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MODELLING AND THEORETICAL ANALYSIS OF THE FLOW OF A ELECTRICALLY CONDUCTING FLUID IN THE PRESENCE OF HIGH FREQUENCY WAVES

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Abstract. *The aim of this work is to model and analyze a compressible flow of a electrically conducting fluid in the presence of an expansional viscosity (i.e. high frequency flows). The non-dimensional form of the governing equations are obtained in terms of the fundamental physical parameters of the flow. The governing equations represent a coupling between hydrodynamics and Maxwell's equations in the context of magnetohydrodynamics, where the density of free charges is invariably very small in electrically conducting fluids. The set of governing equations are then linearized when considering the hydrodynamics and magnetic variables as the sum of their equilibrium value with an associated fluctuation of small amplitude around it. The resulting linearized equations has oscillatory nature and can be written in the form of a set of equations in the frequency-wave number space (i.e. Fourier space), from which, combined in a particular way, we obtain the dispersion relations for Alfvén and magnetoacoustic waves in the presence of a expansional viscosity effect and in terms of the main physical parameters of the flow. By the present analysis, we show how the dispersion relations of the different types of magnetic waves investigated are influenced by the extensional viscosity at high frequency compressible flows. In addition, the stability of the waves are examined, verifying if the fluctuations around the equilibrium state are stable or unstable disturbances. A rigours study in terms of the growth rate and of the phase velocities in Fourier space is presented as well as a discussion on the effects of the expansional high frequency effects in the growth rate and in the phase velocities of the waves. A model to determine or calibrate material constants of the constitutive relations using experimental measures of the wave properties is also proposed.*

Keywords: *Magnetohydrodynamics, Expansional Viscosity, Linear Stability Analysis, Magnetoacoustic Waves.*

1. INTRODUCTION

According to (Davidson, 2017), Magnetohydrodynamics (MHD) is the study of the interaction between magnetic fields and moving, electrically conducting fluids. Several applications can be cited, like the flow of plasma in astrophysics, plasma containment in fusion reactors, materials processing (Davidson, 1999), flows of salt water and much others well known applications.

Magnetohydrodynamics models the flows of electrically conducting fluids coupling Maxwell equations with hydrodynamics equations (Knaepen and Moreau, 2008). In this context, the density of free charges is invariably very small, given that if the flow velocity is considerably smaller than the velocity of electrons in the conductor fluid and the fluid is a good conductor, any unbalance between the electrons charges and the positive charges will be removed almost instantaneously by the electric field created by this unbalance (Cunha, 2012). In this context of very small density of free charges, significantly simplifications can be made in the governing equations.

In the case of magnetohydrodynamic flows of ideal barotropic fluids, linearizing the governing equations, considering the hydrodynamics and magnetic variables as the sum of their equilibrium value with an associated fluctuation of small amplitude around it, lead to the dispersion relations for Alfvén waves and magnetoacoustic waves (slow and fast). The first ones are transverse and incompressible (they do not modify the density of the fluid) and the second are longitudinal and compressible (they modify the density of the fluid). In this sense, the idea of this work is to find and analyze the dispersion relation for Alfvén and magnetoacoustic waves for this ideal MHD flow of barotropic fluid, but in the presence of an expansional viscosity. In addition, we perform a stability analysis of the small disturbances in the MHD flow. For this end we write the governing equations in a appropriated nondimensional form and consider small perturbations of oscillatory nature around the equilibrium, finding a set of equations in the frequency-wave number space (i.e. Fourier space). The terms of small perturbations are considered proportional to an amplitude as a function of time that dictates if the disturbances are stable or unstable. Combining these equations in the same way that is done to find the dispersion

relations in the case without the expansional viscosity effect, we find the dispersion relations modified by a term that comes from the regime of high frequency baratropic flows, i.e. the effect of an expansional viscosity must be present. From the calculations developed, results of the theoretical analysis are presented and discussed.

2. GOVERNING EQUATIONS

As stated before, the governing equations of the problem examined here are given by the coupling between hydrodynamics equations and Maxwell equations.

2.1 Hydrodynamics equations

From hydrodynamics we have the continuity equation and the equation of balance of linear momentum. The first is written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}$$

where ρ is the density of the fluid and $\mathbf{u}$ represents the Eulerian velocity field. The equation that gives the local balance of force per unit of volume in a fluid element is given by the Cauchy's equation:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \nabla \cdot \mathbf{\Sigma} + \mathbf{b}, \tag{2}$$

where $\mathbf{\Sigma}$ is the second order stress tensor and represents the constitutive equation for the fluid and $\mathbf{b}$ denotes the body force per unit of volume.

The stress tensor for a Newtonian Non-Stokesian fluid is given by (Batchelor, 2000):

$$\mathbf{\Sigma} = -p_0\mathbf{I} + \kappa \left(\nabla \cdot \mathbf{u} \right) \mathbf{I} + 2\mu \left[\mathbf{D} - \frac{1}{3} \left(\nabla \cdot \mathbf{u} \right) \mathbf{I} \right], \tag{3}$$

where p_0 is the thermodynamic pressure, $\mathbf{I}$ is the second rank identity tensor, κ is the expansional viscosity (or second viscosity coefficient), μ is the fluid viscosity and $\mathbf{D}$ is the symmetric part of the gradient of velocity field tensor, a second rank tensor related to the strain rate of the flow. For an ideal fluid, in Euler's regime, the stress tensor reduces to:

$$\mathbf{\Sigma} = -p_0\mathbf{I} + \kappa \left(\nabla \cdot \mathbf{u} \right) \mathbf{I}. \tag{4}$$

Substituting Eq. (4) in Eq. (2) results in:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p_0 + \nabla \left(\kappa \nabla \cdot \mathbf{u} \right) + \mathbf{b}, \tag{5}$$

which is Euler's equation with an additional expansional viscosity term.

In this way, Eq. (1) and Eq. (5) are the governing equations given by the hydrodynamic theory.

2.2 Electromagnetic equations

The full set of Maxwell's equations, Gauss' Law, Solenoidal nature of $\mathbf{B}$, Faraday's Law and Ampere-Maxwell equation, are given, respectively, by (Jackson, 1999):

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\varepsilon_0}, \tag{6}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{7}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{8}$$

and

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \tag{9}$$

where $\mathbf{E}$ is the electric field, ρ_e is the density of electric charges, ε_0 is the electric permisivity of free space, $\mathbf{B}$ is the magnetic induction field, μ_0 is the magnetic permeability of free space and $\mathbf{J}$ denotes the current density vector. We also have the continuity of charges equation, Ohm's law and the electromagnetic force per unit of volume (also known as Lorentz force per unit of volume), given, respectively, by (Jackson, 1999):

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{10}$$

$$\mathbf{J} = \sigma \mathbf{E}' = \sigma \left(\mathbf{E} + \mathbf{u} \times \mathbf{B} \right) \tag{11}$$

and

$$\mathbf{f_m} = \rho_e \mathbf{E} + \mathbf{J} \times \mathbf{B}, \tag{12}$$

where σ is the conductivity, $\mathbf{f_m}$ is the Lorentz force per unit of volume and $\mathbf{E}'$ is the electric field measured by a frame traveling with the local velocity of the conductor.

In the context of magnetohydrodynamics, the force term $\rho_e \mathbf{E}$ can be neglected in comparison with $\mathbf{J} \times \mathbf{B}$. The proof for this statement can be shown taking the divergence of Eq. (11) and using Eq. (6) and Eq. (10), this results in:

$$\frac{\partial \rho_e}{\partial t} + \frac{\rho_e}{\tau_e} + \sigma \nabla \cdot (\mathbf{u} \times \mathbf{B}) = 0, \tag{13}$$

where τ_e denotes a electrical relaxation time. According to (Davidson, 2017), for typical metallic conductors it has a value around 10^{-18} seconds, in the way we can neglect the first term in Eq. (13) in comparison with the second one and the equation reduces to

$$\frac{\rho_e}{\tau_e} = -\sigma \nabla \cdot (\mathbf{u} \times \mathbf{B}) . \tag{14}$$

Making a scaling analysis, from Eq. (14) and Eq. (11) we have that

$$|\rho_e \mathbf{E}| \sim \frac{u_c \tau_e J_c B_c}{\ell_c}, \tag{15}$$

where u_c is a characteristic velocity of the flow, ℓ_c is the characteristic length, J_c is a characteristic current density and B_c is a characteristic value for the magnetic induction. Consequently,

$$\frac{|\rho_e \mathbf{E}|}{|\mathbf{J} \times \mathbf{B}|} \sim \frac{u_c \tau_e}{\ell_c} \sim \frac{\tau_e}{t_c}, \tag{16}$$

where t_c is a characteristic time of the flow. So, in the context where the flow velocity is much smaller than the velocity of the charge carries, $t_c \gg \tau_e$, given that τ_e is very small. Thus, in the context of magnetohydrodynamics, the Lorentz force per volume unit reduces to:

$$\mathbf{f_m} = \mathbf{J} \times \mathbf{B}. \tag{17}$$

This form of the Lorentz force enters as a field force in the hydrodynamics equations. In the absence of gravitacional forces, the body force $\mathbf{b}$ in Eq. (5) is equal to $\mathbf{f_m}$. Also, in MHD context, the second term on the right hand side of Ampere-Maxwell equation, Eq. (9), vanishes, given that:

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \sim \frac{\varepsilon_0}{\sigma} \frac{\partial \mathbf{J}}{\partial t} \sim \tau_e \frac{\partial \mathbf{J}}{\partial t} \ll \mathbf{J}. \tag{18}$$

Hence, Eq. (9) reduces to

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{19}$$

Additionally, in MHD it is also needed an advection-diffusion equation for $\mathbf{B}$. Using vector calculus relations and combining Eq. (7), Eq. (8), Eq. (11) and Eq. (19), after some algebraic manipulations, we obtain that (Davidson, 2017):

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}. \tag{20}$$

In this work we will consider flows of fluids with high conductivity (i.e. perfect conductors) so that Eq. (20) is simplified as follows:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) . \tag{21}$$

2.3 Energy equation

In this work it is assumed the fluid element to be adiabatic, so that the energy equation is simply the adiabatic relation, given by:

$$\frac{p_0}{\rho^\gamma} = \text{constant}, \tag{22}$$

which implies that

$$\frac{D}{Dt} \left(\frac{p_0}{\rho^\gamma} \right) = \frac{\partial}{\partial t} \left(\frac{p_0}{\rho^\gamma} \right) + \mathbf{u} \cdot \nabla \left(\frac{p_0}{\rho^\gamma} \right) = 0, \tag{23}$$

where γ is the ratio between the specific heat at constant pressure and the specific heat at constant volume.

2.4 A relation for the expansional viscosity coefficient

A simple model to describe the second viscosity coefficient known in the literature was proposed by (Landau and Lifshitz, 1987). In order to develop a thermodynamic relation for the second viscosity, they assume a decomposition of a physical quantity in its value of equilibrium plus a small fluctuation around it. Hence, the calculation is restrict to small relaxation time of the thermodynamic process. Expanding this quantity in a Taylor series and neglecting second order contribution related to the small relaxation time, they find a first order expression in the relaxation time, τ, for this pseudo-thermodynamic properties as a function of density. Now, following this Landau and Lifshitz (1987) procedure, we use the constitutive relation for the difference of mechanical (p) and thermodynamic pressure (p_0) as suggested by (Batchelor, 2000):

$$p_0 - p = \kappa \nabla \cdot \mathbf{u}, \tag{24}$$

and the continuity equation in the following version:

$$\frac{D}{Dt}(\delta\rho) + \rho \nabla \cdot \mathbf{u} = 0, \tag{25}$$

to find the following expression for the second viscosity coefficient:

$$\kappa = \frac{\tau\rho}{1 - i\omega\tau}\left(c_\infty{}^2 - c_0{}^2\right), \tag{26}$$

where τ is a relaxation time, that physically means the time to return to the state of the equilibrium, c_0 is the speed of sound at equilibrium state and c_∞ is the speed of sound at non-equilibrium state.

It should be important to note that the real part of κ and the imaginary one is given, respectively, by

$$Re(\kappa) = \tau\rho(c_\infty{}^2 - c_o{}^2)f(\omega), \tag{27}$$

and

$$Im(\kappa) = \tau^2\rho(c_\infty^2 - c_o^2)\omega f(\omega), \tag{28}$$

where

$$f(\omega) = \frac{1}{(1 - \omega\tau)^2}. \tag{29}$$

Additionally, the Lagrangian derivative of κ is given simply by:

$$\frac{D\kappa}{Dt} = \tau(c_\infty^2 - c_o^2)g(\omega)\frac{D\rho}{Dt}. \tag{30}$$

Here $g(\omega) = (1 + i\omega)f(\omega)$.

2.5 Full set of governing equations

According to previous sections, making $\mathbf{b} = \mathbf{f_m}$ and using the fact that

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B}, \tag{31}$$

the governing equations of the problem are given by the following set of equations:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0, \tag{32}$$

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u}\right) = -\nabla p_0 + \nabla[\kappa(\rho)\nabla \cdot \mathbf{u}] - \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}), \tag{33}$$

$$\frac{\partial}{\partial t}\left(\frac{p_0}{\rho^\gamma}\right) + \mathbf{u} \cdot \nabla\left(\frac{p_0}{\rho^\gamma}\right) = 0, \tag{34}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \tag{35}$$

and

$$\kappa = \frac{\tau\rho c_0{}^2}{1 - i\omega\tau}\left(\widetilde{c}{}^2 - 1\right), \tag{36}$$

where $\widetilde{c} = c_\infty/c_0$.

2.6 Non-Dimensional form of the governing equations

In order to make the governing equations nondimensional the following new quantities are defined in terms of characteristic quantities of the flow problem:

$$\widetilde{\nabla} = \ell_c \nabla, \quad \widetilde{\mathbf{u}} = \frac{\mathbf{u}}{c_0}, \quad \widetilde{t} = \frac{c_0 t}{\ell_c}, \quad \widetilde{\rho} = \frac{\rho}{\rho_0}, \quad \widetilde{p}_0 = \frac{p_0}{p_{00}}, \quad \text{and} \quad \widetilde{\mathbf{B}} = \frac{\mathbf{B}}{B_0}, \tag{37}$$

where the tilde variables are the nondimensional version of the variables, ρ_0 is the equilibrium value of the density, ℓ_c is the characteristic lenght of the flow, c_0 is the equilibrium value of the speed of sound, p_{00} is the equilibrium value of the thermodynamic pressure and B_0 is the absolute value of the magnetic induction in equilibrium. Note that a scaling for κ is $\tau \rho_o c_o{}^2$. In this way, recalling that $k = k(\rho)$, the nondimensional governing equations can be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{38}$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p_0 + \frac{1}{Re_\kappa} \left(\frac{\widetilde{c}^{\,2} - 1}{1 - i\omega\tau} \right) \left[\nabla \rho \left(\nabla \cdot \mathbf{u} \right) + \rho \nabla \left(\nabla \cdot \mathbf{u} \right) \right] - E_m \mathbf{B} \times \left(\nabla \times \mathbf{B} \right), \tag{39}$$

$$\frac{\partial}{\partial t} \left(\frac{p_0}{\rho^\gamma} \right) + \mathbf{u} \cdot \nabla \left(\frac{p_0}{\rho^\gamma} \right) = 0 \tag{40}$$

and

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{u} \times \mathbf{B} \right), \tag{41}$$

where Re_κ is defined as the expansional Reynolds number and is given by:

$$Re_\kappa = \frac{\ell_c}{\tau c_0}, \tag{42}$$

and E_m is defined as the Euler magnetic number, given by:

$$E_m = \frac{B_0^2}{\mu_0 p_{00}}. \tag{43}$$

The first nondimensional group gives a ratio between the flow time and the relaxation time. The second one gives a ratio between the magnetic pressure and the thermodynamic pressure. Note that in this step the tilde were removed from nondimensional variables for simplicity in notation.

3. LINEARIZED EQUATIONS AND DISPERSION RELATIONS

Assuming an equilibrium magnetic induction vector given by $B_0 \sin \alpha \hat{\mathbf{e}}_\mathbf{x} + B_0 \cos \alpha \hat{\mathbf{e}}_\mathbf{z}$, that in terms of nondimensional quantities is given by:

$$\mathbf{B_0} = \sin \alpha \hat{\mathbf{e}}_\mathbf{x} + \cos \alpha \hat{\mathbf{e}}_\mathbf{z}. \tag{44}$$

Now, considering a decomposition in the hydrodynamics and magnetic quantities as being a sum of their equilibrium value and an associated fluctuation of small amplitude around it, we have that

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1}, \tag{45}$$

$$\mathbf{u} = \mathbf{0} + \mathbf{u_1}, \tag{46}$$

$$p_0 = 1 + p_1 \tag{47}$$

and

$$\rho = 1 + \rho_1. \tag{48}$$

Here the variables with subscript 1 represents the fluctuation of small amplitude of the respective quantity around the equilibrium. Now, substituting Eq. (45) to Eq. (48) into Eq. (38) - Eq. (41) representing the nondimensional set of governing equations and neglecting second order terms in fluctuations, we obtain the linearized set of governing equations given by:

$$\frac{\partial \rho_1}{\partial t} + \nabla \cdot \mathbf{u_1} = 0, \tag{49}$$

$$\frac{\partial \mathbf{u_1}}{\partial t} = -\nabla p_1 + \frac{1}{Re_\kappa} \left(\frac{\widetilde{c}^{\,2} - 1}{1 - i\omega\tau} \right) [\nabla (\nabla \cdot \mathbf{u_1})] - E_m \mathbf{B_0} \times (\nabla \times \mathbf{B_1}), \tag{50}$$

$$\frac{\partial p_1}{\partial t} - \gamma \frac{\partial \rho_1}{\partial t} = 0 \tag{51}$$

and

$$\frac{\partial \mathbf{B_1}}{\partial t} = \nabla \times (\mathbf{u_1} \times \mathbf{B_0}). \tag{52}$$

Assuming plane waves propagating along $\hat{\mathbf{e}}_\mathbf{z}$ direction and a solution to the fluctuation parameters of the type:

$$f_1 = f_a e^{ikz+st}, \tag{53}$$

where k is the nondimensional wavenumber, t is the nondimensional time, f_a denotes the nondimensional amplitude of the fluctuation and s is a complex function given by:

$$s(k) = \xi(k) - i\omega(k), \tag{54}$$

characterizing a modal temporal analysis, where the imaginary part of s represents the wave oscillation in time coordinate (i.e. related to the propagation velocity of the wave, ω/k) and the real part of s defines the rate of the wave amplification according to its physical parameters. If the real part has values greater than 0 for certain physical parameters, the disturbance has amplification and it is unstable for small disturbances. On the other hand, for negative values of the real part of s the disturbance decays in time and the wave configuration is stable. For a null value of s real part the wave amplitude is invariant in time.

Now, substituting the disturbance given by the modal solution, Eq. (45), into Eq. (49) to Eq. (52), after some algebraic manipulations it results in the following set of algebraic equations in the wave numer-frequency space:

$$s\rho_1 + iku_{1z} = 0, \tag{55}$$

$$su_{1x} - ikE_m B_{0z} B_{1x} = 0, \tag{56}$$

$$su_{1y} - ikE_m B_{0z} B_{1y} = 0, \tag{57}$$

$$su_{1z} + ikp_1 + \frac{1}{Re_\kappa} \left(\frac{\widetilde{c}^{\,2} - 1}{1 - i\omega\tau} \right) k^2 u_{1z} + ikE_m B_{0x} B_{1x} = 0, \tag{58}$$

$$sB_{1x} + iku_{1z}\sin\alpha - iku_{1x}\cos\alpha = 0, \tag{59}$$

$$sB_{1y} - iku_{1y}\cos\alpha = 0, \tag{60}$$

$$sB_{1z} = 0, \tag{61}$$

and

$$sp_1 - \gamma s\rho_1 = 0. \tag{62}$$

The consistency condition for Eq. (57) and Eq. (60) gives:

$$s^2 + k^2 E_m \cos^2\alpha = 0. \tag{63}$$

The relation given by Eq. (63) is a quadratic equation representing Alfvén waves, where if the real part of s is 0, it returns exactly the dispersion relation for Alfvén waves (Davidson, 2017). The consistency condition for Eq. (55), Eq. (56), Eq. (58), Eq. (59) and Eq. (62) gives us:

$$\frac{1}{Re_\kappa} \left(\frac{\widetilde{c}^{\,2} - 1}{1 - i\omega\tau} \right) \left(sk^4 E_m \cos^2\alpha + s^3 k^2 \right) + \left(s^2 + k^2 E_m \cos^2\alpha \right) \left(s^2 + k^2\gamma \right) + k^2 s^2 E_m \sin^2\alpha = 0. \tag{64}$$

The relation given by Eq. (64) is a equation that have two pair of roots. They correspond to fast and slow magnetoacoustic waves, where if the real part of s is 0, the equation returns the dispersion relations for these waves. Therefore, from Eq. (63) we have the solutions given by:

$$s = ik\cos\alpha\sqrt{E_m}. \tag{65}$$

In the case of Eq. (64), the analytical solution is too tedious and therefore we have solved it using Python and its library Sympy, a library for symbolic computation. The equation was solved determining the roots of the polynomial numerically with a Sympy's method. The results will be presented in the next section.

4. RESULTS AND DISCUSSIONS

Eq. (65) represents Alfvén waves. In this case, the amplification factor (ξ) is always null, what means that the Alfvén waves have a neutral value for the amplification factor. Under this condition, this kind of wave are always stable in the MHD context considered here. Besides that, its also clear from this relation that the expansional viscosity does not have any influence on the behavior of this type of waves. This was expected as these disturbance are basically incompressible waves. Hence, this kind of disturbance does not exerted any variation on the fluid density, making the dispersion relation obtained for this wave be

$$\omega^2 - k^2 E_m \cos^2 \alpha = 0, \qquad (66)$$

which is completely independent of the extensional Reynolds number.

In contrast, Eq. (64) makes clear that the expansional Reynolds number, the nondimensional frequency and the ratio of sound speed may exert a significant influence on the propagation of magnetoacoustic waves. Consequently, magnetoacoustic disturbances are compressible ones and the expansional effects of the fluid affects directly the wave dispersion relation and its stability.

Also from the equation that represents the magnetoacoustic waves, Eq. (64), we obtain four roots for s, from which the values of the real part, the amplification factor, and the values for the imaginary part, the frequency ω, can be evaluated for different values of the physical parameters and different wavenumbers, showing what is the influence of these physical parameters on the stability and on the phase velocity ω/k of the waves. By varying the physical parameters of the flow system, different plots for ξ and for ω/k are presented in this section.
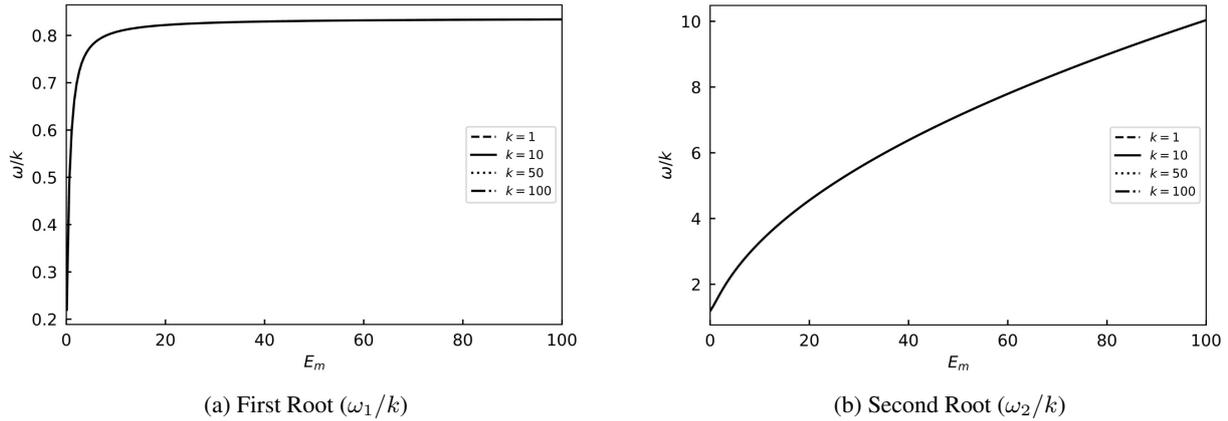

(a) First Root (ω_1/k)

(b) Second Root (ω_2/k)

Figure 1: Phase velocity ω/k as a function of E_m with $Re_k \to \infty$, $\alpha = \pi/4$ and $\gamma = 1.4$.

Analyzing Fig. 1 it is seen that the phase velocity of the waves, ω/k, does not depend on the value of the wavenumber when the high frequency expansional effects are negligible, that means that the propagation velocity at short and long wave of small amplitudes are invariant for any mode of the wave. Indeed, according to (Bittencourt, 2004), the magnetoacoustic waves without expansional high frequency effects are nondispersive waves (i.e. its phase velocity does not depend on k) and its dispersion relations have four roots, one corresponding to fast waves, a second one corresponding to slow waves and the others two ones representing just the first two waves propagating in the opposite direction. This behavior is clearly depicted in Fig. 1. Also, for the case where the high frequency effects term is not present, Eq. (64) is a biquadratic equation that can be solved analytically and used to validate the numerical solution. The analytical solution expressed by:

$$s^2 = -\frac{1}{2}k^2\left(\gamma + E_m\right) \pm \frac{1}{2}k^2\left[\left(\gamma + E_m\right)^2 - 4\gamma E_m \cos^2\alpha\right]^{1/2} \qquad (67)$$

is exactly obtained by the numerical code solving Eq. (64) in the asymptotic limit of $Re_k \to \infty$ as presented in Fig. 1.

Now, varying the values of the Euler magnetic number, E_m, and the α angle, the magnitude of the phase velocity of both slow and rapid waves are affected substantially as shown in Fig. 1 and Fig. 2. For instance, ω/k of the slow waves increases fast with E_m for all wavenumbers and it reaches a constant value for E_m around 20, i.e. as the magnetic pressure is twenty times greater than the thermodynamic one. In contrast, the root corresponding to fast waves shown in Fig. 1 (b) increases with the nondimensional magnetic pressure in the full interval of E_m with a dependence of ω/k following basically the scaling $E_m^{1/2}$. Again, since that the waves in the limit of very high Re_k are nondispersive, the observation here works for all spectrum of wavenumbers (short and long waves). These results indicate that the ratio between magnetic pressure and thermodynamic pressure and the angle α are key physical parameters that determines the phase velocity of the wave. On the other hand, changing the angle α raises or decreases the roots representing the slow
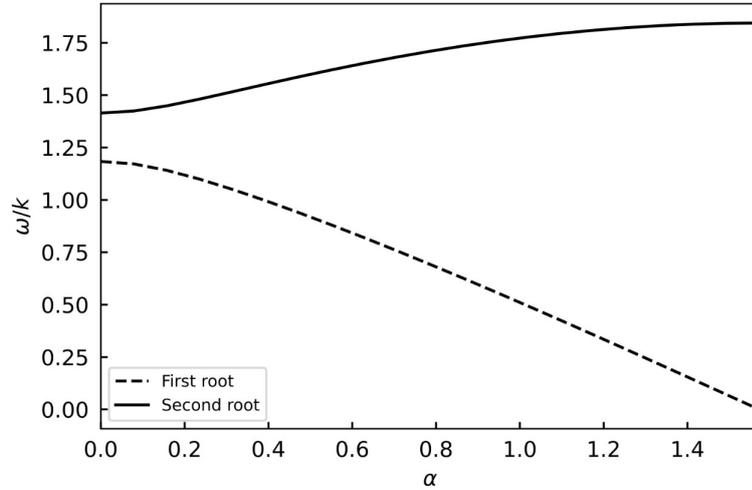
Figure 2: Phase velocity ω/k as a function of the angle α with $Re_k \to \infty$, $Em = 2$ and $\gamma = 1.4$.

magnetoacoustic waves and the fast magnetoacoustic waves, depending on the root. For slow waves raising α decreases the phase velocity and for fast waves raising α increases the phase velocity. So, the orientation of the applied magnetic field with respect to the direction of wave propagation change quite differently the wave speed depending on how fast or slow is its velocity. This suggests that the orientation of a magnetic field can be used to control the propagation velocity of magnetoacoustic wave in a conductor fluid.
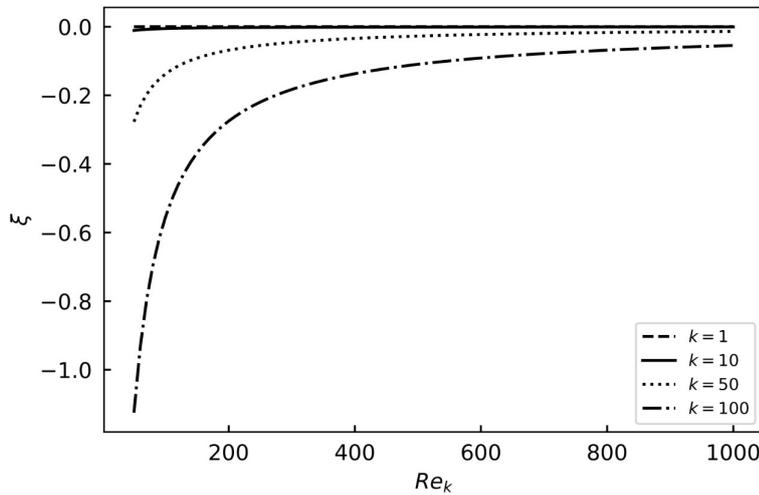

Figure 3: Amplification factor as a function of Re_k with $\widetilde{c} = 1.05$, $\omega\tau = 1$, $\alpha = \pi/4$, $Em = 2$ and $\gamma = 1.4$.

Solving s for the amplification factor (the real part) on the asymptotic case where $Re_k \to \infty$, we get from both analytical and numerical solutions that it is null independently of the values of the wavenumber, of the Euler magnetic number and of the angle α. This shows that this waves are always stable in the absence of expansional high frequency effects. On the other hand, from the numerical solution, for finite values of Re_k (i.e., when the expansional effects at high frequency takes place), these effects change considerable the wave growth rate, as indicated in Fig. 3. They make ξ be more negative, behavior which points out that the expansional high frequency effects tends to attenuate the wave amplitude in any mode of velocity, showing the strong dissipative effects of high frequency flows on this kind of waves. In addition, Fig. 3 shows that at the asymptotic limit of very high Re_k the wave growth rate tent to a null value for both short and long waves (i.e. small and large scales of wavelengths). These results has motivated us to examine the behavior of magnetoacoustic waves in nonlinear regimes in high frequency flows, i.e. for finite values of Re_k. Actually, we plan to explore nonlinear regimes of magnetoacoustic waves in a future work for different wavenumbers, frequencys and values of the physical nondimensional parameters explored in the present linearized MHD flow.

Figure 4 presents a diagram for the phase velocity (ω/k) of the waves as a function of the Re_k, showing its behavior
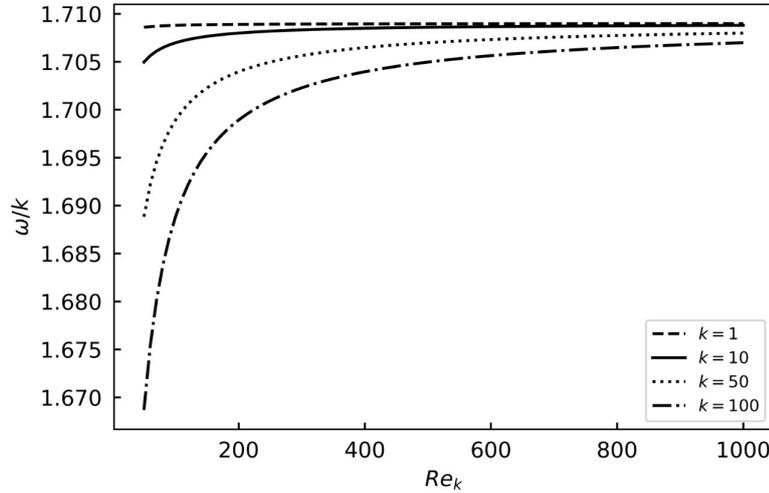
Figure 4: Phase velocity as a function of Re_k with $\widetilde{c} = 1.05$, $\omega\tau = 1$, $\alpha = \pi/4$, $Em = 2$ and $\gamma = 1.4$.

in cases where the expansional high frequency effects assumes finite values. The results indicate the importance of the expansional high frequency effects for varying the wave phase velocity as well. As in the asymptotic limit where $Re_k \to \infty$ the magnetoacoustic waves are nondispersive, from Fig. 4 we see that these high frequency effects acts upon this waves making a dispersive action (i.e. they make the phase velocities have a dependence on the wavenumber) mainly in short waves regimes. Additionally, Fig. 4 corroborates that long waves propagates faster than short waves for finite values of Re_k. It is important to notice that only one root for ω/k was shown in Fig. 4, but the same physical interpretation can be obtained from the other ones.

From the results obtained here, we propose a model to determine or calibrate material constants of the constitutive equations. Making $s = -i\omega$, from Eq. (64) we obtain the dispersion relations for the magnetoacoustic waves in the presence of high frequency effects. It is given by:

$$\frac{i}{Re_\kappa}\left(\frac{\widetilde{c}^2 - 1}{1 - i\omega\tau}\right)\left(\omega k^4 E_m \cos^2 \alpha + \omega^3 k^2\right) - \left(\omega^2 - k^2 E_m \cos^2 \alpha\right)\left(\omega^2 - k^2\gamma\right) + \omega^2 k^2 E_m \sin^2 \alpha = 0. \qquad (68)$$

It should be important to note that as $Re_k \to \infty$, Eq. (68) reduces exactly to a nondimensional version of the dispersion relation for the magnetoacoustic waves in the absence of expansional high frequency effects given by (Bittencourt, 2004). Isolating the Re_k term in Eq. (68), after some algebraic manipulations, we obtain that:

$$\mu_k = \frac{1}{\omega k^4 E_m \cos^2 \alpha + \omega^3 k^2}\left[\left(\omega^2 - k^2 E_m \cos^2 \alpha\right)\left(\omega^2 - k^2\gamma\right) - \omega^2 k^2 E_m \sin^2 \alpha\right]. \qquad (69)$$

Here $\mu_k = \kappa/\rho c_0 \ell_c$ with κ being the second viscosity coefficient. As the phase velocities of the waves are quantities that can be measured in a laboratory and the frequency ω is direct related to it, from Eq. (69) we can obtain values for the expansional viscosity κ in the frequency domain. In this way, this equation represents a model to measure indirectly the second viscosity coefficient if the phase velocities and the other physical parameters of the flow can be measured. It is important to say that this model is limited to small values of the relaxation time τ, as the model for the expansional viscosity used is itself a model only on the first order approximation.

5. CONCLUSIONS

In the context of ideal compressible MHD flows of barotropic fluids with expansional viscosity, we developed the governing equations and made them nondimensional, setting the mathematical model for these kinds of flows. Linearizing the governing nondimensional equations we deduced the equations representing Alfvén waves and Magnetoacoustic waves in the presence of an expansional viscosity effect. From this equations we were able to determine the curves for the amplification factor and for the phase velocities in terms of different physical parameters. It was shown that the expansional viscosity effects have not influenced Alfvén waves, what was expected, as these waves are incompressible and they do not affect the fluid density. On the other hand, we have shown that the magnetoacoustic waves are influenced by the expansional viscosity effects, as its dispersion relation has dependence on the expansional Reynolds number, on the ratio of sound speed and on the product between the frequency and the relaxation time. The amplification factor for the Alfvén waves was shown to be zero, showing that these waves are always stable. The results have indicated that the Euler magnetic number and the angle between the direction of propagation of the disturbances and the direction of the magnetic

induction vector just changes the magnitude of the phase velocities of the magnetoacoustic wave modes. We have shown that the expansional high frequency effects makes a dispersive effect on the magnetoacoustic waves, deviating it from the asymptotic case with a infinite expansional Reynolds number, where these waves behave like nondispersive ones. The amplification factor for the magnetoacoustic waves was also proven to be zero for all wavenumbers in the absence of expansional effects. However, in the presence of expansional effects the amplification factor assumed negative values, showing that high frequency flows have a dissipative effect on this kind of waves. A proposal to a model to measure or calibrate material constants of the constitutive relations were made using the equations developed in this work. This preliminary analysis was a good starting point to perform nonlinear analysis of the magnetoacoustic waves in order to explore the influence of the effects of expansional high frequency flows on a nonlinear regime of wave propagation.

6. ACKNOWLEDGEMENTS

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