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# ANALYSIS OF LAMINATE COMPOSITE PLATES USING MOVING LEAST SQUARE RITZ METHOD

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***Abstract.*** *A semi-analytical model using moving least square Ritz (MLS-Ritz) method is proposed in this paper to evaluate the behavior of laminate composite plates. A symmetrical and balanced laminate stacking sequence is considered here. Conditions of simply supported edges and uniform distribution of flexural loads have been analyzed. The model consists of creating points in a rectangular composite plate, where polynomials functions associated with a weight function are created for each point over the plate to obtain a shape function. Each point is imposed to meet the respective geometric boundary conditions. The basis function to approximate the solution is taken from Pascal triangle in order to ensure the minimum completeness. An appropriate positive weight function is chosen to evaluate the contribution of each node within a circular domain at a specific point. Therefore, we can assemble the Ritz shape function to approximate all the grid points displacement. In order to obtain the stiffness matrix, the classical laminate theory (CLT) is used to express the total potential energy functional. Therefore, the displacement field can be obtained. A convergency study is performed to evaluate the suitable number of nodes and Gauss points. The results are compared to Navier solution.*

***Keywords:*** *Laminate Composite Plates, Moving Least Square Ritz Method, Semi-analytical.*

## 1. INTRODUCTION

The structural response of deformable solids is usually described by a set of differential equations. Exact solutions for these differential equations are limited to particular geometries, materials, boundary conditions and loading cases.

In order to handle different boundary conditions, geometries and load cases, approximated solution methods based on Finite Element, Finite Difference, Boundary Element and Ritz method are usually employed.

The Moving Least Square (MLS) is a popular mathematical method used in data fitting and surfaces construction that uses a series representation of functions. Lancaster and Salkauskas (1981) introduced the method for surface generation problems.

The MLS can be an alternative approach to form meshless approximation. Liu and Gu (2005) describe the formulation and construction of the MLS shape functions. The authors present important characteristics of the method such as the choice of the weight function, consistency, reproduction, partition of the unit and Lack of Kronecker delta function property.

The Ritz method is a semi-analytical method based on a variational, to solve a differential equation. The solution is approximated by minimizing the functional that describes the problem, according to the boundary conditions. In solid mechanics, Ritz method can be applied, for example, to solve dynamic and static problems, where the nontrivial solution of the resultant set of equations may lead to eigenproblem (in structural dynamics) or linear system of equations (in elastostatics), from which the generalized coordinates used to describe the displacement field are obtained.

Zhou (2007) and Zhou and Zheng (2008) applied the MLS combined with Ritz to analyze isotropic square, triangular and skew plates under free vibrations. In this case, the natural frequencies and the vibration modes are found through the eigenvalues and eigenvectors problem. The authors conducted a parametric study to investigate the effects of the Gauss points, mesh size, support radius, polynomial order and the shape of the weight function on the convergence of the method. The semi-analytical models based on MLS-Ritz method available in the open literature are limited to isotropic materials only. Thus, there is clear need to extend the MLS-Ritz formulation to handle material anisotropy effects, particularly composite structures.

The objective of this work consists of implementing the MLS-Ritz method to analyze a rectangular composite plate under uniform distributed load and conditions of simply supported edges. A composite laminate is chosen here, due to the important applications, mainly in the aircraft and aerospace fields, allowing weight savings.

The stiffness matrix for composite materials is considered here according to Classical Laminate Theory or CLT (see Daniel and Ishai, 2006), for a symmetrical and balanced laminate constituted of unidirectional tape carbon/epoxy. A convergency study is performed to evaluate the suitable number of nodes. The results are compared with Navier solution for plates.

## 2. FORMULATION

### 2.1 MLS-RITZ

Figure 1 depicts a schematic of the problem analyzed in this work. A uniform distributed pressure load $q_0$ is applied perpendicularly to the simply supported plate with dimensions *a* x *b*. The reference system is positioned at the center of the plate (*x=0mm* and *y=0mm*).

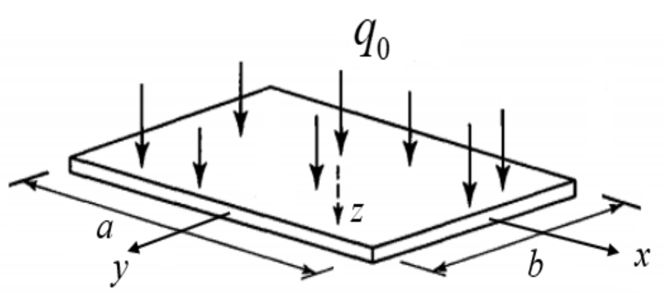


Figure 1: Problem representation

The strain energy for the plate, using classical laminate theory (CLT) is given by:

$$\mathbf{U} = \frac{1}{2}\int_S \boldsymbol{\kappa}^T \cdot \mathbf{D} \cdot \boldsymbol{\kappa}\, dS \tag{1}$$

where $\boldsymbol{\kappa}$ is the strain vector containing the strains along *x, y* and the plate plane and **D** is the laminate bending stiffness matrix (Daniel and Ishai, 2006).

The potential energy of the external pressure load is:

$$V_p = \int_S q_o \cdot w(x,y)\, dS \tag{2}$$

*w (x,y)* is the displacement field.

The displacement predicted by MLS-Ritz method defined as follows,

$$w(x,y) = \sum_{i=1}^{n} R_i(x,y) w_i = \mathbf{R}\mathbf{q} \tag{3}$$

where ***R*** and ***q*** are, respectively, the vector of shape functions $R_i$ and the vector of unknown coefficients (Ritz constants) related to each point of the discretized plate, as shown in the Figure 2.

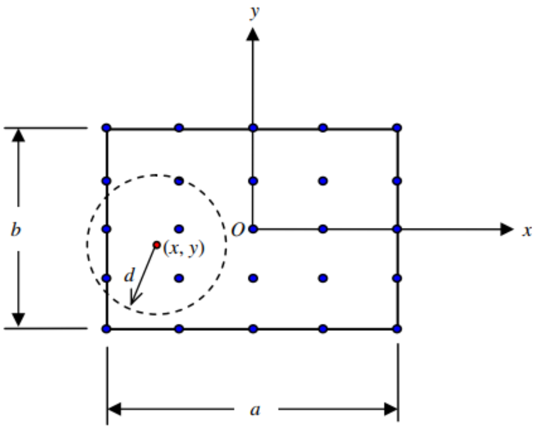


Figure 2: Plate discretization (obtained from Zhou and Zheng, 2005)

The shape functions for the node *i* are given by:

$$R_i = \gamma^T(x,y) g_i(r) \mathbf{p}(x_i, y_i) \tag{4}$$

where:

$$\gamma^T = \mathbf{p}^T(x,y) \cdot \mathbf{A}^{-1}(x,y) \tag{5}$$

***p*** is the vector of basis function formed by the terms from Pascal Tringle, in order to ensure the minimum completeness:

$$\mathbf{p}(x,y) = [p_1(x,y) \;\; p_2(x,y) \; \ldots \; p_m(x,y)]^T \tag{6}$$

***A(x,y)*** is defined as weighted moment matrix, given by:

$$\mathbf{A}(x,y) = \sum_{i=1}^{n} g_i(r) \mathbf{p}(x_i, y_i) \mathbf{p}^T(x_i, y_i) \tag{7}$$

The support given by the circumference of radius *d* (see Figure 2) is used to determine the influence of each node. In this case, a weight function *g* is used. According to Liu and Gu (2005), the weight function can assume different forms, such as cubic and quartic spline and exponential and needs to be sufficient smooth, especially on the boundary. When evaluated at a point, this weight function decreases the value from this point to zero at or out of the border. Here, the weight function used is given by (Zhou and Zheng, 2005):

$$g_i = \begin{cases} (1-r^2)^k & if\; r \le 1 \\ 0 & if\; r > 1 \end{cases} \tag{8}$$

*r* is the normalized distance from the point *(x, y)* to the point *i*, given by:

$$r = \sqrt{(x-x_i)^2 + (y-y_i)^2} \,/\, d \tag{9}$$

*k* is an integer which can be adjusted in order to ensure a weight function sufficient smooth.

Therefore:

$$\mathbf{R} = [R_1(x,y) \;\; R_2(x,y) \;\; R_3(x,y) \cdots R_n(x,y)] \tag{10}$$

Now we can write the total potential energy, involving the plate strain energy and the potential energy of the external pressure load which is given by,

$$\boldsymbol{\pi}_p = \mathbf{U} + \mathbf{V}_p \tag{11}$$

The strains can be obtained by taking the derivatives of the displacements. Therefore, Equation (3) can be used to relate the displacement and strains in equations (1) and (2). By minimizing the functional, the stiffness matrix and Force vector are written as:

$$\mathbf{K} = \frac{1}{2}\int_S \mathbf{R}^{wwT} \cdot \mathbf{D} \cdot \mathbf{R}^{ww} \, dS \tag{12}$$

$$\mathbf{F} = q_o \int_S \mathbf{R}^T \, dS \tag{13}$$

$\mathbf{R}^{ww}$ is a vector containing the second derivatives $\mathbf{R}_{,xx}$, $\mathbf{R}_{,yy}$ and $2\mathbf{R}_{,xy}$.

In order to apply the boundary conditions, we write the displacements for the points on the edges as:

$$w(x_j, y_j) = \sum_{i=1}^{n} R_i(x_j, y_j)\, q_i = 0 \tag{14}$$

Separating the coefficients related to the nodes on the edges in the vector $\mathbf{q}_B$ and the free nodes in the vector $\mathbf{q}_I$ we can solve the linear equation system:

$$\begin{bmatrix} \mathbf{Q} & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{q}_B \\ \mathbf{q}_I \end{bmatrix} = \mathbf{0} \tag{15}$$

Or:

$$\mathbf{q}_B = -\mathbf{Q}^{-1}\mathbf{S}\mathbf{q}_I \tag{16}$$

The boundary conditions are applied by using the point substitution approach. Therefore, we write the restricted nodes in function of free nodes:

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_B \\ \mathbf{q}_I \end{bmatrix} = \begin{bmatrix} -\mathbf{Q}^{-1}\mathbf{S} \\ \mathbf{I} \end{bmatrix} \mathbf{q}_I = \mathbf{T}\mathbf{q}_I \tag{17}$$

Now we obtain the condensed stiffness and load matrices:

$$\overline{\mathbf{K}} = \mathbf{T}^T \mathbf{K} \mathbf{T} \tag{18}$$

$$\overline{\mathbf{F}} = \mathbf{T}^T \mathbf{F} \tag{19}$$

Finally:

$$\mathbf{q}_I = \overline{\mathbf{K}}^{-1} \overline{\mathbf{F}} \tag{20}$$

## 2.2 NAVIER SOLUTION

In order to verify the MLS-Ritz results, the Navier solution for composite plates is used (see Reddy, 2003). This solution consists of a double trigonometric function (Fourier) which is chosen properly to meet the boundary conditions. The double summation uses odd terms *M* and *N* and can easily converge by using few terms.

$$w_0(x, y) = \frac{1}{\pi^4} \sum_{M=1,3,5}^{\infty} \sum_{N=1,3,5}^{\infty} \frac{\dfrac{16 q_0}{\pi^2 MN}}{\left[ D_{11}\left(\dfrac{M}{a}\right)^4 + 2(D_{12} + 2D_{66})\left(\dfrac{M}{a}\right)^2 \left(\dfrac{N}{b}\right)^2 + D_{22}\left(\dfrac{N}{b}\right)^4 \right]} \sin\left(\frac{M\pi(x + a/2)}{a}\right) \sin\left(\frac{N\pi(y + b/2)}{b}\right) \tag{21}$$

## 2.3 TRANSVERSE SHEAR FORCES

When we are interested in obtaining the strains, stresses and internal forces, we need to take the displacement field derivatives. For example, the internal shear force will depend on the third order derivatives of the displacement field. Therefore, it is important to verify the convergence for such cases.

From the equilibrium equations, the shear forces and the moments related to *x, y* and *xy* are given by (Reddy, 2003):

$$Q_x = M_{x,x} + M_{xy,y} \tag{22}$$

$$Q_y = M_{y,y} + M_{xy,x} \tag{23}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} -w_{,xx} \\ -w_{,yy} \\ -2w_{,xy} \end{Bmatrix} \tag{24}$$

## 3. RESULTS AND DISCUSSIONS

In this work, a plate of dimensions *a = 300 mm* and *b = 200 mm* and total thickness of 1.52 *mm* is analyzed. The laminate stacking sequence is $[0° / 90°]_{2s}$. This lay-up avoids bend-twist coupling ($D_{16} = D_{26} = 0$), simplifying the problem. The elastic properties are given in the Table 1, where $E_1$ is the modulus of elasticity parallel to fibers; $E_2$ is the modulus of elasticity transverse to fibers; $\nu_{12}$ is the Poisson's ratio, $G_{12}$ is the in-plane shear modulus.

Table 1: Lamina mechanical properties (Arakaki, 2016)

| $E_1$ (MPa) | $E_2$ (MPa) | $\nu_{12}$ | $G_{12}$ (MPa) | Thickness (mm) |
|---|---|---|---|---|
| 125450 | 9450 | 0.32 | 4700 | 0.19 |

MLS-Ritz method and Navier method were implemented in a computational code. A transversal and uniform distributed load of $q_0 = 10^{-3}\, MPa$ is considered. Table 2 shows the convergency study conducted for Navier solution. The transversal displacement was obtained at the plate center (*x = 0 mm* and *y = 0 mm*). Convergence results is obtained for *M* and *N* between 7 and 15.

In the MLS-Ritz implementation, a support radius of *d = 0.5a* and *k = 10* for the weight function were adopted based on the Zhou (2007) study. Integration using Gauss Quadrature was performed to obtain the stiffness matrix and the load vector (Equations 12 and 13) and a total of 15x15 Gauss points were used. The polynomial vector from Pascal triangle is $\mathbf{p}(x, y) = [1\ x\ y\ x^2\ xy\ y^2]^T$. The number of nodes (equally spaced along direction *x* and *y*) used ranged from 7x7 (49 nodes) to 15x15 (225 nodes). A minimum of 7x7 was necessary, otherwise, the matrix ***A*** will be bad conditioned to calculate its inverse. The results showed an excellent convergency for a number of nodes greater than 9x9 (see table 2) For Navier solution, 11x11 terms are used to compare the results.

Table 2: Convergency study results for the displacement obtained at the center of the plate using MLS-Ritz and Navier solution.

| *n, M x N* | *w(mm), MLS-Ritz* | $w_0$ *(mm), Navier* | *Ratio* ($w / w_0$) |
|---|---|---|---|
| 7x7 | 1.2144 | 1.1913 | 1.0194 |
| 9x9 | 1.1922 | 1.1914 | 1.0007 |
| 11x11 | 1.1910 | 1.1913 | 0.9997 |
| 13x13 | 1.1913 | 1.1913 | 1.0000 |
| 15x15 | 1.1913 | 1.1913 | 1.0000 |

To better evaluate the results of the displacement and internal shear force, a convergence study of the influence of the number of Gauss points and the number of nodes were done along a specific line. For displacement, the results were obtained along the line located at *y = 0 mm.* Results using MLS-Ritz is compared with Navier solution. Figure 3 shows the convergence results for the number of Gauss points, where13x13 nodes were used. We can observe a good agreement with the Navier solution for 13x13 and 15x15 Gauss points. It is possible to observe that, although the order of the basis function be relatively low, the MLS shape functions have high order continuity due to the use of the weight function. Therefore, a very high number of Gauss points were necessary.

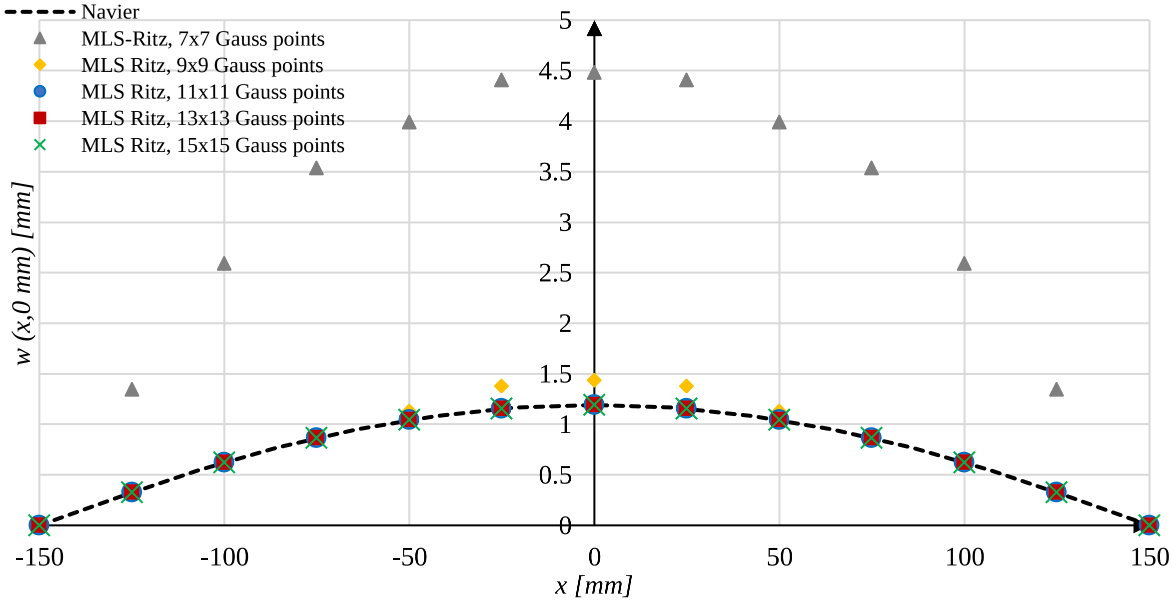


Figure 3: Convergency study results for the influence of the number of Gauss points in the displacement obtained along *y = 0 mm* using MLS-Ritz, compared with Navier solution.

Figure 4 shows the displacement along the line located at *y = 0 mm.* In this case, 15x15 Gauss points were used. The convergency for the number of nodes is very fast, as we can see. As mentioned above, a minimum of 7x7 nodes was necessary, otherwise, the matrix ***A*** will be bad conditioned to calculate its inverse.

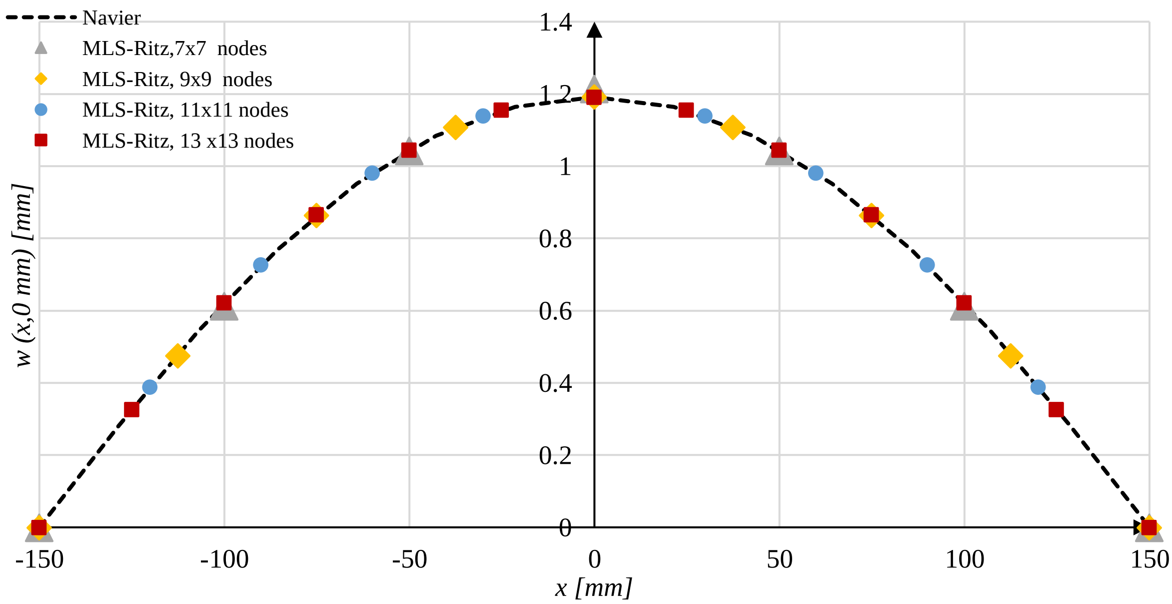


Figure 4: Convergency study results for the influence of the number of nodes in the displacement obtained along *y = 0 mm* using MLS-Ritz, compared with Navier solution.

The internal shear forces $Q_x$ were evaluated along a line located at $x = -a/2$. For a number of nodes equal or less than 11x11 nodes, the results show a bad behavior. For 13x13 number of Gauss points, we observe better results compared with Navier solution. Figure 5 shows the results for 15x15 Gauss points.

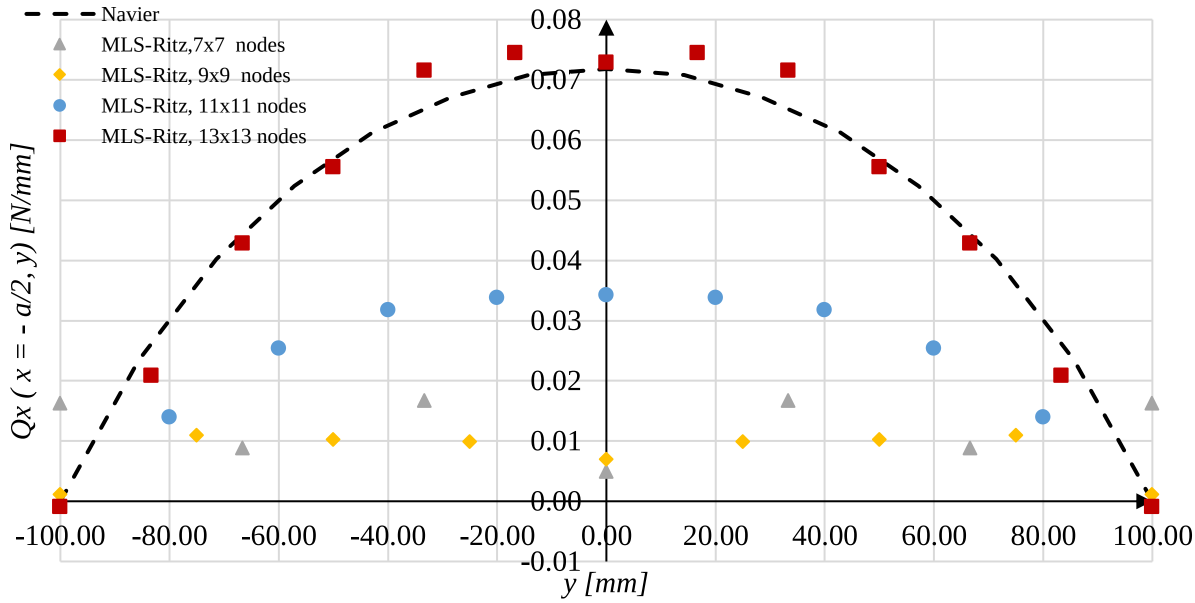


Figure 5: Convergency study results for the influence of the number of nodes in the internal shear force $Q_x$ obtained along the line located at $x = -a/2$ using MLS-Ritz and compared with Navier solution.

Figure 6 shows the influence of the Gauss points, using 13x13 Nodes. The best convergence is reach for 15x15 Gauss points. For number of Gauss points less than 11x11, the error is too large, therefore, it is not presented in the figure.

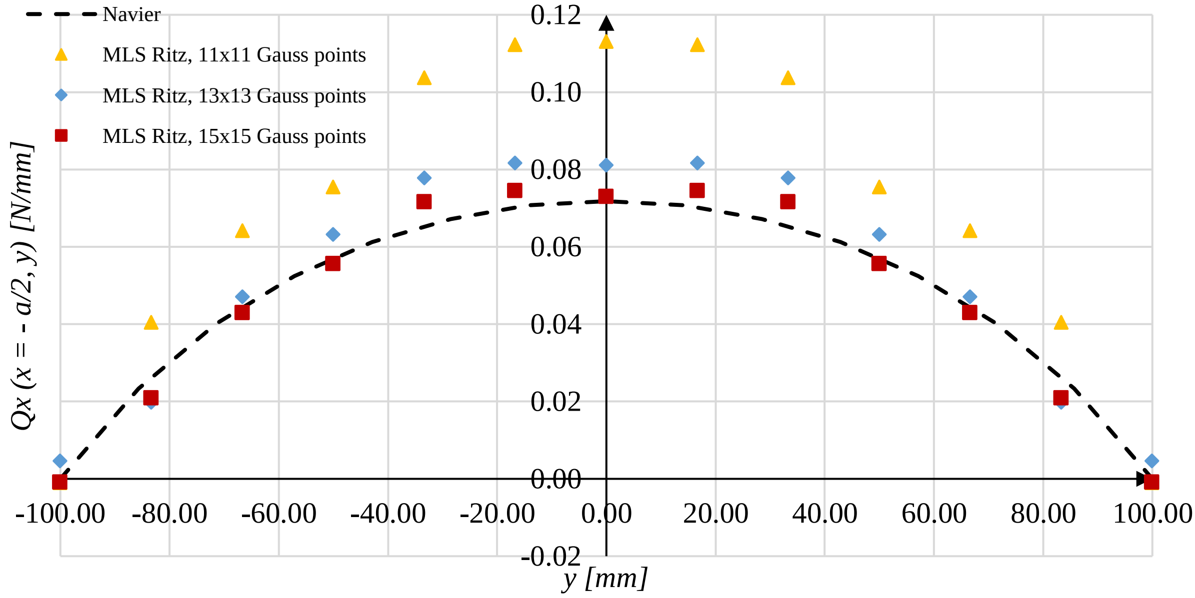


Figure 6: Convergency study results for the influence of the number of Gauss points in the internal shear force $Q_x$ obtained along the line located at $x = -a/2$ using MLS-Ritz and compared with Navier solution.

Finally, a contour plot for MLS-Ritz (Figure 7a), using 13x13 nodes and 15x15 Gauss points, and Navier (Figure 7b) were created to represent the results for the transversal displacement field. Both figures show an excellent agreement between the methods studied here.

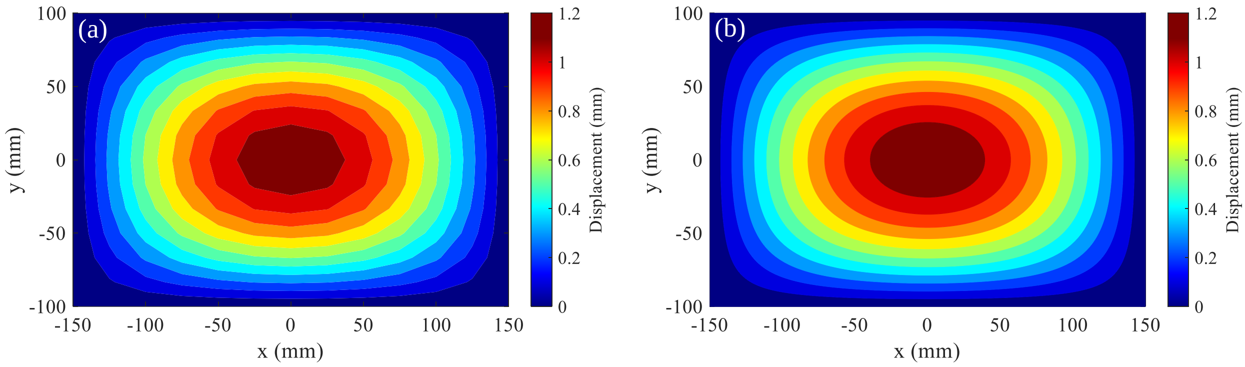


Figure 7: Displacement field obtained using (a) MLS-Ritz and (b) Navier

## 4 CONCLUSIONS

In this work, a semi-analytical model using moving least square Ritz (MLS-Ritz) method was proposed to evaluate the behavior of laminate composite plates simply supported under uniform transversal load. The study showed that the internal shear forces have more difficulties to converge due to the third order derivatives required. Consequently, its necessary to increase the number of nodes, as well as the number of Gauss points. However, a fast convergence and an excellent correlation with the Navier solution was obtained by adopting the new MLS approach for this kind of materials. Characteristics of the MLS-Ritz, such as the facility of applying some kinds of boundary conditions and adaptation to the geometry of the structure, plays an important role for the application of this method.  Therefore, this method is very promising for future studies, such as the consideration of different geometries and boundary conditions as well as the kind of required analysis.

## 5 ACKNOWLEDGEMENTS

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