Dynamic behavior of two-dimensional quasiperiodic plates: achieving high-order wave directionality

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Abstract. *Quasiperiodic configurations, or shortly quasicrystals, have emerged as a potential candidate for unexplored wave phenomena within the architected materials (phononic crystals and metamaterials) field. Their spatial distribution lacks translational symmetry, but they present a long-range order and are unrestricted in rotational symmetries. In this work, three-dimensional quasiperiodic elastic metamaterial plates are investigated with numerical simulations and experimental observations. The numerical and experimental results show a complex dispersion for the quasiperiodic plates, which is related to their fractal nature. At certain frequencies, this behavior becomes highly anisotropic and high-order wave directionality is observed on the dynamic response. Therefore, this work observes the wave beaming and diffraction in high-order fold symmetries (e.g., 10-fold) and stimulates a variety of applications involving this unusual wave directionality previously reported only in the crystallographic fold symmetries (2-, 4- and 6-fold).*

Keywords: *wave propagation, wave beaming, elastic metamaterials, quasicrystals.*

1. INTRODUCTION

Phononic crystals and acoustic-elastic metamaterials are architected materials with unusual properties and functionalities related to wave propagation, which cannot be achieved with conventional materials (Joannopoulos *et al.*, 1997; Hussein *et al.*, 2014; Christensen *et al.*, 2015; Ma and Sheng, 2016; Cummer *et al.*, 2016; Assouar *et al.*, 2016). Among their unusual and extraordinary properties are the negative dynamic mass, negative dynamic stiffness, wave directionality, and more recently, nonreciprocal wave propagation and topological wave guiding. Usually, their configuration is based on building block approach with periodic repeating patterns (i.e., unit cells). However, non-periodic configurations have also been used to obtain other properties and capabilities such as broadband vibration and acoustic attenuation, wave localization and wave trapping (Celli *et al.*, 2019; Beli *et al.*, 2019). In this context, quasiperiodic configurations, or shortly quasicrystals, have emerged as a potential candidate for unexplored wave phenomena. Their spatial distribution lacks translational symmetry, but they present a long-range order and are unrestricted in rotational symmetries that have opened new possibilities for wave manipulation (Shechtman *et al.*, 1984; Levine and Steinhardt, 1984; Vardeny *et al.*, 2013; Kraus and Zilberberg, 2016).

One of the wave phenomena largely explored in phononic materials is the wave directionality or beaming, where the wave is guided in two-dimensional domains using the dispersion anisotropy of the crystals. Although in certain frequencies the wave propagation is highly directional, it is governed by the symmetry of the crystals and patterns that follows their fold symmetry (2-,3-, 4- and 6-fold) are allowed (Ruzzene *et al.*, 2003; Ruzzene and Scarpa, 2005; Phani *et al.*, 2006; Gonella and Ruzzene, 2008; Trainiti *et al.*, 2016; Beli *et al.*, 2018; Foehr *et al.*, 2018; Rosi and Auffray, 2019; Grabec *et al.*, 2020). In this work, quasiperiodic elastic metamaterial plates are investigated with numerical simulations and experimental observations in order to achieve high-order wave directionality beyond the crystallographic symmetries such as the 10-fold.

This paper is organized as follows, after this brief introduction, the design of the quasiperiodic plates is shown in Section 2, next the numerical and experimental methods are presented in Section 3. The dispersion and time response results are depicted in Section 4 and the main conclusions are discussed in Section 5.

2. QUASICRYSTAL DESIGN

The quasicrystalline designs are based on a two-dimensional wave number strategy where the rotational/fold symmetry order is enforced by selecting sharp peaks in a 2D Fourier spectrum (Lubensky, 1988; Widom, 2008; Beli *et al.*, 2021). According to this strategy, a continuum distribution in physical space $\phi(\mathbf{r})$, with $\mathbf{r} = [x, y] \in \mathcal{R}^2$, is defined by directly assigning N Bragg peaks in reciprocal space ($\mathbf{k} = [k_x, k_y] \in \mathcal{R}^2$) as pure points in the two-dimensional Fourier spectra Vardeny *et al.* (2013). These Bragg peaks are angularly spaced by $\theta_N = 2\pi/N$ over a constant circle of fundamental wave number k_0. In this work, only even numbers of peaks are considered to guarantee a real distribution in physical space. Therefore, reciprocal and physical spaces can be expressed, respectively, as:

$$\hat{\phi}(\mathbf{k}) = \sum_{n=0}^{N-1} \delta(\mathbf{k} - \mathbf{k}_n), \tag{1}$$

and

$$\phi(\mathbf{r}) = \sum_{n=0}^{N-1} e^{i\mathbf{k}_n \cdot \mathbf{r}}, \tag{2}$$

where δ is the delta function that locates the wave number $\mathbf{k}_n$ of each Bragg peak, which is given by

$$\mathbf{k}_n = k_0[\cos(n\theta_N), \sin(n\theta_N)], \tag{3}$$

with $n = 0, ..., N-1$ and $k_0 = 2\pi/\lambda_0$. In this design strategy, a single parameter N defines the rotational symmetry of the distribution in physical space, which leads to periodic distributions (1D bilayer for $N = 2$, square pattern for $N = 4$ and hexagonal pattern for $N = 6$) or quasiperiodic distributions with rotationally N-fold symmetry such as the 8-fold and 10-fold. An example of the reciprocal space $\hat{\phi}(\mathbf{k})$ and the physical field $\phi(\mathbf{r})$ for the 10-fold symmetry are depicted in Fig. 1(a).
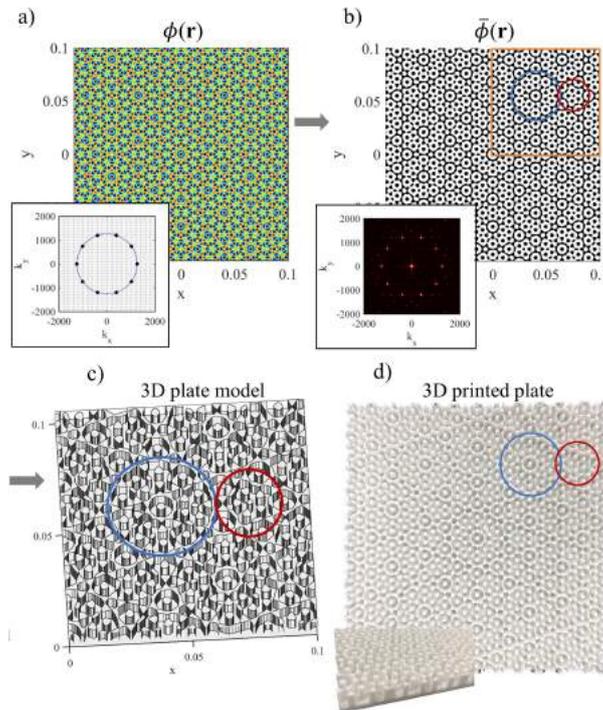

Figure 1. Design strategy for the quasicrystalline elastic metamaterial plates with a volume fraction of 0.30. The spectral content of the design and the correspondent physical field (a), the threshold distribution and its diffraction pattern (b), the quasicrytalline elastic plate with geometric modulation following the threshold distribution (c), and the model manufactured in 3D printing (d).

A threshold procedure is further applied to obtain a two phase distribution as shown in Fig. 1(b) by white and black colors. This threshold distribution have been used to modulate the plate thickness: one of the phases correspond to a

uniform plate and the other phase gives the distribution of the pillars/stubs extruded in one of its sides, which is presented in Fig. 1(c). This configuration is convenient to construct a practical demonstrator in additive manufacturing with a single phase material, as shown in Fig. 1(d), as well as to perform experimental measurements on the flat side of the plate.

The models have a square domain in the xy-plane of size $L = 0.2$ m and $\lambda_0 = 5$ mm, and their modulated thickness in z-axis is given by $h(\mathbf{r}) = h_{\text{A}} + \bar{\phi}(\mathbf{r})(h_{\text{B}} - h_{\text{A}})$, where $h_{\text{A}} = 4$ mm and $h_{\text{B}} = 12$ mm. In addition, they are manufactured using selective laser sintering (SLS) process and polymer nylon 12, with nominal elastic properties: mass density $\rho_n = 1500$ kg/m^3, elastic modulus $E_n = 5$ GPa, Poisson ratio $\nu_n = 0.3$ and structural damping $\eta_n = 0.001$.

3. ANALYSIS METHODS

Numerical simulations are conducted using the finite element approach in `COMSOL Multiphysics` $^{\circledR}$ environment, where three-dimensional elastic solid elements (with 6 degrees of freedom per node) with enforced linear strain are employed. Their dynamics is governed by

$$\mathbf{M}\ddot{\mathbf{u}}(\mathbf{r}, t) + \mathbf{K}\mathbf{u}(\mathbf{r}, t) = \mathbf{F}(\mathbf{r}, t), \tag{4}$$

where $\mathbf{M}$ is the mass matrix carrying kinetic energy terms, $\mathbf{K}$ is the stiffness matrix carrying elastic strain energy terms, $\mathbf{u}$ is the displacement vector field and $\mathbf{F}$ is the externally applied load vector. Its spatial mesh respects 10 elements per wavelength, i.e. $\Delta a = \lambda_0/10$, and for time response simulations, a time-step of $\Delta t = 1/(20 f_e)$ is used to appropriately describe the dynamic behavior, where f_e is the excitation frequency in Hz.
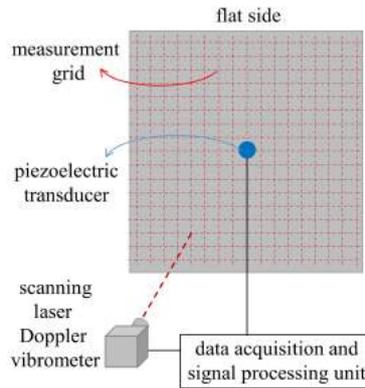

Figure 2. Experimental set-up, where the excitation is imposed by a piezoeletric transducer and the out-of-plane displacement field is measured by the scanning laser Doppler vibrometer.

In the experimental set-up is shown in Fig. 2, the free boundary conditions are emulated by suspending the quasicrystalline plate by nylon strings fixed in a rigid frame. The input excitation is due to a circular piezoelectric transducer (PZT) placed at the center of the flat side of the plate. In this work, only the flexural behavior (i.e., bending waves) is considered and the out-of-plane velocities of a rectangular grid on the flat side of the plate are measured using a scanning laser Doppler vibrometer (SLDV) connected to a data acquisition and signal processing unit.

Due to lack of periodicity, Bloch-Floquet theory cannot be enforced to obtain the dispersion properties. Therefore, an indirect approach based on the time response and its three dimensional Fourier transform (3D-FT) is employed. In both, simulations and experiments, time transient analysis with burst sine excitation are performed, and hence, the displacement field

$$\mathbf{u}_t(x, y, t) \tag{5}$$

obtained. The space-time results are transformed to reciprocal space

$$\hat{\mathbf{U}}_t(k_x, k_y, \omega) \tag{6}$$

(i.e., wave number-frequency domain) by applying the spatial-temporal Fourier transform (3D-FT). However, it leads to a full populated reciprocal space where the wave with low amplitude hide the important wave information. To solve this issue, a filtering procedure is employed to eliminate the low amplitude waves (or noise), which enables to see the wave information in reciprocal space,

$$\tilde{\mathbf{U}}_t(\tilde{k}_x, \tilde{k}_y, \tilde{\omega}) \tag{7}$$

where the *tilde* represents the Fourier points with high amplitude.

4. RESULTS

First, the dispersion results $\tilde{\mathbf{U}}_t(\tilde{k}_x, \tilde{k}_y, \tilde{\omega})$ are shown in Fig. 3 for the 10-fold quasiperiodic plate. Numerical and experimental results, in perfect agreement, show a complex dispersion for the 10-fold quasiperiodic elastic plates, which is related to their fractal nature. The dispersion branches are rotationally disposed in the 10-fold symmetry; moreover, they split in several branches twisted to each other as exemplified in Fig. 3 (d, h), which provides a frequency-dependent high-order wave anisotropy behavior. An almost isotropic wave behavior gives the transition between these twisted bands.
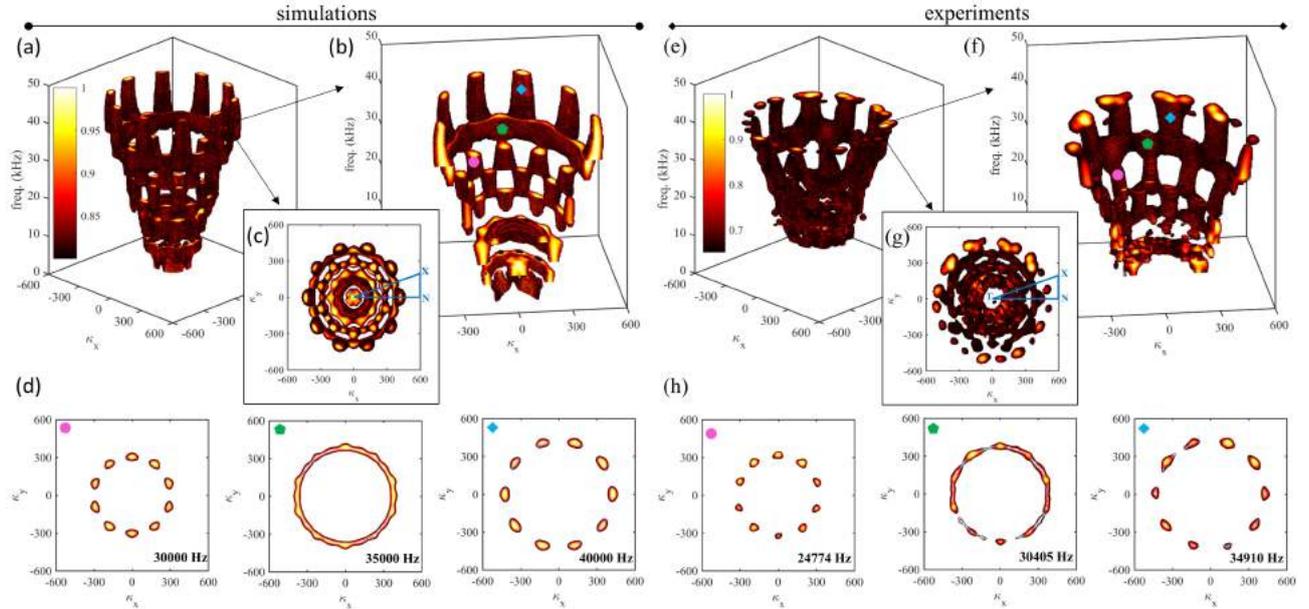

Figure 3. Dispersion properties obtained by numerical simulations (a-d) and experiments (e-h).

Next, the time response for specific frequencies and their wave number contours are shown in Fig. 4. A clear wave directionality is observed in Fig. 4 (a,c), but their are twisted by $\theta_N/2 = 18°$ in relation to each other as predicted by the dispersion results, which reinforce the frequency-dependent beaming behavior. Moreover, the transition is dominated by almost isotropic wave number contours 4 (b,d). These results show that wave beaming can also be created in fold symmetries not allowed in the crystallographic symmetries.
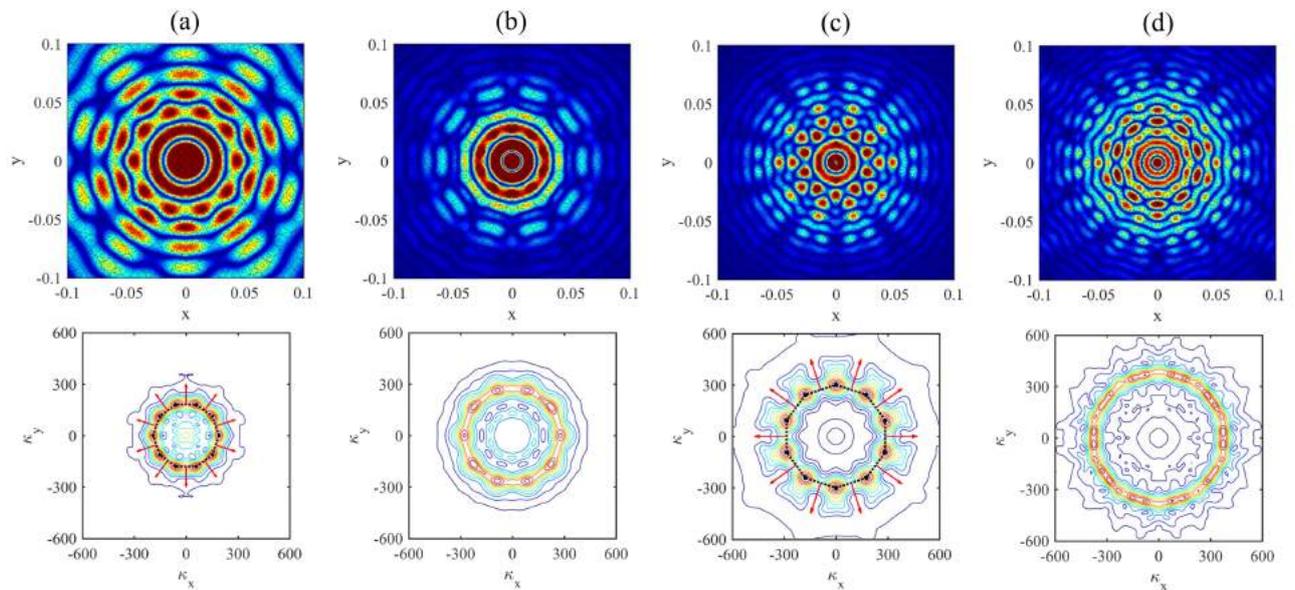

Figure 4. Wave beaming. Time response (top line) and spectral content (bottom line) at 9.3 kHz (a), 22 kHz (b) , 30 kHz (c) and 35 kHz (d).

Finally, the high-order diffraction behavior of these quasiperiodic plates is investigated. Half of the quasicrystal

domain extended on y-direction and combined to a uniform plate (a line excitation source is centered at its bottom) as well as low-reflection boundary conditions are considered. The incident wave reaches the quasicrystal at its center. Two configurations with twisted diffraction pattern are used and the frequency of the first wave beaming is considered. Similar to the wave beaming in Fig. 4(a), the diffraction pattern follows the quasicrystal fold symmetry as shown in Fig. 5. This concept can be used to split the wave front in a high-order fold symmetry, which is not possible by using architected materials with the traditional crystallographic symmetries.
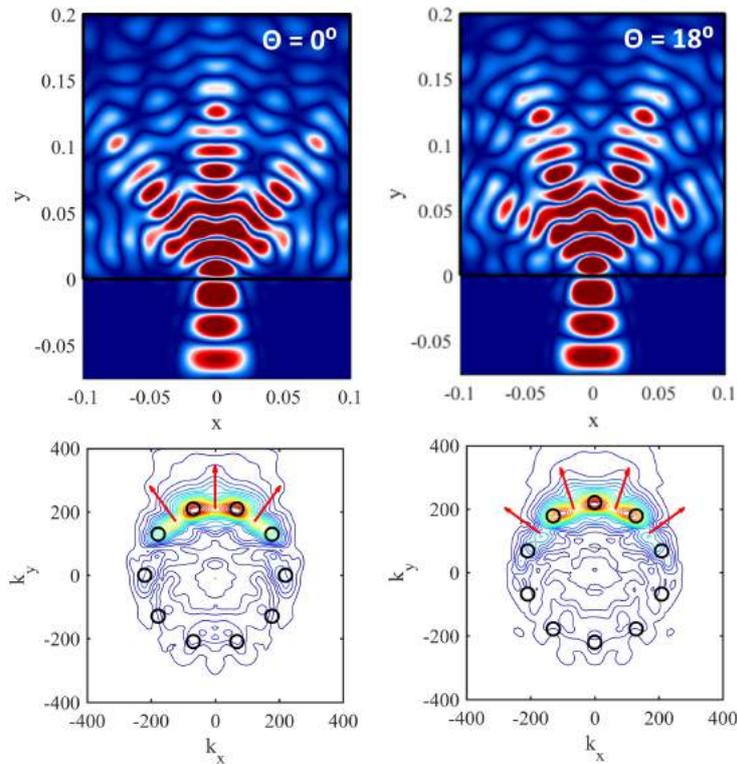

Figure 5. Wave diffraction at 9.3 kHz with the diffraction pattern of the 10-fold quasicrystal aligned (a) and misaligned (b) to the incident wave.

5. CONCLUSIONS

Wave beaming and diffraction have been reported only in the crystallographic fold symmetries (2-, 4- and 6-fold). This work, therefore, expands the wave beaming and diffraction to other fold symmetries (e.g., 10-fold) and stimulates a variety of applications involving this unusual wave directionality such as superior guiding, focusing, sensing and imaging.

6. ACKNOWLEDGEMENTS

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