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COMPUTATION OF MINIMUM-TIME SOLUTION OF A SEALING MACHINE THROUGH LINEAR PROGRAMMING WITH SENSITIVITY AND CLOSED-LOOP ROBUSTNESS ANALYSIS

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Abstract. *Instances of minimum-time path-planning problems occur frequently in several engineering areas. In this paper the problem of temperature control of a sealing machine used in automatic packing lines is considered where the input is the thermal power applied to the jaw and the output is its temperature. A second-order transfer function with real poles is adopted to relate the output to the input. The problem is to transfer in minimum-time the temperature of the jaw between two given initial and final values without overshoot and with null time-derivative of the jaw temperature at the final time. It is shown that a sequence of Linear Programming problems can be used to solve the minimum-time problem. Then, the sensitivity of both the final temperature and of its time-derivative with respect to the parameters of the transfer function are evaluated through the integration of an appropriate system of linear differential equations. Finally, using a SIMC-PID controller to close the loop, a modified version of Monte Carlo simulation is used in order to analyze the robustness of the system when parameter uncertainties are present in the model.*

Keywords: *path-planning, bang-bang control, linear programming, sensitivity, robustness.*

1. INTRODUCTION

Minimum-time path-planning arises commonly in several areas of engineering – industrial robotics, satellite maneuver, crane operations, etc. One of the objectives of this paper is to use a minimum-time optimal control law for temperature computation in the process of heating the jaws of a plastic packaging sealing machine. In addition, the paper also proposes to analyze both the sensitivity of the solution to model parameter uncertainties as well as the robustness of its closed-loop tracking. The optimization considered refers to the heating of the jaws when the machine is turned on. Both the excessive delay to reach the desired temperature or exceeding it during the heating process – which, when it occurs, requires waiting for a natural cooling of the jaws, which is usually a slow process – cause a loss of productivity of the machine.

It is known that the search for solutions of optimal control problems subject to constraints is usually an intricate task. Indirect methods involve the use of the Pontryagin Minimum Principle to formulate a Two-Point Boundary Value Problem (TPBVP), whose solution is far from trivial (Dhanda and Franklin, 2010), (Kamien and Schwartz, 1981), (Seiersted and Sydsaeter, 1977); direct methods involve the use of nonlinear programming algorithms to find a discrete-time approximation to the numerical solution of optimal control problems (Böhme and Frank, 2017). Rao (2009) contains a good survey of numerical methods for optimal control.

The first point of this work is to use a method of solving a sequence of maximum range problems – each one of them with a fixed final time – to find the minimum-time solution. The final time is then adjusted in order to reach the desired final temperature. This strategy has been used for crane control during loading and unloading of ships (Cruz and Leonardi, 2013). The dynamic model of the continuous time process is discretized and then a Linear Programming (LP) problem is solved at each step in the sequence.

Among all the mathematical programming problems, LP are certainly the simplest ones that can be solved. Usually LP algorithms are not iterative, they do not require an initial guess for the solution and they do not have a high computational cost in practice. Furthermore, they are always conclusive as to whether or not the solution to the problem exists, and when the solution exists, it is found in a finite number of steps. This in general is not the case neither with most TPBVP's solution algorithms nor with nonlinear programming algorithms, which, after all, still need a good initial guess of the solution.

Obtaining the minimum-time solution is a typical path-planning problem, being thus based on a mathematical model of the process dynamics. Hence, from a practical point of view, the sensitivity of the solution with respect to uncertainties in the model parameters is a key issue. This study allows us to analyze which model parameters have the largest impact on the optimal solution and, therefore, which are the ones whose accuracy is the most critical for the implementation of the planned path. This is the second point of this work.

In practice, carrying out the planned path requires a closed-loop control system to follow it since the mathematical model is always an approximate representation of the real system – either because of the action of external disturbances or because of uncertainties in the model. In this paper, a PID-SIMC (Skogestad and Postlethwaite, 2005) compensator with anti-windup is used to close the control loop.

In the specific case of the minimum-time problem, the optimal solution is of bang-bang type where the control variable switches between its maximum and the minimum values at very-well defined time instants. This characteristic of the control law causes the performance of the closed-loop controller to be degraded when uncertainties are present since there is no margin for the control variable to correct deviations from the planned path. This occurs when the value of the control variable determined by the closed-loop controller violates the extreme value constraint. In the case of a sealing machine it is intuitive that when path-planning is based on a heating power that is lower than the maximum one available at the heater, there will be some margin for closed-loop corrections and hence there will be some robustness to uncertainties; however, this must occur at the expense of an increase in the time required for heating the jaws. Using a modified version of the Monte Carlo simulation, a study is carried out to assess the robustness of the closed-loop tracking of the optimal trajectory. The study is done given both the amount of uncertainty in the parameters of the system dynamic model and the margin between the real heater maximum power and the lower power used in path-planning. This is third point of this work.

2. SYSTEM MODEL

Considering the heater power as input (u) and the difference between the jaw and the room temperatures as the output (θ), the process model adopted was

$$G(s) = \frac{\theta(s)}{u(s)} = \frac{K}{(\tau_1 s+1)(\tau_2 s+1)}\,, \tag{1}$$

where K represents the gain and τ_1 and τ_2 are the time constants.

Defining the vector

$$\boldsymbol{x}(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}, \tag{2}$$

the model can be immediately written in state form as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \tag{3}$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t), \tag{4}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\tau_1\tau_2} & -\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right) \end{bmatrix}, \tag{5}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ \frac{K}{\tau_1\tau_2} \end{bmatrix} \tag{6}$$

e

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{7}$$

Assuming that the jaw is initially at rest at room temperature, the initial conditions are null. The general solution of the state equation (Eq. (3)) is then given by

$$\boldsymbol{x}(t) = \int_0^t e^{\boldsymbol{A}(t-\xi)}\boldsymbol{B}u(\xi)d\xi\ . \tag{8}$$

In order to later use the strategy of solving the optimal control problem by using LP, the system must be discretized in time. The discretization step is defined as

$$\Delta t = t_f/n, \tag{9}$$

where t_f is the final time and n is the number of steps.

Considering that $u(t)$ is stepwise constant, that is,

$$u(t) = u_i \qquad (1 \leq i \leq n) \tag{10}$$

for $(i-1)\Delta t \leq t < i\Delta t$, and defining

$$\boldsymbol{\Gamma}_i = \int_{(i-1)\Delta t}^{i\Delta t} \mathrm{e}^{A(t_f-\xi)} d\xi \, \boldsymbol{B} \qquad (1 \leq i \leq n), \tag{11}$$

it follows that

$$\boldsymbol{x}(t_f) = \sum_{i=1}^{n} \boldsymbol{\Gamma}_i u_i. \tag{12}$$

Obviously, since the dynamical model is linear, $\boldsymbol{x}(t_f)$ is also linear in $u_1, u_2, \ldots, u_n$, which is essential for the use of LP in Section 3 to solve the minimum-time problem.

3. THE MINIMUM-TIME PATH-PLANNING PROBLEM

3.1 Problem formulation

It is desired that, through the action of the heater, the jaw reaches a given final temperature $\theta_f > 0$, that is,

$$\theta(t_f) = \theta_f \tag{13}$$

and that it has a tendency to locally remain at this value, that is,

$$\dot{\theta}(t_f) = 0. \tag{14}$$

Denoting by $\boldsymbol{U} = [u_1, u_2, \ldots, u_n]$ and $u_{\max}$, respectively, the vector of problem unknowns and the maximum power of the heater, a set of natural constraints is

$$0 \leq \boldsymbol{U} \leq \boldsymbol{U}_{\max}, \tag{15}$$

where

$$\boldsymbol{U}_{\max} = [u_{\max}, u_{\max}, \ldots, u_{\max}]. \tag{16}$$

The objective is then to minimize t_f with respect to $\boldsymbol{U}$, i.e,

$$\min_{\boldsymbol{U}} t_f \tag{17}$$

subject to the terminal conditions given by Eqs. (13) and (14) as well as to the constraints of Eq. (15).

3.2 Problem solution through Linear Programming

In the problem formulated in the previous section, the value of t_f is obviously unknown. Considering a sequence of linear problems of maximum range – that is, in which the objective is to maximize $\theta(t_f)$ – with a fixed t_f and the same constraints, it is possible to use LP to plan the minimum-time path, since $\theta(t_f)$ is an increasing function of t_f. By means of a simple search algorithm (like binary search), the value of t_f can be adjusted such that $\theta(t_f) = \theta_f$.

The basic reasoning of the method is shortly presented in the following. For more details, see (Cruz and Leonardi, 2013).

Denoting by $\bar{\dot{\theta}}(0, t_f)$ the average value of the rate of change of temperature in the interval $(0, t_f)$, the relationship between the final temperature of the jaws $\theta(t_f)$ and t_f can be written as

$$\theta(t_f) = \bar{\dot{\theta}}(0, t_f)\, t_f. \tag{18}$$

Consider initially that $\theta(t_f)$ is given in Eq. (18) and the objective is to minimize t_f. In this case, of course, the value of $\bar{\bar{\theta}}(0,t_f)$ must be maximum. On the other hand, if the value of $\bar{\bar{\theta}}(0,t_f)$ is maximum and t_f is fixed in Eq. (18), then $\theta(t_f)$ must be maximum.

The algorithm to find the minimum-time solution can now be described as a sequence of maximum range problems where, at each step, t_f is assumed to be given and the following LP problem is solved:

$$\max_{\boldsymbol{U}} \theta(t_f) \tag{19}$$

subject to

$$\dot{\theta}(t_f) = 0 \tag{20}$$

$$0 \leq \boldsymbol{U} \leq \boldsymbol{U}_{\max} . \tag{21}$$

Denoting by $\theta^*(t_f)$ the maximum value obtained for $\theta(t_f)$, then:

1) if $\theta^*(t_f)>\theta_f$, the value of t_f is reduced and a new maximum range problem is solved;
2) If $\theta^*(t_f)<\theta_f$, the value of t_f is increased and a new maximum range problem is solved.

To adjust t_f the binary search algorithm can be used with the following stop criterion

$$|\theta^*(t_f)-\theta_f| \leq \beta, \tag{22}$$

where $\beta>0$ is given and sufficiently small.

4. SENSITIVITY

As it has been seen, the terminal conditions $\theta(t_f)$ and $\dot{\theta}(t_f)$ are key elements in the solution of the problem of minimum-time path-planning by the method proposed in Section 3. The objective of this section is to compute the sensitivities of these terminal conditions with respect to the model parameters, namely, $\partial\theta(t_f)/\partial\tau_1$,$\partial\theta(t_f)/\partial\tau_2$, $\partial\theta(t_f)/\partial K$, $\partial\dot{\theta}(t_f)/\partial\tau_1$, $\partial\dot{\theta}(t_f)/\partial\tau_2$ and $\partial\dot{\theta}(t_f)/\partial K$.

4.1 Variations in process model parameters

As a first step towards the computation of sensitivities, variations of the values of the model parameters are considered. Then the limits are taken for these variations tending to zero.

If $\Delta\boldsymbol{x}$ denotes the variation of the state vector $\boldsymbol{x}$ in Eq. (3) in correspondence to variations $\Delta\boldsymbol{A}$ and $\Delta\boldsymbol{B}$ in matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively, a linear approximation gives

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}\Delta\boldsymbol{x} + \Delta\boldsymbol{A}\boldsymbol{x} + \Delta\boldsymbol{B}u. \tag{23}$$

From Eqs. (4) and (5), the first-order approximations of $\Delta\boldsymbol{A}$ and $\Delta\boldsymbol{B}$ are given, respectively, by

$$\Delta\boldsymbol{A} = \frac{\partial\boldsymbol{A}}{\partial\tau_1}\Delta\tau_1 + \frac{\partial\boldsymbol{A}}{\partial\tau_2}\Delta\tau_2 \tag{24}$$

and

$$\Delta\boldsymbol{B} = \frac{\partial\boldsymbol{B}}{\partial\tau_1}\Delta\tau_1 + \frac{\partial\boldsymbol{B}}{\partial\tau_2}\Delta\tau_2 + \frac{\partial\boldsymbol{B}}{\partial K}\Delta K, \tag{25}$$

where all partial derivatives are calculated at the point (τ_1, τ_2, K).

Replacing Eqs. (24) and (25) in Eq. (23) and rearranging, it follows that

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}\Delta\boldsymbol{x} + \left(\frac{\partial\boldsymbol{A}}{\partial\tau_1}\boldsymbol{x} + \frac{\partial\boldsymbol{B}}{\partial\tau_1}u\right)\Delta\tau_1 + \left(\frac{\partial\boldsymbol{A}}{\partial\tau_2}\boldsymbol{x} + \frac{\partial\boldsymbol{B}}{\partial\tau_2}u\right)\Delta\tau_2 + \left(\frac{\partial\boldsymbol{B}}{\partial K}u\right)\Delta K. \tag{26}$$

4.2 Computation of sensitivities

Given that $\boldsymbol{x}(t_f) = [\theta(t_f) \ \dot{\theta}(t_f)]^T$, to compute the sensitivities, it is obvious that one must simply evaluate the partial derivatives of $\boldsymbol{x}(t_f)$ with respect to each of the model parameters, namely, $\partial\boldsymbol{x}(t_f)/\partial\tau_1$,$\partial\boldsymbol{x}(t_f)/\partial\tau_2$ and $\partial\boldsymbol{x}(t_f)/\partial K$.

4.2.1 Evaluation of $\partial \boldsymbol{x}(\boldsymbol{t_f})/\partial \boldsymbol{\tau_1}$

Initially the problem of computing $\Delta\boldsymbol{x}(t_f)/\Delta\tau_1$ is considered. Then the limit corresponding to $\Delta\tau_1 \to 0$ is taken.

Obviously in this case $\Delta\tau_2 = \Delta K = 0$. Hence Eq. (26) reduces to

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}\Delta\boldsymbol{x} + \left(\frac{\partial \boldsymbol{A}}{\partial\tau_1}\boldsymbol{x} + \frac{\partial \boldsymbol{B}}{\partial\tau_1}u\right)\Delta\tau_1. \quad (27)$$

Since $\Delta\boldsymbol{x}(0) = 0$, the solution of Eq. (27) is given by

$$\Delta\boldsymbol{x}(t_f) = \int_0^{t_f} e^{\boldsymbol{A}(t_f-\xi)}\left(\frac{\partial \boldsymbol{A}}{\partial\tau_1}\boldsymbol{x}(\xi) + \frac{\partial \boldsymbol{B}}{\partial\tau_1}u(\xi)\right)d\xi\,\Delta\tau_1. \quad (28)$$

Noticing that $\Delta\boldsymbol{x}(t_f)$ is linear in $\Delta\tau_1$, it follows immediately from Eq. (28) that

$$\frac{\partial \boldsymbol{x}(t_f)}{\partial\tau_1} = \int_0^{t_f} e^{\boldsymbol{A}(t_f-\xi)}\left(\frac{\partial \boldsymbol{A}}{\partial\tau_1}\boldsymbol{x}(\xi) + \frac{\partial \boldsymbol{B}}{\partial\tau_1}u(\xi)\right)d\xi. \quad (29)$$

Defining $\boldsymbol{S}_{\boldsymbol{\tau_1}}$ as the sensitivity of $\boldsymbol{x}(t_f)$ with respect to τ_1, that is

$$\boldsymbol{S}_{\boldsymbol{\tau_1}} = \frac{\partial \boldsymbol{x}(t_f)}{\partial\tau_1}, \quad (30)$$

Eq. (29) shows that $\boldsymbol{S}_{\boldsymbol{\tau_1}}$ can easily be computed by integrating

$$\dot{\boldsymbol{S}}_{\boldsymbol{\tau_1}} = \boldsymbol{A}\boldsymbol{S}_{\boldsymbol{\tau_1}} + \left(\frac{\partial \boldsymbol{A}}{\partial\tau_1}\boldsymbol{x} + \frac{\partial \boldsymbol{B}}{\partial\tau_1}u\right) \quad (31)$$

with the initial condition $\boldsymbol{S}_{\boldsymbol{\tau_1}}(0) = 0$.

4.2.2 Evaluation of $\partial \boldsymbol{x}(\boldsymbol{t_f})/\partial \boldsymbol{\tau_2}$

Similarly the sensitivity $\boldsymbol{S}_{\boldsymbol{\tau_2}} = \partial\boldsymbol{x}(t_f)/\partial\tau_2$ is the solution of

$$\dot{\boldsymbol{S}}_{\boldsymbol{\tau_2}} = \boldsymbol{A}\boldsymbol{S}_{\boldsymbol{\tau_2}} + \left(\frac{\partial \boldsymbol{A}}{\partial\tau_2}\boldsymbol{x} + \frac{\partial \boldsymbol{B}}{\partial\tau_2}u\right) \quad (32)$$

with the initial condition $\boldsymbol{S}_{\boldsymbol{\tau_2}}(0) = 0$.

4.2.3 Evaluation of $\partial \boldsymbol{x}(\boldsymbol{t_f})/\partial \boldsymbol{K}$

In a similar way, taking into account that $\partial\boldsymbol{A}/\partial K = 0$ (see Eq. (4)), $\boldsymbol{S_K}$ is the solution of

$$\dot{\boldsymbol{S}}_{\boldsymbol{K}} = \boldsymbol{A}\boldsymbol{S}_{\boldsymbol{K}} + \frac{\partial \boldsymbol{B}}{\partial K}u \quad (33)$$

with the initial condition $\boldsymbol{S_K}(0) = 0$.

4.3 Normalization of sensitivities

The sensitivities defined above are expressed in engineering units. For example, $\dot{\theta}$ and τ_1 are given in ºC/s and s, respectively; consequently $\partial\dot{\theta}/\partial\tau_1$ will be given in ºC/s^2, which is a unit of not very common use and that may require some effort to be interpreted.

Therefore, it is convenient to normalize sensitivities by adopting a reference value for each variable. Table 1 contains a suggestion for the choice of these values.

Table 1. Reference values for normalization

Variable	Reference
θ	Value of θ_f
$\dot{\theta}$	Value of $\bar{\theta}[0, t_f]$
τ_1	Value of τ_1
τ_2	Value of τ_2
K	Value of K

The symbol $\bar{\theta}[0, t_f]$ represents the average value of θ in the interval $[0, t_f]$.

Using an over hat to indicate the normalized value of a variable, the normalized sensitivities are then given by

$$\frac{\partial\hat{\theta}(t_f)}{\partial\hat{\tau}_1} = \frac{\partial\theta(t_f)}{\partial\tau_1}\frac{\tau_1}{\theta_f} \quad (34)$$

$$\frac{\partial\hat{\theta}(t_f)}{\partial\hat{\tau}_2} = \frac{\partial\theta(t_f)}{\partial\tau_2}\frac{\tau_2}{\theta_f} \quad (35)$$

$$\frac{\partial\hat{\theta}(t_f)}{\partial\hat{K}} = \frac{\partial\theta(t_f)}{\partial K}\frac{K}{\theta_f} \quad (36)$$

$$\frac{\partial\hat{\dot{\theta}}(t_f)}{\partial\hat{\tau}_1} = \frac{\partial\dot{\theta}(t_f)}{\partial\tau_1}\frac{\tau_1}{\bar{\theta}[0,t_f]} \quad (37)$$

$$\frac{\partial\hat{\dot{\theta}}(t_f)}{\partial\hat{\tau}_2} = \frac{\partial\dot{\theta}(t_f)}{\partial\tau_2}\frac{\tau_2}{\bar{\theta}[0,t_f]} \quad (38)$$

$$\frac{\partial\hat{\dot{\theta}}(t_f)}{\partial\hat{K}} = \frac{\partial\dot{\theta}(t_f)}{\partial K}\frac{K}{\bar{\theta}[0,t_f]} \quad (39)$$

To illustrate, assume that $\tau_1 = 180$ s and $\theta_f = 120$ ºC, for example, and that a variation of 3.6 s in τ_1 corresponds to a variation of 1.2 ºC in θ $\theta(t_f)$. In this case, $\partial\theta(t_f)/\partial\tau_1 = 0.333$ ºC/s, which means that a variation of 1 s in τ_1 corresponds to a variation of 0.333 ºC in θ_f. On the other hand, $\partial\,\hat{\theta}(t_f)/\partial\hat{\tau}_1 = 0{,}5$, which means that a variation of 1% in τ_1 corresponds to a variation of 0.5% in $\theta(t_f)$. The latter is clearly a measure of immediate interpretation, which may not the case with sensitivity in engineering units.

5. TRACKING ROBUSTNESS IN CLOSED-LOOP

First of all, it should be recalled that path-planning is generally based on a nominal model of the system to be controlled. In order to guarantee that the system operates satisfactorily despite the uncertainties to which it is subject, the tracking of the planned path is usually carried out using a closed-loop control system. Since external disturbances and errors in the dynamic model in general are not taken into account in the planning, the robustness of the system performance is thus a key point in practice.

The procedure for assessing the robustness of the closed-loop control system when subject to uncertainties in the model parameters is introduced in this section.

A closed-loop with a PID controller with anti-windup is considered in this section, since it is widely used in industry (Astrom and Hagglund, 1995). More specifically, we adopted the PID-SIMC tuning rules proposed by Skogestad and Postlethwaite (2005), which claim that they are "probably the best simple PID tuning rules in the world".

In the case of minimum-time path-planning, the resulting control law is of bang-bang type, which means that the control variable switches between its minimum and maximum allowable values at well defined instants of time. Thus, the closed-loop controller has no margin of action when the sum of the control variable computed by it and the optimal nominally planned one tends to violate such extreme allowable values.

Using a modified version of Monte Carlo simulations as described in the following, the objective here is to assess the robustness of the closed-loop control system when subject to uncertainties in the model parameters. Uncertainties of different magnitudes (ε) are considered to affect the three parameters of the model, namely, K, τ_1 and τ_2. More specifically, this means that if $\bar{p}$ represents the nominal value of a parameter p, its value is a pseudo-random number in the range $[(1-\varepsilon)\bar{p}, (1+\varepsilon)\bar{p}]$. In addition to the pseudo-random points, the modified form of the Monte Carlo method used here considers also points uniformly distributed in the same interval for each parameter.

For the study of closed-loop robustness, paths are planned considering maximum values of heating power lower than the one available in the heater. Let δ, $0 < \delta < 1$, be such power margin. Then path-planning is computed considering that the maximum heating power is $u_{max,p}$,

$$u_{max,p} = (1-\delta)\, u_{max} \,, \tag{40}$$

where u_{max} is the maximum power of the heater.

It is expected that as δ increases, the larger can be the uncertainties that affect the process model without compromising the system performance – at the expense, of course, of longer final times. In addition, it is also expected that the larger the uncertainty ε in the parameter values, the larger will be the control effort of closed-loop controller.

6. SIMULATIONS AND RESULTS

Both, the minimum-time control law computation script and closed-loop simulator were created in the MATLAB/Simulink® environment.

Table 2 shows the numerical values of the parameters used in what follows. Additionally, the following values were used: for the power margin δ: 5%, 10%, 15% and 20% and, for parametric uncertainty, ε: 1%, 5%, 10% and 15%.

Table 2. Numerical values of the parameters

Parameter	Value
K	0.5 ºC/W
τ_1	30 s
τ_2	180 s
n	100
θ_f	120 ºC
$u_{máx}$	500 W

6.1 Minimum-time path-planning

Using the procedure presented in Section 3, the minimum-time path was computed. The time required to reach the desired temperature was 162.11 s. Figure 1 shows the optimal control law obtained, which is clearly a bang-bang one.

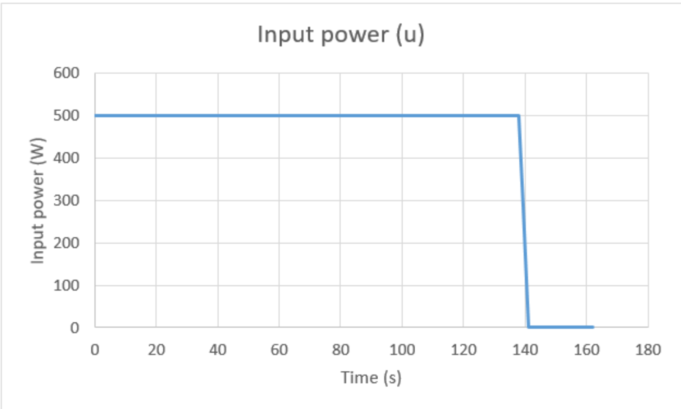

Figure 1. Optimal control law $u(t)$.

The plots of the jaw temperature and of its rate of change versus time are shown in Figure 2, respectively at left and right. As one can see, the terminal conditions given by Eqs. (13) and (14) were both satisfied.

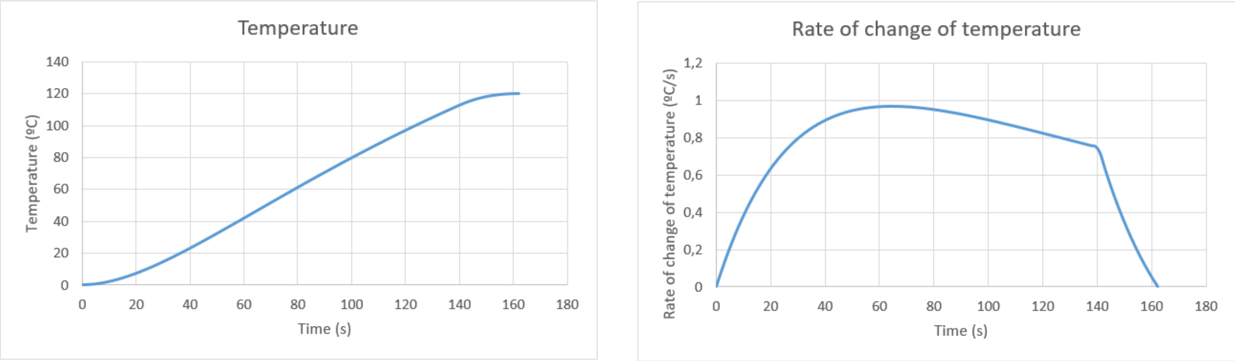

Figure 2. Optimal temperature path (left); rate of change of temperature (right).

6.2 Sensitivities computation

Normalized sensitivities were computed according to the procedure described in Section 4. The results are shown in Table 3.

Table 3. Numerical values of normalized sensitivities

Sensitivities	Value
$\partial\hat{\theta}(t_f)/\partial\hat{\tau}_1$	-0.14
$\partial\hat{\theta}(t_f)/\partial\hat{\tau}_2$	-0.65
$\partial\hat{\theta}(t_f)/\partial\widehat{K}$	1
$\partial\hat{\dot{\theta}}(t_f)/\partial\hat{\tau}_1$	0.74
$\partial\hat{\dot{\theta}}(t_f)/\partial\hat{\tau}_2$	0.58
$\partial\hat{\dot{\theta}}(t_f)/\partial\widehat{K}$	~ 0

From Table 3 it can be noticed that the final temperature $\hat{\theta}(t_f)$ is on the order of 5 times more sensitive to variations in the largest time constant of the system $\hat{\tau}_2$ than to its smallest one $\hat{\tau}_1$. Table 3 also shows that variations in the gain K are fully reflected in the final temperature.

It can also be noticed from Table 3 that the sensitivities of the final rate of change in temperature $\hat{\dot{\theta}}(t_f)$ with respect to both $\hat{\tau}_1$ and $\hat{\tau}_2$ are relatively close to each other, whereas $\hat{\dot{\theta}}(t_f)$ is practically insensitive to variations in the gain K.

6.3 Closed-Loop Simulations

Initially, different values were considered for the power margin δ – namely, 5%, 10%, 15% and 20% – and the minimum-time values were computed for each one of them (see Figure 3).

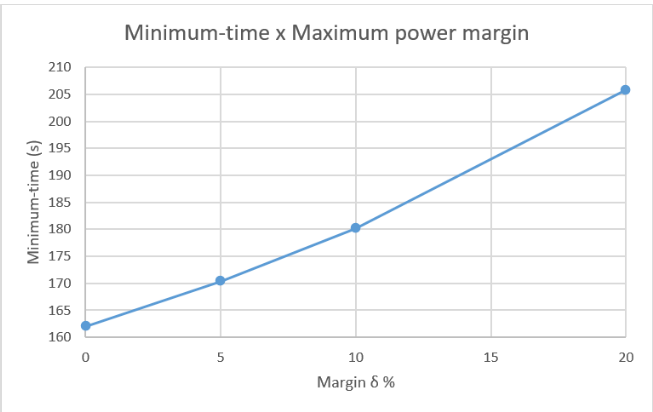

Figure 3. Minimum-time as a function of the power margin δ.

As expected, the minimum time is an increasing function of the power margin δ. Notice, for example, that, for $\delta =$ 5%, the increase in minimum time is close to 5%.

To assess closed-loop robustness the four values of δ above for the path-planning and various levels of parametric uncertainties ε – namely, 1%, 5%, 10% and 15% – were considered.

First of all, to simplify the exposition, in correspondence to each value of ε let π_ε denote the parallelepiped centered at the nominal point (K, τ_1, τ_2) with edges of lengths $2\varepsilon K$, $2\varepsilon\tau_1$ and $2\varepsilon\tau_2$, respectively.

For each pair (δ, ε), the following simulations were carried out: i) with 3 points uniformly distributed points at each edge of the parallelepiped π_ε; ii) the same procedure was repeated for 4, 5 and 6 points at each edge; iii) additionally 13 distinct Monte Carlo simulations containing from 9 to 250 pseudo-random points in π_ε were also executed. This procedure is what we have called modified Monte Carlo simulation.

Figure 4 (left) shows the results obtained for the maximum error (that is, the worst case) between the final temperature reached and the desired one. Figure 4 (right) shows the rate of change of temperature at the final instant (which, ideally, should be zero) for the worst case.

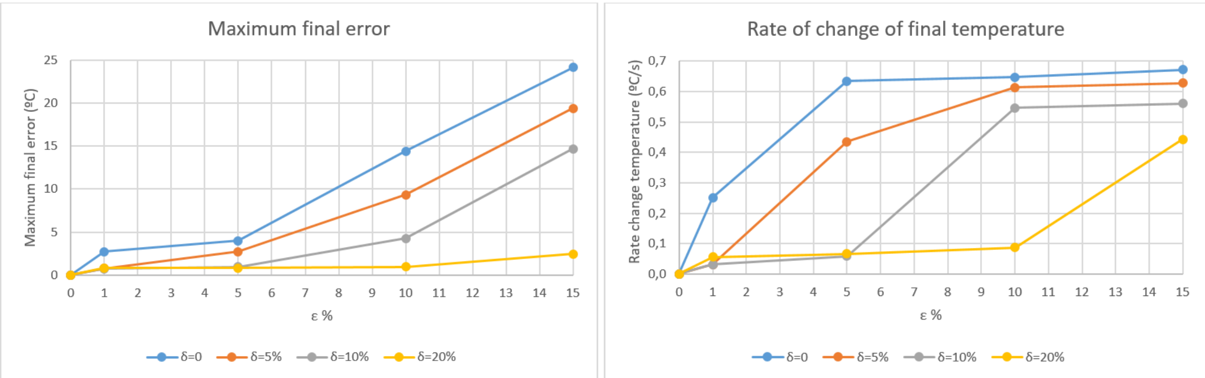

Figure 4. Maximum final error (left) and rate of change of final temperature (right) for different power margins and errors in the model parameters.

As expected, Figure 4 (left) shows that the final errors become smaller as δ increases, which points out an improvement in the robustness of the system's performance. As expected too, it can also be seen that when $\delta = 0$, due to the absence of power margin, the performance of the closed-loop system is severely affected by errors in the values of the model parameters. In other words, the closed-loop system exhibits a low performance robustness. It is interesting to notice that, for both $\varepsilon = 1\%$ and $\varepsilon = 5\%$, the maximum final errors for δ =10% and δ =20% are practically the same and relatively small. This fact means that a path-planning carried out with a power margin δ of 10% presents a good robustness of performance for errors in the parameters of up to 5%. The price to be paid for this robustness, however, is an increase of around 11% in the value of the minimum-time.

The conclusions taken from Figure 4 (right) are the same as the previous ones obtained from the observation of Figure 4 (left).

7. CONCLUSIONS

The numerical tests carried out showed that the procedure of computing the minimum-time control law by using LP can be simple and effective to solve the optimal path-planning problem for the temperature control of a sealing machine. In addition to the efficiency of the LP algorithm, the convergence in a few steps of the binary search method –usually less than 10 steps – contributed to this.

The computation of the normalized sensitivities as proposed can be useful since it allows the analyst to objectively assess which are the parameters of the model that must be identified with greater accuracy.

Due to the fact that the optimal control law is of bang-bang type – hence always assigning extreme values to the control variable –, the study of closed-loop robustness carried out through a modified Monte Carlo simulation clearly showed the implications of such characteristic in the performance degradation of the planned path tracking as the parameters of the model become more uncertain. The present work also showed that the path-planning carried out with a power margin at the heater can significantly reduce this degradation at the expense of an increase in the minimum-time planned.

To close this section, it is clear that the proposed procedure for sensitivity evaluation could be used in general whenever the dynamical model is linear. In the same way, closed-loop robustness assessment for tracking of minimum-time planned paths could be done following the same lines of Section 5. Furthermore it should be emphasized that the computation of the minimum-time control law through the LP based method could be extended to other problems where the maximum value of the controlled variable that can be reached – maximum range – is an increasing function of the time span.

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9. RESPONSIBILITY NOTICE

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