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Unsteady time-domain analysis of finite wings using the lifting-line theory via a finite element approach

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Abstract. *Over the past few years, aeronautical entities have been dedicating themselves to reduce the carbon footprint of aircraft operations, production, and maintenance. Among the advances in design concepts, it is possible to highlight some technologies that achieved drag and weight reductions. The combination of high aspect ratio wings with advanced materials used on modern aircraft leads to structural systems susceptible to large displacements when subjected to aerodyamic loads. This behavior must be considered in the aircraft dynamic response and in its performance evaluations. The analysis of these complex phenomena in the time domain requires an unsteady aerodynamic model to predict spanwise wing loading history. Therefore, this article aims to present a method to compute the unsteady lift and pitch moment of a rigid thin finite wing undergoing step or oscillatory motions in plunge or pitch. Since Wagner's function describes the airfoil's unsteady behavior, its solution can be extended to finite wings when combined with the lifting-line theory. This combination leads to a second kind Volterra-Fredholm integrodifferential equation with a singular kernel. Herein, the model is written on its weak form and is applied an approach based on the finite element method. Hence, the problem is described by a set of equations that one can differentiate with respect to time using the Leibniz integral rule yielding an initial value problem. Runge-Kutta algorithm performs the time integration. A Python routine was implemented using an object-oriented paradigm. The developed code is applied to investigate the aerodynamic behavior of a finite wing for four cases: (i) a step in pitch, (ii) a step in plunge; (iii) an oscillating pitch motion; and (iv) an oscillating plunge motion. The results are compared with reported literature data presenting a good agreement.*

Keywords: *unsteady aerodynamics, lifting-line, Wagner function, finite element method*

1. INTRODUCTION

The transient response of aerodynamic surfaces may be evaluated with either time or frequency domain analysis. For some specific geometries, as elliptical wings, it is possible to obtain a closed-form solution of the aerodynamic loads, as presented by Jones (1939). However, arbitrary geometries require more sophisticated models, such as the Doublet-Lattice Method (DLM) (Giesing *et al.*, 1972; Albano and Rodden, 1969; Hedman, 1966), the Vortex-Lattice Method (VLM) (Murua *et al.*, 2012b), and lifting-line theory based models (Boutet and Dimitriadis, 2018).

The DLM provides discrete solutions in the frequency domain of load distribution with a modal approach (Murua *et al.*, 2012a). Thus, a continuous frequency solution may be obtained by several interpolated frequency responses (Vepa, 1977; Roger, 1977; Edwards, 1977), which consumes a lot of computational resources when it comes to performing repetitive aeroelastic analysis. Karpel (1982) developed a minimum state approach that reduces those computational costs, but the continuous solution remains an approximation. Another limitation of DLM is that its formulation assumes small out-of-plane motions, which restricts its applicability. In this regard, an unsteady Vortex-Lattice Method (UVLM) is suitable for applications that require the capture of large structural displacements. Unlike DLM, UVLM constitutes a time-marching scheme, and it has been recently applied to several problems, such as flutter suppression (Hall *et al.*, 2001), morphing wings (Obradovic and Subbarao, 2011), and coupled aeroelasticity and flight dynamic analysis (Murua *et al.*, 2012b; Hang *et al.*, 2020). The theory and numerical implementation of the UVLM is extensively discussed in Katz and Plotkin (2001).

An alternative model for the analysis of a finite wing in the time domain is proposed by Boutet and Dimitriadis (2018). For the unsteady aerodynamic model, the authors used the Wagner function combined with the lifting-line theory, describing the circulation distribution from a Fourier series. Usually, the lifting-line models are solved using a description of the circulation field based on a Fourier series (Anderson Jr, 2010). When it comes to an aeroelastic coupling, the difference between the approximating functions makes the interpolation between the structural domain (polynomial nature), and aerodynamic one (trigonometric functions) necessary. In order to bring together the approaches used to solve both problems, the research group of the Department of Aeronautical Engineering at the São Carlos School of Engineering has been working in recent years on the development of a solution approach via Galerkin Method for the aerodynamic problem (Liorbano *et al.*, 2017; Liorbano, 2019; Pelegrineli *et al.*, 2017; Pelegrineli, 2019). Regarding this approach, displacements, rotations, and circulation fields are described by the same basis functions. The model developed by the research group was validated for stationary aerodynamic cases of planar and nonplanar wings with data from the literature (Liorbano *et al.*, 2017). Therefore, the present article presents a time domain, 3D unsteady aerodynamic model based on the Wagner function and the Prandtl's lifting-line theory solved via a finite element method approach.

2. UNSTEADY LIFTING-LINE MODEL

Based on the potential theory, Wagner (1925) developed a method that describes the indicial built-up of the circulatory lift, i.e., the circulation due to a unitary change in the angle of attack. Thus, one can obtain an expression for the time-dependent sectional circulatory lift coefficient of a wing section submitted to a step-change $\Delta w << U_\infty$ in downwash, so that:

$$c_\ell^c(t,y) = a_0(y)\,\Phi(t)\frac{\Delta w}{U_\infty} \tag{1}$$

where y is the position of the wing section, $a_0(y)$ is the lift curve slope of the local airfoil, U_∞ is the free stream airspeed and $\Phi(t)$ is Wagner's function, approximated by Jones (1939) as:

$$\Phi(t) = 1 - \Psi_1 \exp\left(-\frac{2\,\epsilon_1\,U_\infty}{c}\,t\right) - \Psi_2 \exp\left(-\frac{2\,\epsilon_2\,U_\infty}{c}\,t\right) \tag{2}$$

with $\Psi_1 = 0.165$, $\Psi_2 = 0.335$, $\epsilon_1 = 0.0455$ and $\epsilon_1 = 0.3$.

Duhamel's principle (Chopra, 2012) can be applied in Eq. (1) to obtain a continuous response through the time integral of infinitesimal step responses. In addition, applying integration by parts yields:

$$c_\ell^c(t,y) = a_0(y)\left[\frac{w(t,y)}{U_\infty}\,\Phi(0) - \int_0^t \frac{1}{U_\infty}\frac{\partial\Phi(t-\tau)}{\partial\tau} w(\tau,y)\,d\tau\right] \tag{3}$$

Herein, the downwash is decomposed into three-dimensional flow component and motion components. In this article, it is assumed a rigid wing with local pitch, $\alpha(t,y)$, and plunge, $h(t,y)$ (Fig. 1). The downwash components that depend on the wing kinematics are provided by Katz and Plotkin (2001) via the development of the low amplitude oscillatory motion equations of thin airfoils, so that the resulting downwash is described as:

$$w(t,y) = w_y(t,y) + w_k(t,y) = w_y(t,y) + U_\infty\,\alpha(t,y) + \dot{h}(t,y) + \dot{\alpha}(t,y)\,d \tag{4}$$

where $d = c/4 - x_e$, with x_e being the distance between the pitch axis and the mid-chord, as defined by Theodorsen (1935). The variable $w_k(t,y)$ describes the downwash due to the wing kinematics. It is important to note that x_e is negative when the pitch axis is upstream of the mid-chord. The first term, $w_y(t,y)$, is described here with the use of Prandtl's lifting-line theory (Prandtl, 1923), such that the downwash due to three-dimensional flow effects in a section of a planar wing is given by:

$$w_y = -\frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{\mathrm{d}\Gamma/\mathrm{d}y_0}{y-y_0}\,\mathrm{d}y_0 \tag{5}$$

where b is the wing span.

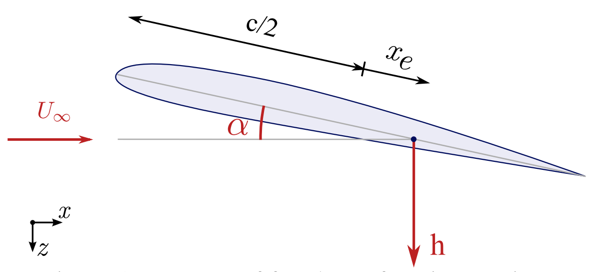


Figure 1: Degrees of freedom of a wing section.

Furthermore, an unsteady Kutta-Joukowski equation is obtained from the unsteady Bernoulli equation, as detailed by Katz and Plotkin (2001), such that the unsteady lift coefficient of a wing section must satisfy the following:

$$c_\ell^c(t,y) = \frac{2\,\Gamma}{U_\infty c(y)} + \frac{2\,\dot{\Gamma}}{U_\infty^2} \tag{6}$$

Introducing Eq. (4) into Eq. (3) and replacing it in Eq. (6) yields an integrodifferential Fredholm-Volterra equation of the second kind that relates the circulation distribution, $\Gamma(t,y)$, and its time rate, $\dot{\Gamma}(t,y)$, with the degrees of freedom $\alpha(t,y)$ and $h(t,y)$. Hence:

$$\frac{2\,\Gamma}{a_0(y)\,c(y)}+\frac{2\,\dot{\Gamma}}{U_\infty\,a_0(y)}=\left[w_y(t,y)+w_k(t,y)\right]\Phi(0)\,+\int_0^t\frac{\partial\Phi(t-\tau)}{\partial\tau}\left[w_y(\tau,y)+w_k(t,y)\right]\,d\tau \tag{7}$$

In this manner, the evaluated circulation $\Gamma(t,y)$ is introduced in the Eq. 6 to compute the circulatory lift coefficient of a wing section. The circulatory moment coefficient of this section around the pitch axis, c_m^c, is given by:

$$c_m^c(t,y)=\frac{c(y)/4+x_e(y)}{c(y)}c_\ell^c(t,y) \tag{8}$$

Moreover, the lift and moment coefficients due to non-circulatory effects, c_ℓ^i and c_m^i respectively, also known as added mass effects, are described by Theodorsen (1935), such that:

$$c_\ell^i(t,y)=\frac{\pi c}{2U_\infty^2}(\ddot{h}-\,x_e\,\ddot{\alpha})+\frac{\pi c}{2U_\infty}\dot{\alpha} \tag{9}$$

$$c_m^i(t,y)=\frac{\pi c}{4U_\infty^2}\left[\frac{2x_e}{c}\ddot{h}-\left(\left(\frac{2x_e}{c}\right)^2+\frac{1}{8}\right)\frac{c}{2}\ddot{\alpha}\right]-\left(\frac{1}{2}-\frac{2x_e}{c}\right)\frac{\pi c}{4U_\infty}\dot{\alpha} \tag{10}$$

Thus, the circulatory lift and moment coefficients of the entire wing are then obtained via an integration over the wing domain of the sum of circulatory and non-circulatory sectional components.

3. NUMERICAL APPROACH

In order to apply the Galerkin's method to solve the unsteady aerodynamic case, Eq. (7) is rewritten in its weak form:

$$\begin{aligned}\int_{-b/2}^{b/2}\delta\Gamma\,\frac{2\,\Gamma}{a_0(y)\,c(y)}\,dy+\int_{-b/2}^{b/2}\delta\Gamma\,\frac{2\,\dot{\Gamma}}{U_\infty\,a_0(y)}\,dy=-\frac{\Phi(0)}{4\pi}\int_{-b/2}^{b/2}\int_{-b/2}^{b/2}\frac{\delta\Gamma}{y-y_0}\frac{d\Gamma(t,y_0)}{dy_0}\,dy_0\,dy-\\ -\int_0^t\frac{1}{4\pi}\frac{\partial\Phi(t-\tau)}{\partial\tau}\int_{-b/2}^{b/2}\int_{-b/2}^{b/2}\frac{\delta\Gamma}{y-y_0}\frac{d\Gamma(t,y_0)}{dy_0}\,dy_0\,dy\,d\tau\,+\int_{-b/2}^{b/2}\delta\Gamma\,w_k(t,y)\Phi(0)\,dy+\\ +\int_{-b/2}^{b/2}\delta\Gamma\int_0^t\frac{\partial\Phi(t-\tau)}{\partial\tau}w_k(t,y)\,d\tau\,dy\end{aligned} \tag{11}$$

where $\delta\Gamma$ is an arbitrary weight function homogeneous on domain boundaries, i.e. $\delta\Gamma(-b/2)=\delta\Gamma(b/2)=0$.

Towards a finite element method implementation, the domain $\Omega=[-b/2,\,b/2]$ can be partitioned into subdomains Ω_e, such that $\Omega_1\cup\Omega_2...\cup\Omega_e\cup...\Omega_n=\Omega$ and $\Omega_1\cap\Omega_2...\cap\Omega_e\cap...\Omega_n=\emptyset$, with $e=1,...,n$. Therefore, integrals over Ω_e are rewritten as a sum of integrals over the subdomains Ω_e. So, the Eq. (11) becomes:

$$\begin{aligned}\sum_{e=1}^{N}\int_{\Omega_e}\delta\Gamma^{(e)}\,\frac{2\,\Gamma^{(e)}}{a_0(y)\,c(y)}\,dy+\sum_{e=1}^{N}\int_{\Omega_e}\delta\Gamma^{(e)}\,\frac{2\,\dot{\Gamma}^{(e)}}{U_\infty\,a_0(y)}\,dy=\\ -\frac{\Phi(0)}{4\pi}\sum_{e=1}^{N}\int_{\Omega_e}\sum_{k=1}^{N}\int_{\Omega_k}\frac{\delta\Gamma^{(e)}}{y-y_0}\frac{d\Gamma^{(k)}(t,y_0)}{dy_0}\,dy_0\,dy-\\ -\int_0^t\frac{1}{4\pi}\frac{\partial\Phi(t-\tau)}{\partial\tau}\sum_{e=1}^{N}\int_{\Omega_e}\sum_{k=1}^{N}\int_{\Omega_e}\frac{\delta\Gamma^{(e)}}{y-y_0}\frac{d\Gamma^{(k)}(t,y_0)}{dy_0}\,dy_0\,dy\,d\tau+\\ +\sum_{e=1}^{N}\int_{\Omega_e}\delta\Gamma^{(e)}\,w_k(t,y)\,\Phi(0)\,dy\,+\sum_{e=1}^{N}\int_{\Omega_e}\delta\Gamma^{(e)}\int_0^t\frac{\partial\Phi(t-\tau)}{\partial\tau}w_k(t,y)\,d\tau\,dy\end{aligned} \tag{12}$$

A solution approximation is obtained by describing the circulation distribution and the weight function via p orthogonal functions denoted by $\mathbf{N}^{(e)}(y)=[N_1(y),...,N_p(y)]$, so that:

$$\Gamma^{(e)}(t,y)=\mathbf{N}^{(e)}(y)\;\boldsymbol{\Gamma}^{(e)}(t)\quad\text{and}\quad\delta\Gamma^{(e)}(y)=\mathbf{N}^{(e)}(y)\;\delta\boldsymbol{\Gamma}^{(e)} \tag{13}$$

where $\mathbf{\Gamma}^{(e)}$ and $\delta\mathbf{\Gamma}^{(e)}$ are local circulation and weight function vectors. As a characteristic of the Galerkin method, the same basis functions $\mathbf{N}^{(e)}$ is used for the circulation and weight functions. Furthermore, the local degree of freedom can be expressed with respect to global ones using a gathering matrix $\mathbf{L}^{(e)}$, in such a way that:

$$\mathbf{\Gamma}^{(e)} = \mathbf{L}^{(e)}\,\mathbf{\Gamma} \quad \text{and} \quad \delta\mathbf{\Gamma}^{(e)} = \mathbf{L}^{(e)}\,\delta\mathbf{\Gamma} \tag{14}$$

Therefore, Eq. (12) is rewritten as:

$$\begin{aligned}
\delta\mathbf{\Gamma}^T \left(\sum_{e=1}^{N} \mathbf{L}^{(e)^T} \int_{\Omega_e} \frac{2}{a_0(y)\,c(y)}\,\mathbf{N}^T\,\mathbf{N}\,dy\;\;\mathbf{L}^{(e)} \right) \mathbf{\Gamma} + \delta\mathbf{\Gamma}^T \left(\sum_{e=1}^{N} \mathbf{L}^{(e)^T} \int_{\Omega_e} \frac{2}{U_\infty\,a_0(y)}\,\mathbf{N}^T\,\mathbf{N}\,dy\;\mathbf{L}^{(e)} \right) \dot{\mathbf{\Gamma}} = \\
= -\,\delta\mathbf{\Gamma}^T\,\frac{\Phi(0)}{4\pi} \left(\sum_{e=1}^{N}\sum_{k=1}^{N} \mathbf{L}^{(e)^T} \int_{\Omega_e}\int_{\Omega_k} \frac{\mathbf{N}^T}{y-y_0}\,\frac{d\mathbf{N}(y_0)}{dy_0}\,dy_0\,dy\,\mathbf{L}^{(k)} \right) \mathbf{\Gamma} - \\
-\,\delta\mathbf{\Gamma}^T \int_0^t \frac{1}{4\pi}\,\frac{\partial\Phi(t-\tau)}{\partial\tau} \left(\sum_{e=1}^{N}\sum_{k=1}^{N} \mathbf{L}^{(e)^T} \int_{\Omega_e}\int_{\Omega_e} \frac{\mathbf{N}^T}{y-y_0}\,\frac{d\mathbf{N}(y_0)}{dy_0}\,dy_0\,dy\;\mathbf{L}^{(k)} \right) \mathbf{\Gamma}(\tau)\;d\tau + \\
+\,\delta\mathbf{\Gamma}^T \left(\sum_{e=1}^{N} \mathbf{L}^{(e)^T} \int_{\Omega_e} \mathbf{N}^T\,w_k(t,y)\,\Phi(0)\,dy \right) + \delta\mathbf{\Gamma}^T \left(\sum_{e=1}^{N} \mathbf{L}^{(e)^T} \int_{\Omega_e} \mathbf{N}^T \int_0^t \frac{\partial\Phi(t-\tau)}{\partial\tau} w_k(t,y)\,d\tau\,dy \right)
\end{aligned} \tag{15}$$

Since the solution must not depend on the choice of the weight function $\delta\mathbf{\Gamma}$, Eq. (15) provides a system of equations that can be expressed in its matrix form, such that:

$$\mathbf{A}\,\dot{\mathbf{\Gamma}} + \Big(\mathbf{M} + \mathbf{K}\,\Phi(0)\Big)\,\mathbf{\Gamma} + \int_0^t \frac{\partial\Phi(t-\tau)}{\partial\tau}\mathbf{K}\,\mathbf{\Gamma}(\tau)\,d\tau = \Phi(0)\,\mathbf{G}(t) + \int_0^t \frac{\partial\Phi(t-\tau)}{\partial\tau}\,\mathbf{G}(\tau)\,d\tau \tag{16}$$

where

$$\mathbf{A} = \sum_{e=1}^{N} \mathbf{L}^{(e)^T} \int_{\Omega_e} \frac{2}{U_\infty\,a_0(y)}\,\mathbf{N}^T\,\mathbf{N}\,dy\;\mathbf{L}^{(e)} \tag{17}$$

$$\mathbf{M} = \sum_{e=1}^{N} \mathbf{L}^{(e)^T} \int_{\Omega_e} \frac{2}{a_0(y)\,c(y)}\,\mathbf{N}^T\,\mathbf{N}\,dy\;\;\mathbf{L}^{(e)} \tag{18}$$

$$\mathbf{K} = \frac{1}{4\pi}\sum_{e=1}^{N}\sum_{k=1}^{N} \mathbf{L}^{(e)^T} \int_{\Omega_e}\int_{\Omega_k} \frac{\mathbf{N}^T}{y-y_0}\,\frac{d\mathbf{N}(y_0)}{dy_0}\,dy_0\,dy\,\mathbf{L}^{(k)} \tag{19}$$

$$\mathbf{G}(t) = \sum_{e=1}^{N} \mathbf{L}^{(e)^T} \int_{\Omega_e} \mathbf{N}^T\,w_k(t,y)\,dy \tag{20}$$

The shape functions $\mathbf{N}$ are polynomials built according to the approximation desired for the circulation into the lifting-element. In particular, linear and quadratic approximations were implemented on the in-house software and are denoted as LL2 and LL3 lifting-elements, respectively. Towards a more accurate global approximation of circulation, the mesh discretization is performed using a geometric progression, so that N is the number of elements and r is the ratio of tip and root element lengths. Because of the higher gradients of the circulation closed to the wingtips, it is recommended a mesh refinement on these regions.

Introducing the Wagner function (Eq. (2)) into Eq. (16):

$$\begin{aligned}
\mathbf{A}\,\dot{\mathbf{\Gamma}} + \Big(\mathbf{M} + \mathbf{K}\,\Phi(0)\Big)\,\mathbf{\Gamma} - \mathbf{K}\left(\frac{2\,\psi_1\,\epsilon_1\,U_\infty}{c}\,\mathbf{z_1} + \frac{2\,\psi_2\,\epsilon_2\,U_\infty}{c}\,\mathbf{z_2} \right) = \\
= \Phi(0)\,\mathbf{G}(t) + \left(\frac{2\,\psi_1\,\epsilon_1\,U_\infty}{c}\,\mathbf{z_3} + \frac{2\,\psi_2\,\epsilon_2\,U_\infty}{c}\,\mathbf{z_4} \right)
\end{aligned} \tag{21}$$

where

$$\mathbf{z_1} = \int_0^t e^{\frac{-2\,\epsilon_1\,U_\infty}{c}(t-\tau)}\,\mathbf{\Gamma}(\tau)\,d\tau; \qquad \mathbf{z_2} = \int_0^t e^{\frac{-2\,\epsilon_2\,U_\infty}{c}(t-\tau)}\,\mathbf{\Gamma}(\tau)\,d\tau \tag{22}$$

$$\mathbf{z_3} = \int_0^t e^{\frac{-2\,\epsilon_1\,U_\infty}{c}(t-\tau)}\,\mathbf{G}(\tau)\,d\tau; \qquad \mathbf{z_4} = \int_0^t e^{\frac{-2\,\epsilon_2\,U_\infty}{c}(t-\tau)}\,\mathbf{G}(\tau)\,d\tau$$

The unknown variables $\mathbf{z_i}$ (i=$1,\cdots,4$) are named aerodynamic states (Boutet and Dimitriadis, 2018). Making use of Leibniz's integral rule in Eq. (22) yields the following ordinary differential equations:

$$\dot{\mathbf{z}}_\mathbf{1} = \mathbf{\Gamma}(t) - \frac{2\,\epsilon_1\,U_\infty}{c}\mathbf{z_1}; \qquad \dot{\mathbf{z}}_\mathbf{2} = \mathbf{\Gamma}(t) - \frac{2\,\epsilon_2\,U_\infty}{c}\mathbf{z_2} \tag{23}$$

$$\dot{\mathbf{z}}_\mathbf{3} = \mathbf{G}(t) - \frac{2\,\epsilon_1\,U_\infty}{c}\mathbf{z_3}; \qquad \dot{\mathbf{z}}_\mathbf{4} = \mathbf{G}(t) - \frac{2\,\epsilon_2\,U_\infty}{c}\mathbf{z_4}$$

Hence, Eqs. (21) and (23) compose a system of ordinary differential equations given by:

$$\dot{\mathbf{y}} = \mathbf{A_{rk}}\,\mathbf{y} + \mathbf{g_{rk}} \tag{24}$$

where:

$$\mathbf{y} = \begin{bmatrix}\mathbf{\Gamma} & \mathbf{z_1} & \mathbf{z_2} & \mathbf{z_3} & \mathbf{z_4}\end{bmatrix}^T \tag{25}$$

$$\mathbf{A}_{rk} = \begin{bmatrix} \mathbf{A}^{-1}[-\mathbf{M} - \mathbf{K}\Phi(0)] & \mathbf{A}^{-1}\mathbf{K}\zeta_1 & \mathbf{A}^{-1}\mathbf{K}\zeta_2 & \mathbf{A}^{-1}\zeta_1 & \mathbf{A}^{-1}\zeta_2 \\ \mathbf{I} & -\zeta_3\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & -\zeta_4\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\zeta_3\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\zeta_4\mathbf{I} \end{bmatrix} \tag{26}$$

$$\mathbf{g}_{rk} = \begin{bmatrix}\mathbf{A}^{-1}\,\mathbf{G}(t)\,\Phi(0) & \mathbf{0} & \mathbf{0} & \mathbf{G}(t) & \mathbf{G}(t)\end{bmatrix}^T \tag{27}$$

and:

$$\zeta_1 = \frac{2\,\psi_1\,\epsilon_1\,U_\infty}{c}; \qquad \zeta_2 = \frac{2\,\psi_2\,\epsilon_2\,U_\infty}{c}; \qquad \zeta_3 = \frac{2\,\epsilon_1\,U_\infty}{c}; \qquad \zeta_4 = \frac{2\,\epsilon_2\,U_\infty}{c} \tag{28}$$

Therefore, Eq. (24) represents an initial value problem and its solution is here obtained via a 4th order Runge-Kutta method.

4. EXAMPLES

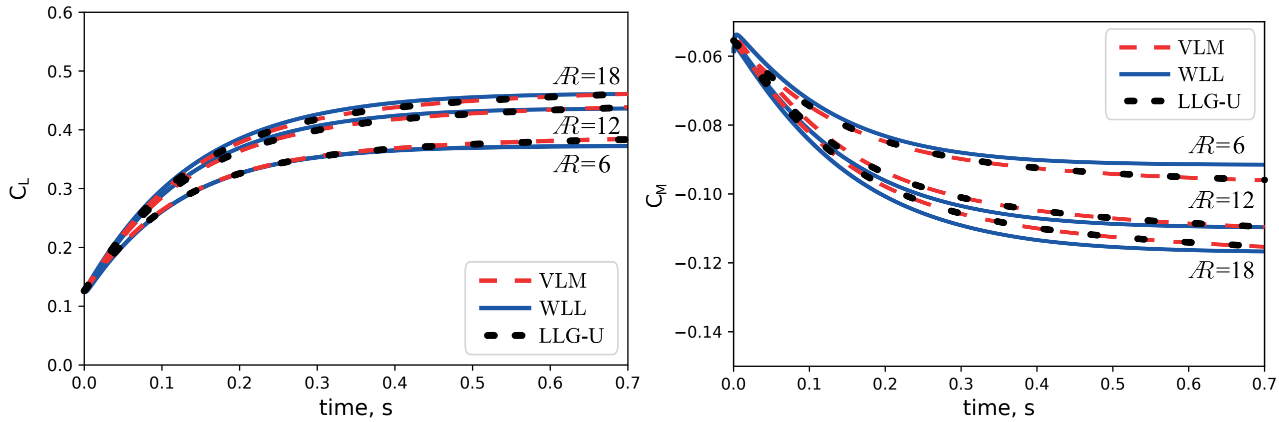
The unsteady aerodynamic model based on the lifting-line theory and Galerkin's method was implemented in a Python routine using an object-oriented paradigm. This model is referenced as LLG-U in Sec. 5. and is applied here to rectangular wings with unitary chords and no geometric or aerodynamic twist. The input function, f, of pitch or plunge was modeled as $f_s = A(1 - e^{-10t})$ and $f_h = A\cos(\frac{2U_\infty k}{c}t)$, for step and oscillatory motion, respectively, where A is the amplitude of motion and k is the reduced frequency.

Hence, the wings were subjected to (i) a step motion in pitch, with $A = 5°$; (ii) a step motion in plunge, with $A = -0.1$ m; (iii) an oscillatory motion in pitch, with $A = 5°$ and $k = 0.1$; (iv) an oscillatory motion in plunge, with $A = -0.1$ m and $k = 0.1$. Aiming to compare with Boutet and Dimitriadis (2018), wings of aspect ratio Æ $= 6$, 12 and 18 were analyzed for step and oscillatory input cases. The mesh parameters were set with a number of elements $N = 70$ and a ratio between tip and root elements lengths of $r = 0.1$. Moreover, the Runge-Kutta's solver was configured with an initial step of 10^{-9}, a maximum step of 10^{-3} and maximum and relative tolerances of 10^{-9} and 10^{-7}, respectively.

5. RESULTS

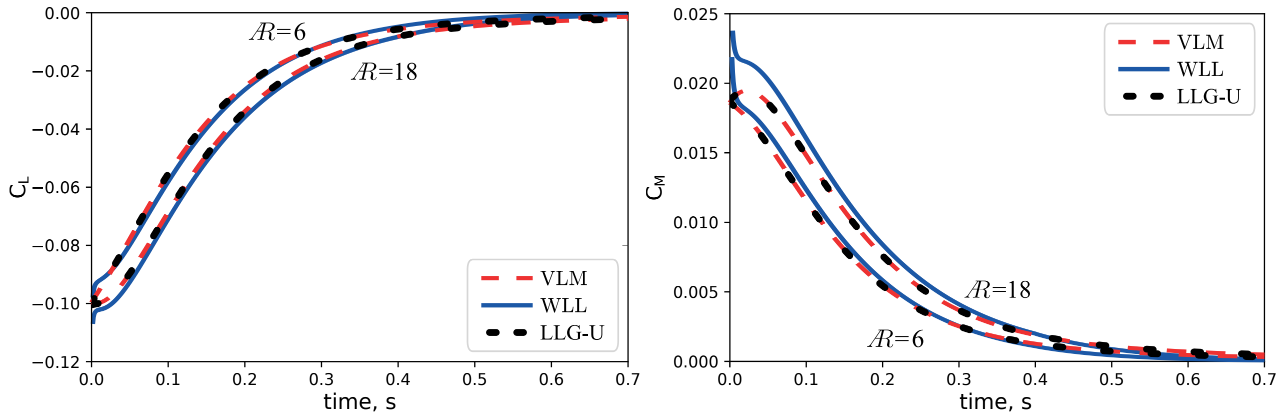
The results obtained from the LLG-U model were compared against those from VLM (Vortex Lattice Method) and WLL (Wagner Lifting-Line Method) models, both described and published by Boutet and Dimitriadis (2018). The results for wings with different aspect ratios subjected to step motion in pitch can be seen in Fig. 2. There is a good agreement between LLG-U and VLM for all the analyzed aspect ratios. The agreement holds for the results of a step motion in the plunge degree of freedom (Fig. 3). In contrast, the WLL seems to predict lift and moment coefficients not as good, showing discordance in transient and permanent regions of both cases.

The lift and moment coefficient results for the oscillatory motion in pitch are plotted in Figs. 4a and 4b. The three models showed good agreement, though LLG-U and WLL indicate a high-frequency component of the response in the very first steps of the simulation that is not predicted by VLM. Similarly, the results for oscillatory motion in the plunge degree of freedom are plotted in Figs. 5a and 5b. VLM prediction disagrees with the other two methods only at the transient region. Nevertheless, there is good agreement between LLG-U, VLM, and WLL predictions of the permanent oscillatory response.



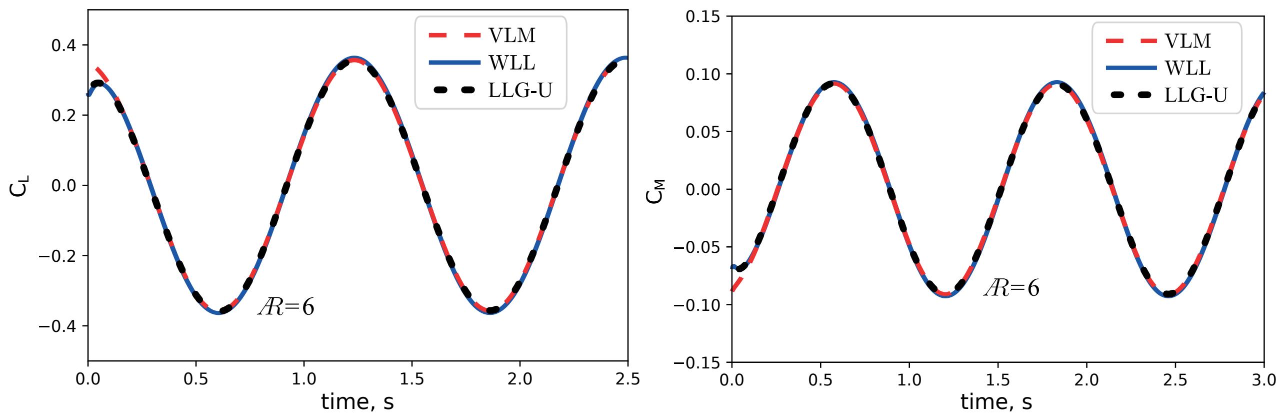
(a) Lift coefficient. (b) Moment coefficient.

Figure 2: Comparison between VLM, WLL and LLG-U (present work) results for rectangular wings of $\mathit{AR} = 6$, $\mathit{AR} = 12$ and $\mathit{AR} = 18$ subjected to step motion in pitch, with $A = 5°$.



(a) Lift coefficient. (b) Moment coefficient.

Figure 3: Comparison between VLM, WLL and LLG-U (present work) results for rectangular wings of $\mathit{AR} = 6$ and $\mathit{AR} = 18$ subjected to step motion in plunge, with $A = -0.1$ m.

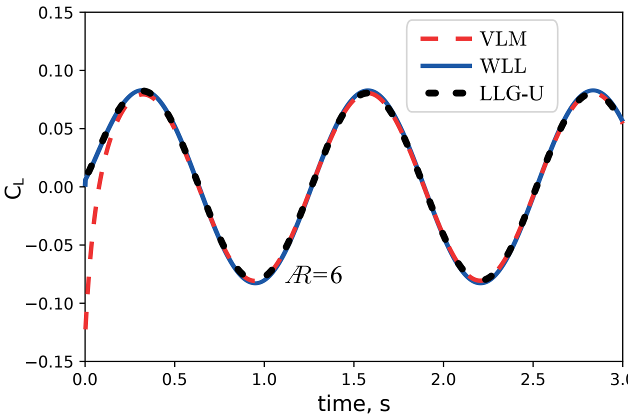


(a) Lift coefficient. (b) Moment coefficient.

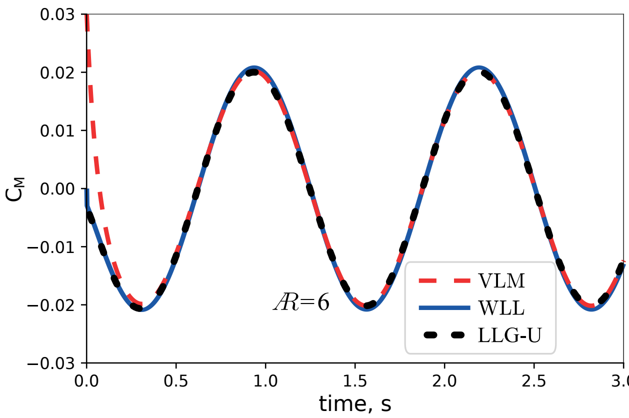
Figure 4: Comparison between VLM, WLL and LLG-U (present work) results for rectangular wing of $\mathit{AR} = 6$ subjected to oscillatory motion in pitch, with $A = 5°$ and $k = 0.1$.

6. CONCLUSIONS

The LLG-U represents a closed-form solution of the unsteady aerodynamic problem of three-dimensional wings under motion in pitch and plunge degrees of freedom under attached incompressible flow conditions. Due to its approach via the finite element method, its final form yields a state-space representation that can be easily coupled with a structural FEM-based model into an aeroelastic one.



(a) Lift coefficient.



(b) Moment coefficient.

Figure 5: Comparison between VLM, WLL and LLG-U (present work) results for rectangular wing of $Æ = 6$ subjected to oscillatory motion in plunge, with $A = -0.1$ m and $k = 0.1$.

The resultant model combines Wagner's function, Prandtl's lifting-line theory, the unsteady Kutta-Joukowski theorem, Galerkin's method, and the added mass terms from Theodorsen's analysis. The time-marching solution was obtained via a Runge-Kutta schema. Simulations of several aspect ratios of wings under step and oscillatory motions were performed. When compared against reported literature data, LLG-U results presented good agreement.

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