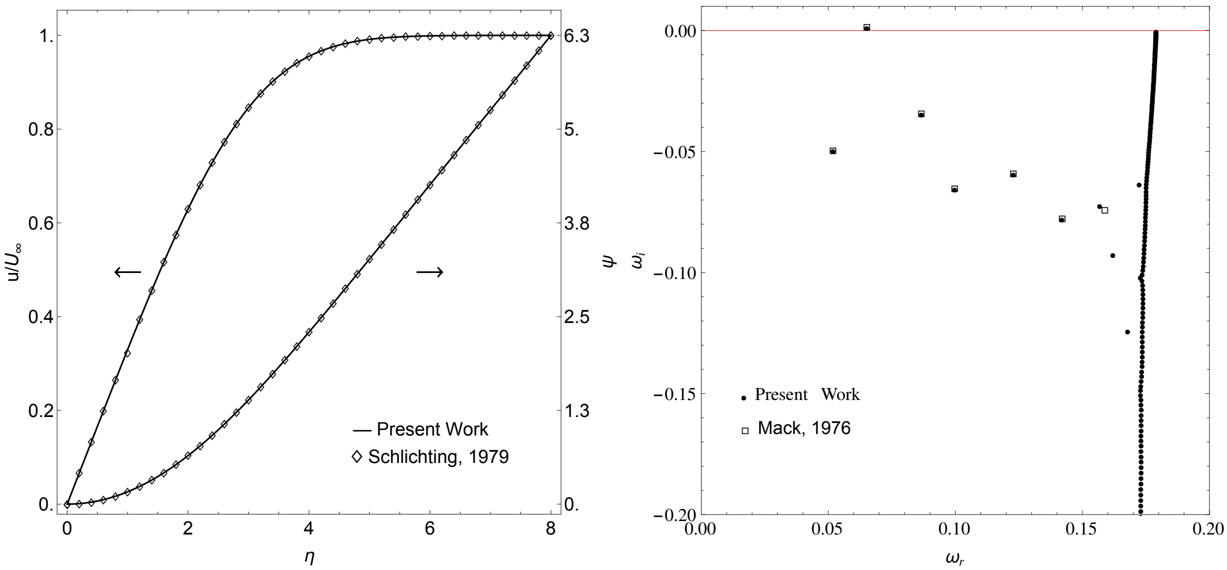

Figure 1. Base flow (left) and eigenvalues spectrum (right) verification. $\alpha = 0.179$, $Re = 580$, $N = 500$.

As can be seen, the base flow and spectrum results shows good numerical accuracy. After that the discrete approach was verified, the continuous approach was compared with this one. The results for the spectrum can be seen in the Figures 2 and 3. The spectrum was calculated using different matrix order for both methods and the results shown good agreement.

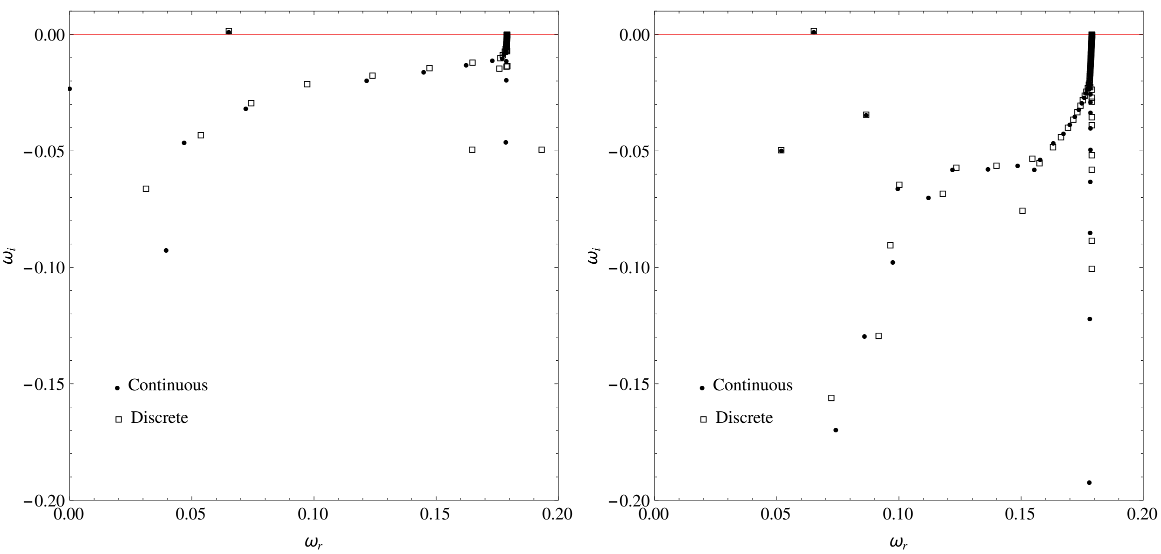

Figure 2. Plane Boundary Layer Spectrum. Matrix order N^2. $N = 50$ (left), $N = 100$ (right). $\alpha = 0.179$, $Re = 580$, $\beta = 0$.

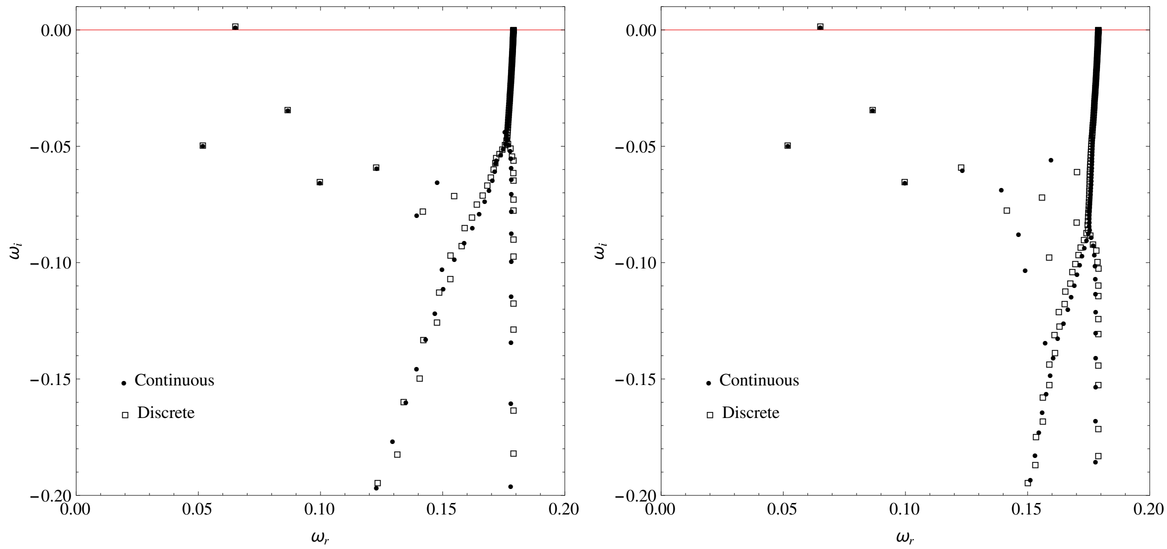

Figure 3. Plane Boundary Layer Spectrum. Matrix order N^2, $N = 150$ (left) and $N = 200$ (right). $\alpha = 0.179$, $Re = 580$, $\beta = 0$.

It is possible to note the spectrum similarity between both methodologies, even thought the continuous spectrum is not completely converged. This results show the equivalence of these two spectral approximations. Since was need 500 points for a completely converged spectrum with the discrete approach, is expected that with equal order will be possible recovery all spectrum. Unfortunately that was not possible because of the large CPU time required. The Figure 4 shows the time necessary to construct and solve the algebraic generalized eigenvalue problem.

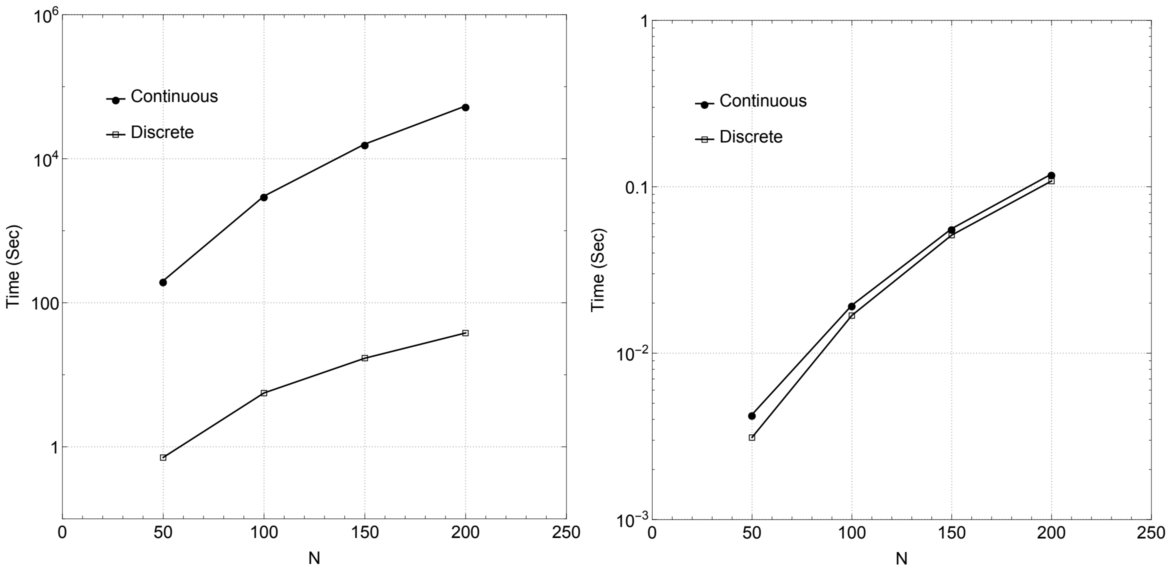

Figure 4. Eigenvalue problem matrix order (N^2) versus time to construct the matrix (left) and eigenvalue problem matrix order (N^2) versus time to solve the eigenvalue problem (right).

Although the large difference between discrete and continuous methodologies when compared the time to construct the matrix, the time to solve is nearly equal, it too was show in a previous work by Silva *et al.* (2020). The large time necessary to construct the eigenvalue problem not resulted in a larger error decrease rate as may be seen in the Figure 5, the Figure show the absolute error in the eigenvalue calculated for the three least stable eigenvalue.

As can be seen in the Figure 5, the discrete methodology had advantage, although the decrease error rate was the same since both methodologies are spectral, the absolute error with the discrete metodology was smaller. Another point is the error propagation when is need to evaluate the eigenfunctions to realize the numerical integration in the continuous approach, it was solved increasing the machine precision that results a increase of the time to construct the eigenvalue problem when using this one.

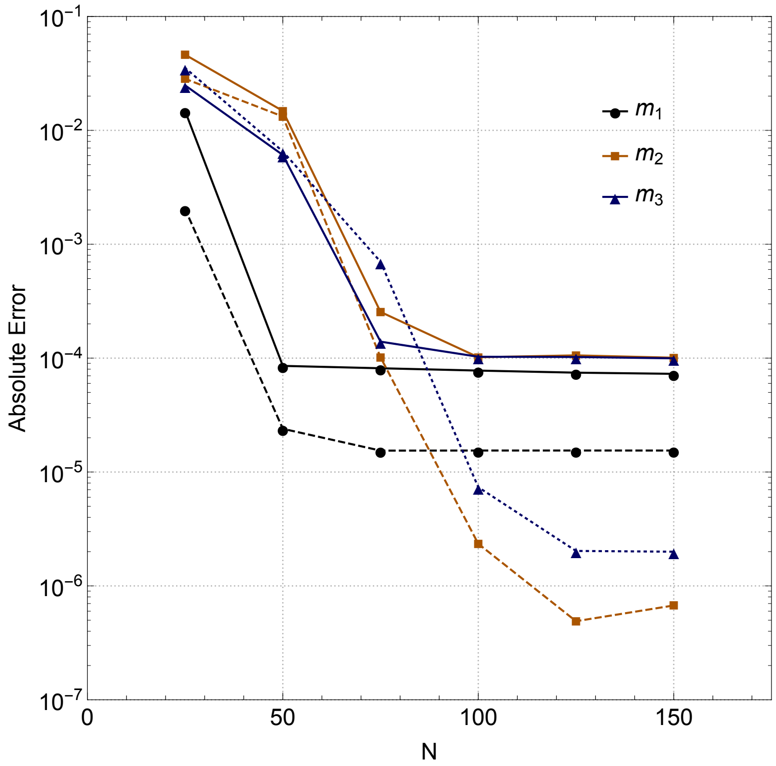

Figure 5. Absolute error for three least stables eigenvalues. The continuous lines represent the continuous methodology while the dashed lines represent the discrete methodology.

4. CONCLUSION

In this work, two different spectral methods were developed and applied to boundary layer stability. Both methods given accurate numerical results for the eigenvalue spectrum. The time to construct the algebraic eigenvalue problem, i.e. the matrix $\mathbf{A}$ and $\mathbf{B}$, was very worse when used the continuous approach (Chebyshev Galerkin Method). However, the time to solution the eigenvalue problem with the *Wolfram Mathematica* function *Eigenvalues* was nearly equal.

Although the higher time to construct the the algebraic eigenvalue problem through continuous ways, it is not represent a higher error decrease rate, therefore, the discrete approach (Chebyshev Collocation method) is the better choice when the base flow is not analytical solution.

Besides that, no difference was show on the fully spectrum solution between discrete and continuous methodologies, but was not possible to recover a fully converged spectrum with the continuous methodology because of its time was prohibitive with the computational resources available for this work.

In a future work other eigenfunctions of Sturm-Liouville problem will be used to developed new spectral operators and a new performance analysis between chebyshev and others eigenfunction will be done

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