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QUARTER-WAVELENGTH RESONATOR METAMATERIAL MODELING VIA WAVE FINITE ELEMENT METHOD

Campos, B. V. L.
Pacheco, F. B.
Dos Santos J. M. C.
State University of Campinas - UNICAMP
brenno.campos@fem.unicamp.br, flaiannypacheco@gmail.com.

Abstract. *In the last couple of years, studies about metamaterials, whose the aim is ensure the occurrence of local resonances to block the acoustics wave to propagate in some frequency range, called bandgaps, have increased. New theoretical and experimental works have been developed due its applicability in waveguides, acoustic filters, cavities and sensors. In this work locally quarter-wavelength resonator mounted in a duct is designed for a certain resonance frequency that can be used in noise reduction problems. The duct-resonator acoustic system is modeled using the Transfer Matrix (TM) method and the Wave Finite Element (WFE) method, which has inherent advantages in solving wave propagation problems and periodic schemes. Dispersion diagram, forced response and sound transmission loss (STL) are calculated and the results are compared between methods and verified by the conventional Finite Element (FE) method.*

Keywords: *quarter-wavelength resonator, metamaterials, acoustic system.*

1. INTRODUCTION

The concern with the control of noise impulses in studies on system applications periodicals such as phononic crystals and acoustic metamaterials (Xiao et al., 2011). This noise can be defined as any unwanted sound that interferes with the fluency of an activity. By this reason, over the years, ways have been sought to alleviate noise through development theoretical and experimental studies of acoustic systems. In the last two decades, the new concepts of metamaterials and phononic crystals, opened the possibility of applying the effect of bandgaps as a tool to help reduce them (Farooqui et al., 2016).

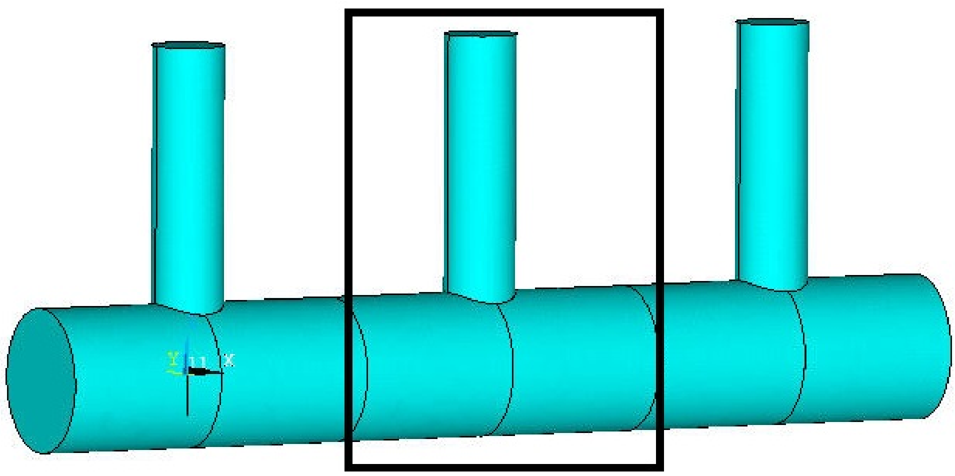

Figure 1. Periodic duct with quarter-wavelength resonator.

Among the characteristics of acoustic metamaterials, which are structures composed of a set of spatially distributed resonators (periodic or not), there is the effect of local resonance that produces forbidden bands. When distributed periodically throughout the space, these produce forbidden bands of the Bragg type, which are generated by incompatibility of impedance (Bragg effect), which prevent the wave propagation in certain ranges of frequencies (Hussein *et al.*, 2014).

This work investigates the bandgaps created in the duct-side pipe periodic system using the Transfer Matrix Method

Table 1. Geometric parameters and material properties.

Geometry/Properties	Value
Periodic part length (m)	1.026
Unit-cell length (m)	0.342
Duct diameter (m)	0.15
Tube diameter (m)	0.075
Tube length (m)	0.342
Sound velocity (m/s)	343.24
Air density (kg/m^3)	1.2041

(TMM), which is quite used to evaluate the performance of acoustic resonators (Campos *et al.*, 2019). This method is based on the acoustic formulation of the plane wave. The results obtained with the TMM are verified by the Finite Element Method.

2. THEORETICAL FOUNDATION

The Transfer Matrix Method (TMM), also known as the four-pole method or transmission line method, has been widely used to calculate and optimize transmission and sound absorption of acoustic systems (Munjal, 1987) and (Beranek *et al.*, 1992). The acoustic duct metamaterial with side branching tubes can be modeled by the TMM, as it is based on the formulation of the transfer matrix of a uniform circular duct.

The transfer matrix of a duct can be obtained from a 1D acoustic model. The non-dissipative wave equation can be written as:

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{1}$$

where, p is the acoustic pressure, c is the speed of sound, x is space and t is time. The solution of the equation Eq. (1) can be written as (Kinsler *et al.*, 1999):

$$p(x) = Ae^{-jkx} + Be^{jkx} \tag{2}$$

where, A and B are the complex coefficients to be determined by boundary conditions, j is the imaginary unit, $k = \omega/c$ is the wave number and ω is the angular frequency. Substituting equation Eq. (2) into Euler's equation, $u(x) = -\frac{1}{\rho}\int \frac{\partial p}{\partial x}$, the particle velocity can be obtained as:

$$u(x) = -\frac{1}{\rho c}(Ae^{-jkx} - Be^{jkx}) \tag{3}$$

where ρ is the air density and rewriting as volume velocity we get:

$$U(x) = \frac{S_d}{\rho c}(Ae^{-jkx} - Be^{jkx}) \tag{4}$$

where S_d is the section area of the duct. From equations Eq. (2) and (4) the transfer matrix of a duct with uniform circular section area S_d and length l, we can write it as (Munjal, 2014):

$$\begin{Bmatrix} U_n \\ p_n \end{Bmatrix} = \begin{bmatrix} cos(\hat{k}l) & j\frac{S_d}{\rho c}sin(\hat{k}l) \\ j\frac{\rho c}{S_d}sin(\hat{k}l) & cos(\hat{k}l) \end{bmatrix} \begin{Bmatrix} U_{n-1} \\ p_{n-1} \end{Bmatrix} \tag{5}$$

where $\hat{k} = k(1 + j\eta)$ is the complex wave number that has been formulated to include damping in the system and η is the loss factor. The indices, n and $n-1$, represent the ends of the sides of the duct, inlet and outlet, respectively. And now, considering the duct inlet (source) at $x = 0$ and outlet (open) at $x = L$, and the effective length of the duct, then equation Eq. (5) becomes:

$$\begin{Bmatrix} U_0 \\ p_0 \end{Bmatrix} = \begin{bmatrix} cos(\hat{k}l_e) & j\frac{S_d}{\rho c}sin(\hat{k}l_e) \\ j\frac{\rho c}{S_d}sin(\hat{k}l_e) & cos(\hat{k}l_e) \end{bmatrix} \begin{Bmatrix} U_L \\ p_L \end{Bmatrix} \tag{6}$$

The effective length is $l_e = L + 0.6a$, which is the sum of the physical duct length, L, and the final correction factor, $0.6a$. This final correction factor is for a duct with an unadjusted open end, where a is the radius of the duct.

The method consists of relating the acoustic pressure and the speed of the particle (or mass velocity, or velocity of volume) for two states of the quarter-wavelength resonator. The TMM assumes the hypothesis of plane wave propagation in the duct-tube system, which limits the range of frequencies of application of the method (Campos *et al.*, 2020).

$$\mathbf{T}\mathbf{q} = e^{\mu}\mathbf{q} \tag{7}$$

where $\mathbf{T}$ is the transfer matrix, $\mathbf{q}$ is the state vector, e is the exponential function and d the unit-cell length.t. Applying periodicity condition Eq. (7) can be rewritten in a more convenient form as:

$$\mathbf{q}_r = \mathbf{T}\mathbf{q}_L \tag{8}$$

where $\mathbf{q}_r$ and $\mathbf{q}_L$ now are the state vector at the right and left position of the unit-cell, respectively. The volume velocity and acoustic pressure compatibility and continuity condition produces ${\mathbf{q}_r}^{(n)} = {\mathbf{q}_L}^{(n+1)}$. By substituting in Eq. (8), produces:

$${\mathbf{q}_L}^{(n+1)} = \mathbf{T}{\mathbf{q}_r}^{(n)} \tag{9}$$

the Floquet-Bloch theorem for wave propagation in an infinite periodic system applied to consecutive unit-cells, generates:

$${\mathbf{q}_L}^{(n+1)} = e^{\mu}{\mathbf{q}_L}^{(n)} \tag{10}$$

substituting Eq. (9) in Eq. (10) and rearranging, it has:

$$\mathbf{T}\mathbf{q}_L = e^{\mu}\mathbf{q}_L \tag{11}$$

which is the Bloch wave eigenvalue/vector problem, where e^{μ} is the Bloch wave number and $\mathbf{q}_L$ are the corresponding wave vectors.

By applying continuity and compatibility conditions for the volume velocity and the acoustic pressure in the presented model we obtain:

$$U_1(x) = U_{QH}(x) + U_2(x) \tag{12}$$

$$p_1(x) = p_{QH}(x) = p_2(x) \tag{13}$$

Rewriting equations Eq. (12) and Eq. (13) in matrix form we have:

$$\left\{ \begin{array}{c} U_1 \\ p_1 \end{array} \right\} = \left[\begin{array}{c} U_{QH} + U_L \\ p_L \end{array} \right] \tag{14}$$

.

$$\left\{ \begin{array}{c} U_1 \\ p_1 \end{array} \right\} = \left[\begin{array}{cc} 1 & U_{QH}/U_L \\ 0 & 1 \end{array} \right] \left\{ \begin{array}{c} U_L \\ p_L \end{array} \right\} \tag{15}$$

from Eq. (14), $p_L(x) = p_{QH}(x)$, then we have, $\frac{p_{QH}}{U_{HR}} = Z_{QH}$ which is the acoustic impedance at the quarter-wavelength resonator. Then, also from Eq. (14) and (15) it can be rewritten as:

$$\left\{ \begin{array}{c} U_0 \\ p_0 \end{array} \right\} = \left[\begin{array}{cc} 1 & 1/Z_{QR} \\ 0 & 1 \end{array} \right] \left\{ \begin{array}{c} U_L \\ p_L \end{array} \right\} \tag{16}$$

Equation (16) can be rewritten in compact form as:

$$\mathbf{q}_0 = \mathbf{T}_{QH}\mathbf{q}_L \tag{17}$$

where $\mathbf{T}_{QR}$ is the transfer matrix of the quarter-wavelength resonator and $\mathbf{q}_0$ and $\mathbf{q}_L$ are the state vectors at position $x = 0$ and L, respectively. Now, the two elements, the duct and the quarter wavelength resonator, can be put together to obtain the transfer matrix of the unit cell of the acoustic system of each element which is given as:

$$\mathbf{q}_0 = \mathbf{T}_{duct}\mathbf{T}_{QH}\mathbf{q}_L \qquad or \qquad \mathbf{q}_0 = \mathbf{T}_{DQH}\mathbf{q}_L \tag{18}$$

where $\mathbf{T}_{DqH}$ is the transfer matrix of the unit cell of the quarter-wavelength resonator system. Before applying the periodicity condition Eq. (18) can be more conveniently rewritten as:

$$\mathbf{q}_r = \mathbf{T}_{DQH}\mathbf{q}_L \tag{19}$$

where $\mathbf{q}_r$ and $\mathbf{q}_L$ are now the state vector at the right and left position of the unit cell, respectively. Consider consecutive unit cells, n and $n+1$, in the structure. The condition of compatibility and continuity of the volume velocity and acoustic pressure yields $q_r{}^{(n)} = q_L{}^{(n+1)}$. Substituting into Eq. (19) yields:

$$\mathbf{q}_L{}^{(n+1)} = \mathbf{T}_{DQH}\mathbf{q}_L{}^{(n)} \tag{20}$$

The Floquet-Bloch theorem for wave propagation applied to an infinite periodic system of infinite unit cells, generates:

$$\mathbf{q}_L{}^{(n+1)} = e^{\mu}\mathbf{q}_L{}^{(n)} \tag{21}$$

where $\mu = -jkL$ is the attenuation constant. And substituting Eq. (21) into Eq. (20) and rearranging, we obtain:

$$\mathbf{T}_{DQH}\mathbf{q}_L{}^{(n+1)} = e^{\mu}\mathbf{q}_L{}^{(n)} \tag{22}$$

where the eigenvalue/autovector problem by the Floquet-Bloch wave theorem, where e^{μ} is the Bloch wave number and e^{q_L} are the corresponding wave vectors.

The equation shown below gives the estimate of the ratio of acoustic pressure p_L at the duct exit to the input volume velocity v_0. Z_L is the radiation impedance of an unflanged open end of the duct. If v_0 is set to unity $(1m^3/s)$, then p_L would be:

$$p_L = \frac{Z_L}{A_L + B_L Z_L} \tag{23}$$

Equation Eq. (23) represent the estimates of acoustic pressure level at the duct exit described by Singh et al (2008).

The Transmission Loss (TL) can be described simply as the difference between the sound power incident before the silencer and the sound power that continues to be transmitted after the silencer. As for Munjal (2014), the TL is not dependent on the source and assumes that the acoustic system has an anechoic ending at its exit end, and can be described symbolically like:

$$TL = 20log_{10}\left|\frac{A + \frac{S_d}{c}B + \frac{c}{S_d}C + D}{2}\right| \tag{24}$$

where S_d is the area of the duct section, c is the sound velocity.

Wave Finite Element Model

WFE has been used for free and forced wave propagation in vibration analysis with applications to one and two-dimensional structural models (Cook *et al.*, 2002; Duhamel *et al.*, 2006; Mace and Maconi, 2008). In this section the finite element method starts from the assembly of the mass $\mathbf{M}_a$, and stiffness $\mathbf{K}_a$ acoustical matrices extracted from ANSYS software. Then, we assemble the dynamic stiffness matrix described by:

$$\mathbf{D} = \mathbf{K}_a - \omega^2\mathbf{M}_a \tag{25}$$

The dynamic acoustic stiffness matrix can be rewritten by partitioning the degrees of internal freedom, right and left. So we have:

$$\begin{bmatrix} \mathbf{D_{ii}} & \mathbf{D_{il}} & \mathbf{D_{ir}} \\ \mathbf{D_{li}} & \mathbf{D_{ll}} & \mathbf{D_{lr}} \\ \mathbf{D_{ri}} & \mathbf{D_{rl}} & \mathbf{D_{rr}} \end{bmatrix} \begin{Bmatrix} \mathbf{p_i} \\ \mathbf{p_l} \\ \mathbf{p_r} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0_i} \\ \mathbf{f_l} \\ \mathbf{f_r} \end{Bmatrix} \tag{26}$$

Applying the periodicity condition in a harmonic perturbation propagating through the duct-tube system (Ichchou *et al*, 2005). By rearranging the Eq. (26) as the transfer matrix the formulation yields:

$$\begin{Bmatrix} \mathbf{p_r} \\ \mathbf{-f_r} \end{Bmatrix} = \begin{bmatrix} -\mathbf{D_{lr}}^{-1}\mathbf{D_{ll}} & -\mathbf{D_{lr}^{-1}} \\ \mathbf{D_{rl}} - \mathbf{D_{rr}}\mathbf{D_{lr}}^{-1}\mathbf{D_{ll}} & -\mathbf{D_{rr}}\mathbf{D_{lr}^{-1}} \end{bmatrix} \begin{Bmatrix} \mathbf{p_l} \\ \mathbf{f_l} \end{Bmatrix} \quad \textbf{or} \quad \mathbf{q}_r = \mathbf{T}_{WFE}\mathbf{q}_l \tag{27}$$

where, $\mathbf{T}_{WFE}$ is transfer matrix obtainded by WFE method.

Applying to the Floquet-Bloch theorem solves the similar eigenvalue/eigenvector problem by the equation:

$$\mathbf{T}_{WFE}\mathbf{q}_L = e^{\mu}\mathbf{q}_L \tag{28}$$

3. RESULTS AND DISCUSSION

Figure 2 present the dispersion diagrams for the quarter-wavelength resonator system, which present the real part (above) and the negative imaginary part (below). Bandgaps can be identified by regions where the real part of the wavenumber is equal to Bragg limit or zero the imaginary part of the wavenumber is non-zero (non-propagative waves). The bandgap appears approximately in the range of 270-310 Hz and can be identified as a locally resonant bandgap.

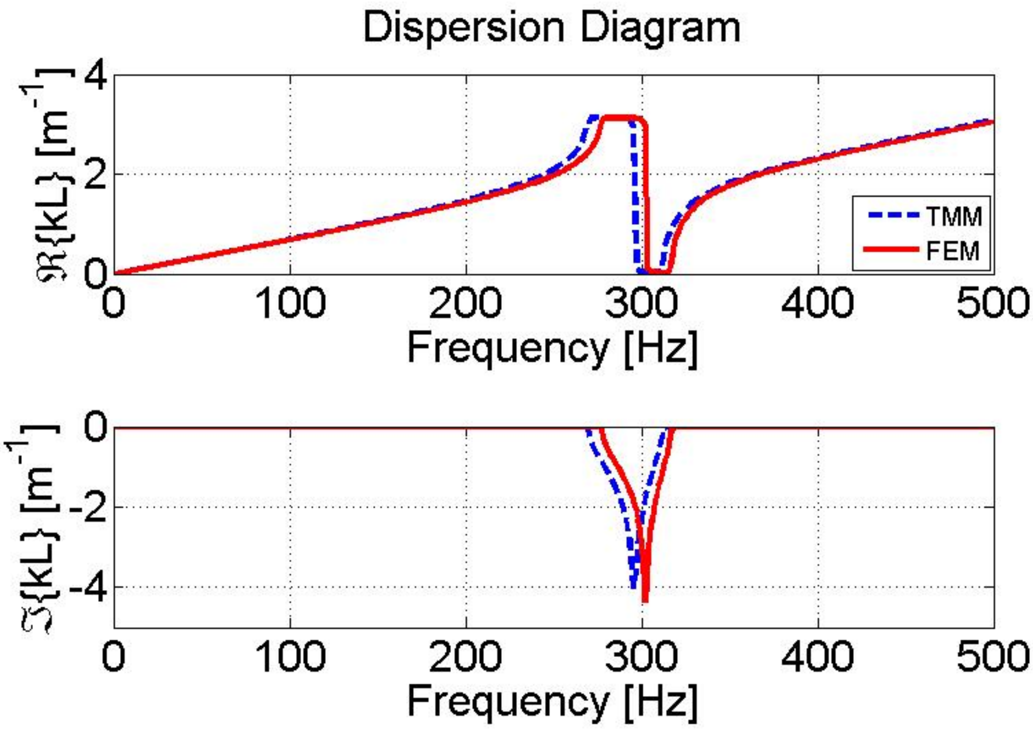

Figure 2. Dispersion diagram of the quarter-wavelength resonator system (real and imaginary parts).

The results show that both the TMM and WFE methods are able to identify bandgaps generated by periodicity (Bragg effect) and locally by quarter-wavelength resonator, the TL calculation points out the efficiency of the acoustic model.

Figure 3 shows the forced response for quarter-wavelength resonator system. It is possible to observe the present intervals between the frequency bands between 270 up to 310 Hz approximately, which agrees with the result found in the dispersion diagram.

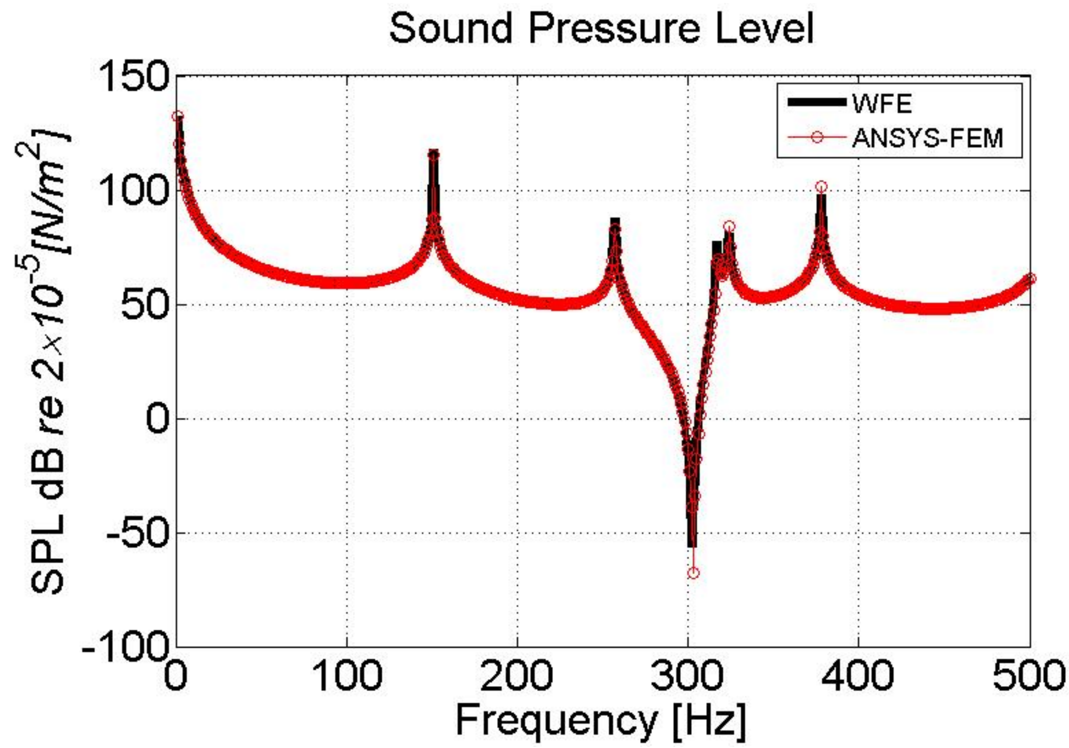

Figure 3. Sound pressure level of the quarter-wavelength resonator system (3 unit-cells).

To measure the efficiency of the analyzed model, it was calculated as an acoustic parameter for comparison purposes the transmission loss of the quarter-wavelength resonator system by the transfer matrix method and finite elements (ANSYS). It can be noted that the responses obtained by both methods significantly converge throughout the analyzed frequency range.

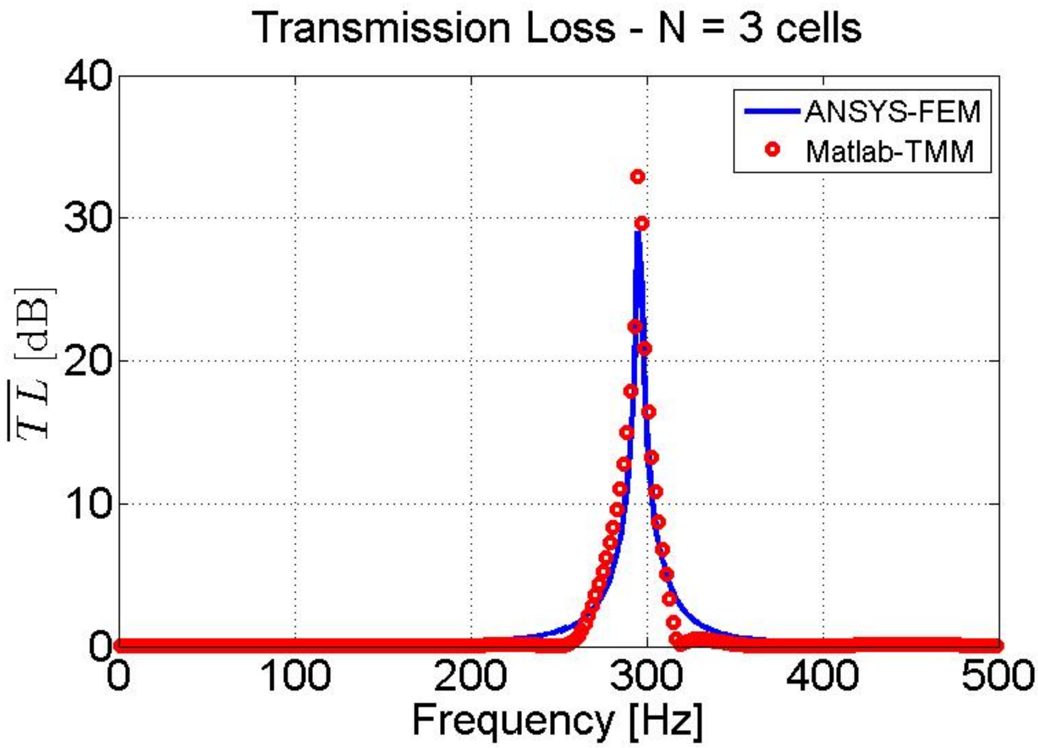

Figure 4. Transmission loss of the quarter-wavelength resonator system.

The methodology used in this paper to determine the sound attenuation in ducts using quarter-wavelength resonator that can be used to estimate and improve noise control in acoustic devices.

An important issue for the analysis of the results obtained by the transfer matrix method was the inclusion of correction factors in addition to the actual dimensions of the quarter-wavelength resonator.

In accordance with the plane wave assumption in all the acoustic formulations evaluated, as the frequency band increases, the results of the dispersion diagram and forced responses can show numerical uncertainties.

For the results obtained from the forced response analysis for three-cell periodic systems using the TMM, WFE and FE methods were very close throughout the evaluated frequency range, which proves that the WFE method is efficient for calculating harmonic responses with much less computational processing.

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