COBEM 2021 26th International Congress of Mechanical Engineering
November 22-26, 2021, Virtual Congress, Brazil

COB-2021-0419
COMPARISON AND ENHANCEMENT OF TURBULENCE CLOSURE MODELS THROUGH PIPES USING WALL FUNCTIONS

U. C. Bitencourt
I. F. Pinheiro
Universidade Federal Fluminense (UFF). Rua Passo da Pátria, 156, São Domingos, Niteroi – RJ.
umbertobitencourt@id.uff.br
isabelaflorindo@id.uff.br

Abstract. *A comparison investigation is proposed in this paper through the study and prediction of the pressure drop along a circular pipe by means of different turbulence closure models. In this study, the performance of the Prandtl's Mixing Length and the k-ω model are studied in an incompressible fully developed turbulent pipe flow, where the equations derived in the former are implemented with a MATLAB code, while the latter is carried out by the ANSYS Fluent software. A wall function covering the viscous sublayer is brought into play to improve the accuracy of the Prandtl Mixing Layer Model. Furthermore, as a way of comparing and validating the results, the Colebrook's equation for the Darcy's friction factor will be taken as a reference. Errors as low as 1 % can be obtained if the damped mixing length is used alongside a wall function and an appropriate thickness of the viscous sublayer.*

Keywords: *Turbulent Flows, Wall Functions, Turbulence closure models, ANSYS Fluent.*

1. INTRODUCTION

Turbulent flows can be observed in everyday phenomena. They are characterized for random fluctuations of velocity, pressure and temperature. The rising of a cigarette plume, the air around an airfoil and the flow of a fluid inside a pipe are examples of turbulent flows.

There are many different turbulence models available nowadays, analytical, empirical and semi-empirical, each one with its own advantages and disadvantages (Sadrehaghighi, 2018). Lim et al (2018) made a comparison between the RNG k-ε, the SST k-ω, the RSM and the analytical Colebrook's equation, using ANSYS Fluent for all three methods and concluded that the RNG gets the closest to the analytical value. Zagustin and Zagustin (1969) solved analytically the velocity profile of a turbulent flow in a pipe making use of a pulsation energy balance to find the mixing length, achieving results that fit very well to experiments, however without using wall functions. Sabdenova and Erzada (2013) presented a complex analysis of the Prandtl Mixing Length model in cartesian coordinates using semi-empirical equations.

All the turbulence models are subject to errors because they are approximations of the real flow. To improve the accuracy of these methods, some functions can take advantage of the near-laminar behavior of the fluid very close to the walls. The literature mentions how well the wall functions fit to the experimental data, especially when it comes to the k-ε model (Lars, 2020). Applications of this technique to the k-ε model are widely available in the literature yet papers with the implementation of this approach to the Prandtl's Mixing Length Model are sparse in the. For that reason, they are worth a try.

Based on previously numerical studies into turbulent models, the purpose of this paper is to estimate the pressure drop of an incompressible fully developed turbulent flow along a circular pipe using two turbulence models, the Prandtl Mixing Length and k-ε model. There two methods are appropriate for an analysis because the first is very simple and its potential is always neglected and the latter is the most classic one. The Prandtl Mixing Length is implemented and discretized by a MATLAB code-based finite difference method and k-ε model is simulated in the ANSYS Fluent. In order to improve the numerical data given by both methods, wall functions are applied to a portion of the flow. Furthermore, the comparison is established in terms of a relative error by calculating the pressure drop of the two models evaluated with the friction factor given by the Colebrook's equation.

The approximations involved in the development of the Prandtl Mixing Length itself are limited. It's not valid at the wall and in the region very close to it, for example. Nevertheless, it's interesting to see how much the wall functions and different mixing lengths expressions can decrease the error associated with this approximation. Its main attractive feature is definitely its very low time-consuming simulations, with plausible errors depending on the mixing length expression used.

2. SIMPLIFICATIONS AND BOUNDARY CONDITIONS

All mathematical formulation will be applied to the problem shown in Fig. 3. Therefore, it's already expected that the equations will be derived in cylindrical coordinates. The average velocities in the θ and r directions will be set to zero. For the boundary conditions, it will be considered no slip at the walls (Eq.4) and symmetry of the velocity profile at $r = 0$ (Eq. 5). The flow will be assumed to be completely developed and at steady state. Finally, to determine the pressure drop along the tube, an iterative procedure is used. This procedure consists in adjusting the velocity profile found via any method, and consequently the pressure drop, until it meets the required Eq. 6.

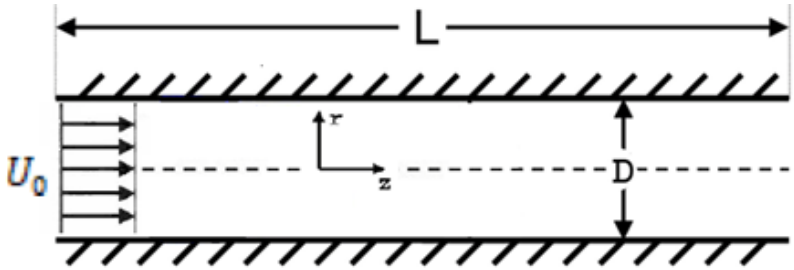

Figure 1: Geometry used to apply the models developed

$$U_z(r = R) = 0 \tag{1}$$

$$\left.\frac{\partial U_z}{\partial r}\right|_{r=0} = 0 \tag{2}$$

$$2\pi \int_0^R U_z r dr = U_0(\pi R^2) \tag{3}$$

3. MATHEMATICAL MODELS

The two turbulence closure models that will be presented in the following subsections are the Prandtl Mixing Length and the k-e models. For the Prandtl Mixing Length the Equations are derived and solved numerically, although for the k-ε model the ANSYS Fluent will be used. Wall functions will also be presented as a way of improving the accuracy of the Prandtl Mixing Length and for the k- ε model, which will be solved using ANSYS Fluent, the standard wall functions will be utilized.

3.1. PRANDTL MIXING LENGTH MODEL

The Prandtl Mixing Length model is the most basic one and is often introduced as a first look at turbulence modeling. Prandtl used Boussinesq's idea of thinking that the turbulence effects of momentum transfer would increase the effect of the viscosity. He found inspiration in how successfully the kinetic theory of gases could predict their viscosity. The only difference is that instead of using the mean free path between molecules, Prandtl created the idea of a "Mixing Length (l)". The Mixing Length is defined as the length that produces the velocity fluctuations given in a real turbulent flow (Souza et al, 2011). To find the velocity profile of a turbulent flow using this approach, the differential equation that has to be solved is:

$$-Kr = \frac{d}{dr}\left(\nu r \frac{dU_z}{dr} - \left(\frac{dU_z}{dr}\right)^2 r l^2\right) \tag{4}$$

Where K is a positive constant defined as $-\frac{1}{\rho}\frac{dp}{dz}$ and it's related to the pressure (p) drop along the length of the pipe, ρ and ν are the fluid density and kinematic viscosity respectively, u is the averaged velocity in the z direction.

The literature recommends different approaches to calculate the Mixing Length (l). Davidson (2014) suggests Eq. 5, where k is the Von Karman constant and it's equals to 0.41. Van Driest proposed that the mixing length given by Prandtl's Eq. should have a dampening effect (Wilcox, 2006) as given by Eq. 6. Where $y^+ = V_*\left(\frac{R-r}{\nu}\right)$ is the nondimensionalized distance from the wall and $V_*^2 = \frac{\tau_w}{\rho}$, where τ_w is the shear stress at the wall and V_* is called the friction velocity.

$$l = k(R - r) \tag{5}$$

$$l = k(R - r)(1 - e^{-0.0385y^+}) \tag{6}$$

The main difference between these two models is that Eq. 5 is not valid very close to the wall, while Eq. 6 is. After a certain distance from the wall, both equations are the same.

3.2. WILCOX k-ω MODEL

The $k-\omega$ model can be partially derived from the Navier-Stokes Equations. It is also a statistical method, like the Prandtl Mixing Length, since it uses Eq. 1 and a time average for its building up. This method consists of solving two equations simultaneously. An equation for k, which is the kinetic energy of the turbulent part of the flow and an equation for ω, which is the specific energy dissipation. The Equation for ω is completely empirical, but its form is guided by some physical reasoning (Wilcox, 2006). This is one of the most used models for engineering simulation of flows and its definition is more appropriate for flow in pipes when compared to the $k-\varepsilon$ model.

The turbulence viscosity in this model is defined as:

$$\nu_t = \frac{k}{\widetilde{\omega}}, \qquad \widetilde{\omega} = \max\left(\omega, C_{lim}\sqrt{\frac{2\boldsymbol{S}:\boldsymbol{S}}{\beta^*}}\right), \qquad C_{lim} = \frac{7}{8} \tag{7}$$

The Eqs. for k and ω are defined as:

$$\frac{\partial k}{\partial \mathrm{t}} + \boldsymbol{U}\nabla k = -tr\left(\overline{\boldsymbol{u}'\otimes\boldsymbol{u}'}\nabla\boldsymbol{U}\right) - \beta^* k\omega + \nabla\cdot\left(\left(\nu + \sigma^*\frac{k}{\omega}\right)\nabla k\right) \tag{8}$$

$$\frac{\partial \omega}{\partial \mathrm{t}} + \boldsymbol{U}\nabla\omega = -\alpha\frac{\omega}{k}tr\left(\overline{\boldsymbol{u}'\otimes\boldsymbol{u}'}\nabla\boldsymbol{U}\right) - \beta\omega^2 + \frac{\sigma_d}{\omega}\nabla k\cdot\nabla\omega + \nabla\cdot\left(\left(\nu + \sigma\frac{k}{\omega}\right)\nabla\omega\right) \tag{9}$$

The closure coefficients are given by:

$$\alpha = \frac{13}{25}, \qquad \beta = \beta_0 f_\beta, \qquad \beta^* = \frac{9}{100}, \qquad \sigma = \frac{1}{2}, \qquad \sigma^* = \frac{3}{5}, \qquad \sigma_{do} = \frac{1}{8} \tag{10}$$

$$\sigma_d = \begin{cases} 0, & \nabla k\cdot\nabla\omega \leq 0 \\ \sigma_{do}, & \nabla k\cdot\nabla\omega > 0 \end{cases} \tag{11}$$

$$\beta = 0.0708, \qquad f_\beta = \frac{1+85\chi_\omega}{1+100\chi_\omega}, \qquad \chi_\omega = \left|\frac{\left(\Omega_{\mathrm{ij}}\Omega_{\mathrm{jk}}S_{ki}\right)}{(\beta^*\omega)^3}\right|, \qquad l = \frac{k^{1/2}}{\omega} \tag{12}$$

$$\boldsymbol{\Omega} = \frac{1}{2}(\nabla\boldsymbol{U} - (\nabla\boldsymbol{U})^T), \qquad \boldsymbol{S} = \frac{1}{2}(\nabla\boldsymbol{U} + (\nabla\boldsymbol{U})^T) \tag{13}$$

3.3. WALL FUNCTIONS

In a turbulent flow, the existence of walls causes interesting effects. The most well-known is the no-slip condition that has to be imposed to the fluid at a stationary wall. Another feature at the wall is that it possesses the highest velocity gradient and consequently the wall is the location of the highest shear stress. For those distinct features, the near-wall effects require special treatment (Pope, 2000). The idea consists of solving the basic model together with the wall functions, each of them used for a different region of the flow.

The literature (Davidson, 2014) suggests the following:

$$U = V_*^2\frac{R-r}{\nu}, \ for\ y^+ < s \tag{14}$$

Where $\eta = \frac{R-r}{R}$ is another nondimensionalized function for the distance from the wall. To this portion of the flow is given the name of Viscous Sublayer and there is no common agreement for the value of s in the Eq. 14. What is understood is that it represents a region of a turbulent flow very close to the wall at which the turbulent effects are negligible and its velocity profile can be considered to be linear (as expressed in the Eq. 14). Wood (2007) suggests that s must be somewhere in between 5 – 8, Pope (2000) says that the viscous sublayer extends itself from the wall until $y^+ = 5$. Numerical experiments (Kim et al, 1987) by using the DNS (Direct Numerical Simulation) method, which is done by solving the Navier Stokes Eq. without further simplifications, theorize that s is approximately equals to 4.

3.4. COLEBROOK EQUATION FOR THE PRESSURE DROP

The most accepted method to evaluate the pressure drop along a pipe is the Colebrook Equation. It is used for turbulent flows and its results fit very well with the experimental data (Çengel and Cimbala, 2015). Since this expression is a standard, it will be used to compare the results found using the Prandtl Mixing Length and the $k-\omega$ models. The Colebrook's Equation is:

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{\epsilon/D}{3{,}7} + \frac{2{,}51}{Re\sqrt{f}}\right) \tag{15}$$

Where f is Darcy's friction factor, ϵ is the equivalent rugosity of the pipe and Re is the Reynolds number of the flow. After finding f, the pressure drop (ΔP) can be found by:

$$\Delta P = \frac{fL}{D}\frac{\rho U_0^2}{2} \tag{16}$$

3. PROBLEM SETUP

The solutions for the flow in a circular pipe will be obtained for a frictionless pipe with various inlet velocities. The fluid considered is water at a constant temperature of 25 °C. The approach used for each method is given in the following subsections. The computer utilized for the simulations is a Lenovo, with 16GB of RAM and 4 processors. This was the maximum computational power at hand.

4.1. ANSYS SETUP AND MESH

The ANSYS Fluent, student version of 2019 was used to simulate the flow. The mesh of a cross section is shown in Fig. 2. The double precision option was selected and all 4 processors were dedicated to the calculations. The portion of the pipe next to the wall is biased, which means that an inflation was applied and the mesh is finer in that region. An inflation is necessary to capture the increased velocity gradients at the wall without making the mesh in the whole geometry high and prejudice the computational time. The maximum length of a meshing element was defined as 0.01 m and the number of layers next to the walls was set as 20. The stop criteria is an error of 10^{-5}.

At the inlet of the pipe, a uniform velocity profile was used as a boundary condition. This velocity profile has a single component in the longitudinal direction. The turbulence intensity in the inlet was defined as 3%. Additionally, the pipe was assumed to be smooth so as to be consistent with the Prandtl Mixing Length Theory.

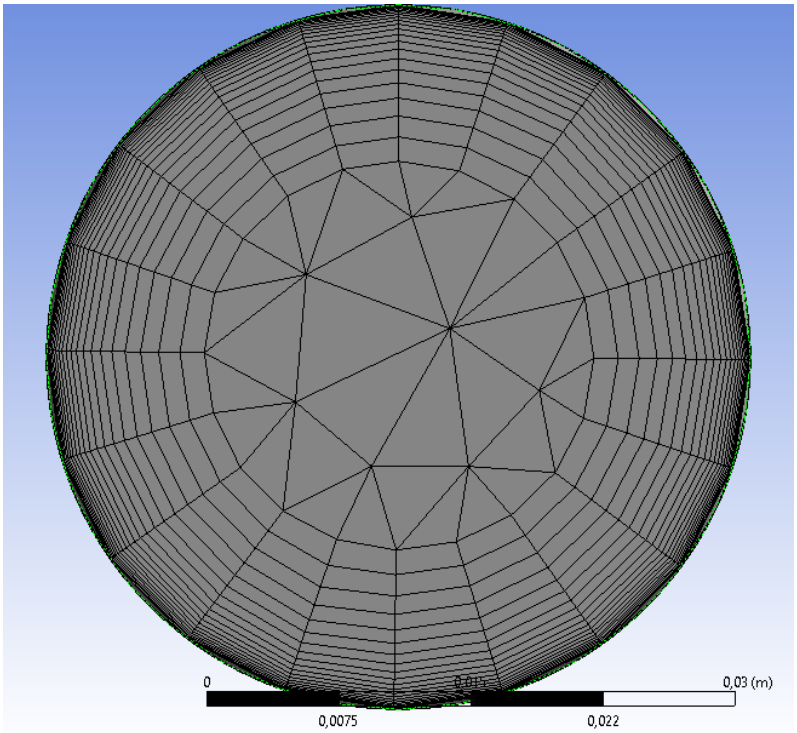

Figure 2: Pipe mesh

4.2. MATLAB SETUP

Before creating the numerical code to solve Eq. 4 some manipulation must be done to decrease the computational power required for its solution. Integrating both sides of Eq. 4:

$$-\frac{Kr^2}{2}+c_1=\nu r\frac{dU_z}{dr}-\left(\frac{dU_z}{dr}\right)^2 rl^2 \tag{17}$$

The constant c_1 can be determined by making $r=0$:

$$c_1=0 \tag{18}$$

Substituting Eq. 17 into 18:

$$-\frac{Kr^2}{2}=\nu r\frac{dU_z}{dr}-\left(\frac{dU_z}{dr}\right)^2 rl^2 \tag{19}$$

Eq. 19 can be solved for $\frac{dU_z}{dr}$ as:

$$\frac{dU_z}{dr}=\frac{\nu}{2l^2}\pm\frac{1}{2}\sqrt{\frac{\nu^2}{l^4}-2\frac{Kr}{l^2}}=\frac{\nu}{2l^2}\left(1\pm\sqrt{1-2\frac{Krl^2}{\nu^2}}\right) \tag{20}$$

Where the symmetry boundary condition (Eq. 2) can be satisfied only if the solution with the minus signal is chosen:

$$\frac{dU_z}{dr}=\frac{\nu}{2l^2}\left(1-\sqrt{1-2\frac{Krl^2}{\nu^2}}\right) \tag{21}$$

The mixing length (l) may be equal to zero (at the wall for example), which means that, at the wall, Eq. 21 has a singularity. To remove it, Antonialli and Silveira (2018) suggest the following manipulation:

$$\frac{dU_z}{dr}=\frac{\nu}{2l^2}\left(1-\sqrt{1+2\frac{Krl^2}{\nu^2}}\right)\frac{\left(1+\sqrt{1+2\frac{Krl^2}{\nu^2}}\right)}{\left(1+\sqrt{1+2\frac{Krl^2}{\nu^2}}\right)} \tag{22}$$

Simplifying Eq. 22:

$$\frac{dU_z}{dr}=\frac{-\frac{\nu}{l^2}\frac{Krl^2}{\nu^2}}{\left(1+\sqrt{1+2\frac{Krl^2}{\nu^2}}\right)}=\frac{-Kr}{\nu\left(1+\sqrt{1+2\frac{Krl^2}{\nu^2}}\right)} \tag{23}$$

Eq. 23 is a nonlinear ordinary differential equation that doesn't have an analytical solution, for this reason it must be solved numerically. The method that will be utilized here is Euler's Method. The numerical accuracy of this method increases if one uses a very low length step (dr), which means that the number of divisions of the grid will be high.

Forward differences are used to discretize the left side of Eq. 23:

$$\frac{dU_z}{dr}=\frac{U_{z,i+1}-U_{z,i}}{dr} \tag{24}$$

Eq. 24 is iterated from the wall to the center of the pipe. For this reason:

$$U_{z,i}=U_{z,i+1}+dr\frac{Kr_i}{\nu\left(1+\sqrt{1+2\frac{Kr_i l_i^2}{\nu^2}}\right)} \tag{25}$$

Finally, to satisfy the inlet condition, and consequently the mass conservation, the integral of the velocity profile must be performed and the value of K has to be adjusted until the criterion in Eq. 26 is met.

$$2\pi\int_0^R U_z r dr=U(\pi R^2) \tag{26}$$

The finite difference algorithm applied to solve Eq. 4 was the Euler method and it was implemented on Matlab. The value of dr utilized was 10^{-6}. The convergence was achieved very easily and depending on the expression for the mixing length the simulation can take more time. A summary of the time taken at each procedure is shown in Table 1.

To start the simulation the Eq. 20 is solved for a given l expression. With the velocity profile obtained, wall functions come into play. A portion of the velocity profile very close to the wall is said to behave linearly (as given by Eq. 14). This way the flow has a linear part and the rest of the domain must be obtained once again using Eq. 25, however, for a region that excludes the portion covered by the wall function. The boundary condition for Eq. 25 is now a little bit different. Since Eq. 25 it's not applied at the wall anymore, the specified velocity boundary condition is now given at $r = R - s$. Its value is:

$$U(r = R - s) = V_*^2 s/\nu \tag{27}$$

Ultimately, the condition given by the Eq. 26 must be satisfied once again. Now, the velocity profile is given by Eq. 14 for the region comprised from $r = R - s$ to $r = R$ and it's represented by the solution of Eq. 25 from $r = 0$ to $r = R - s$. Thus, the integration must be performed for these two regions separately and then added together. Since there are no agreements at which the value of s is appropriate, in this paper all the values which the literatures read include will be tested out and their correspondent errors will be reported. These values are $s = 4, 5, 6, 7$ and 8.

4. RESULTS AND DISCUSSIONS

Figs. 3 and 4 show a comparison of the results obtained using different methods at different Reynolds Numbers. The geometry used has a diameter of $D = 40\ mm$ and a length of $L = 2000\ mm$. As mentioned before, the standard value considered to make the comparisons is the one found with the Colebrook's equation.

The first feature of the Prandtl Mixing Length model that must be observed is how the accuracy improves at higher Reynolds numbers, as it can be seen in Figure 3. The same is not true about the $k - \omega$ model.

The results obtained by the Prandtl Mixing Length expression itself (Eq. 5) are very far away from the reference value. That was actually expected since rough approximations are involved in its definition. The application of a wall function to cover the near wall region decreases that error, as it can be seen in Figures 5 and 6. A possible explanation for this improvement is the non-validity of the Prandtl mixing length expression (Eq. 5) next to the wall and on it. Thus, the wall function covers this existent gap, bringing more reliable results to the very simplistic Eq. 5.

Using Eq. 6 to describe the mixing length, instead of using Eq. 5 gives remarkable results. Errors up to around 10 % were observed when Eq. 6 is employed on solving Eq. 4. The function of the damping is exactly to fulfill the void left by the Eq. 5 in the region very close to the walls. Therefore, a polishing up of the results are expected. Furthermore, the wall functions applied to the velocity profile, found with Eq. 5, decrease the error over again.

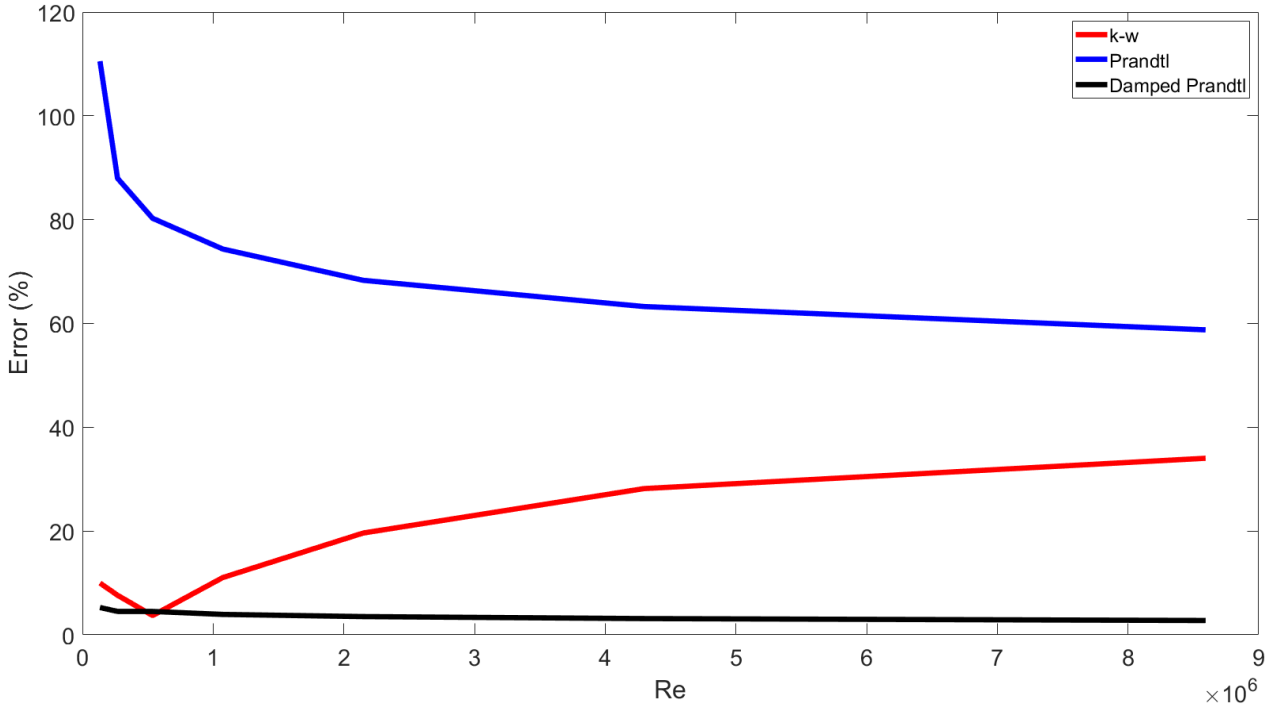

Figure 3: Comparison of different methods at different Reynolds numbers

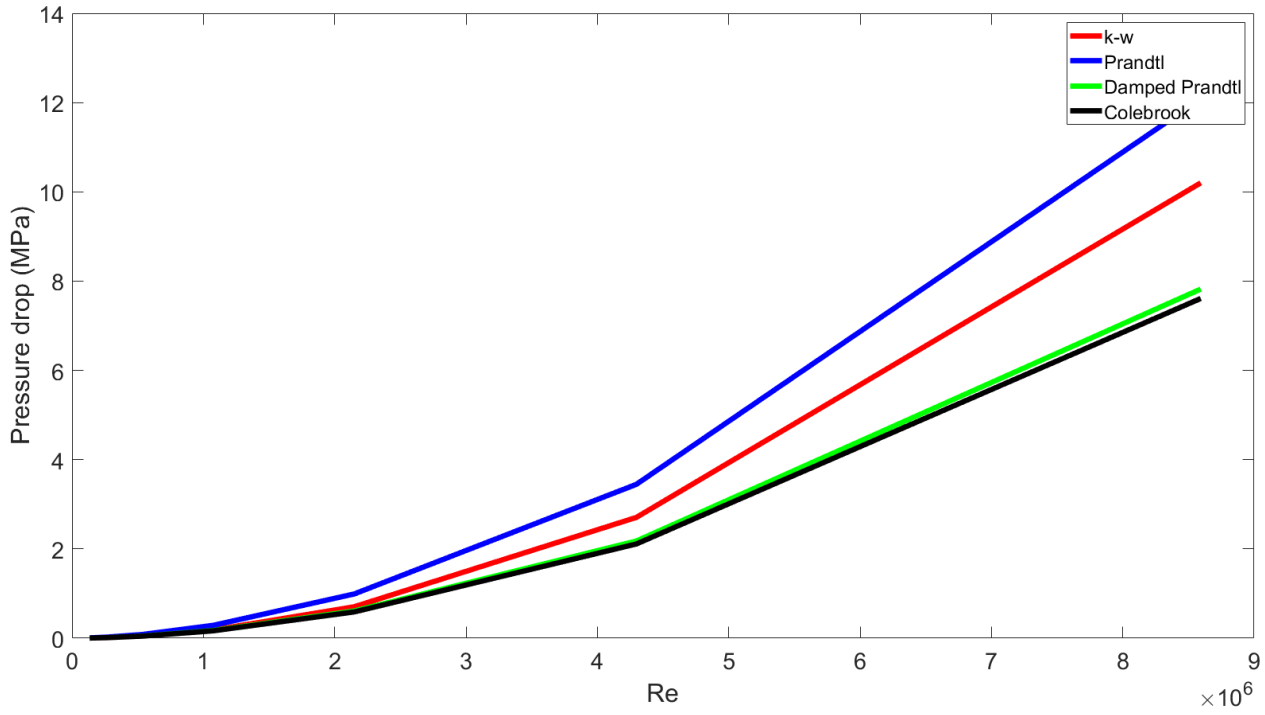

Figure 4: Pressure drop for different Reynolds numbers and models

Table 1 shows how fast it is to find the velocity profile (and consequently the pressure drop) using the Mixing Length method. On the other hand, the $k-\omega$ method solved on ANSYS Fluent can take even 15 minutes to complete their calculation routines for a single Reynolds Number. Also, $k-\omega$ models tend to be very sensitive to the grid mesh (Balaji and Prakash, 2016) which can over predict the pressure drops in the problem analyzed if the mesh is not adequate. The insertion of a wall function doesn't affect effectively the time of the simulations on Matlab.

Method	**Time (s)**
Prandtl	13
Damped Prandtl	21
$k-\omega$	780

Table 1: Comparison of the time taken to perform simulation for a single Reynolds Number

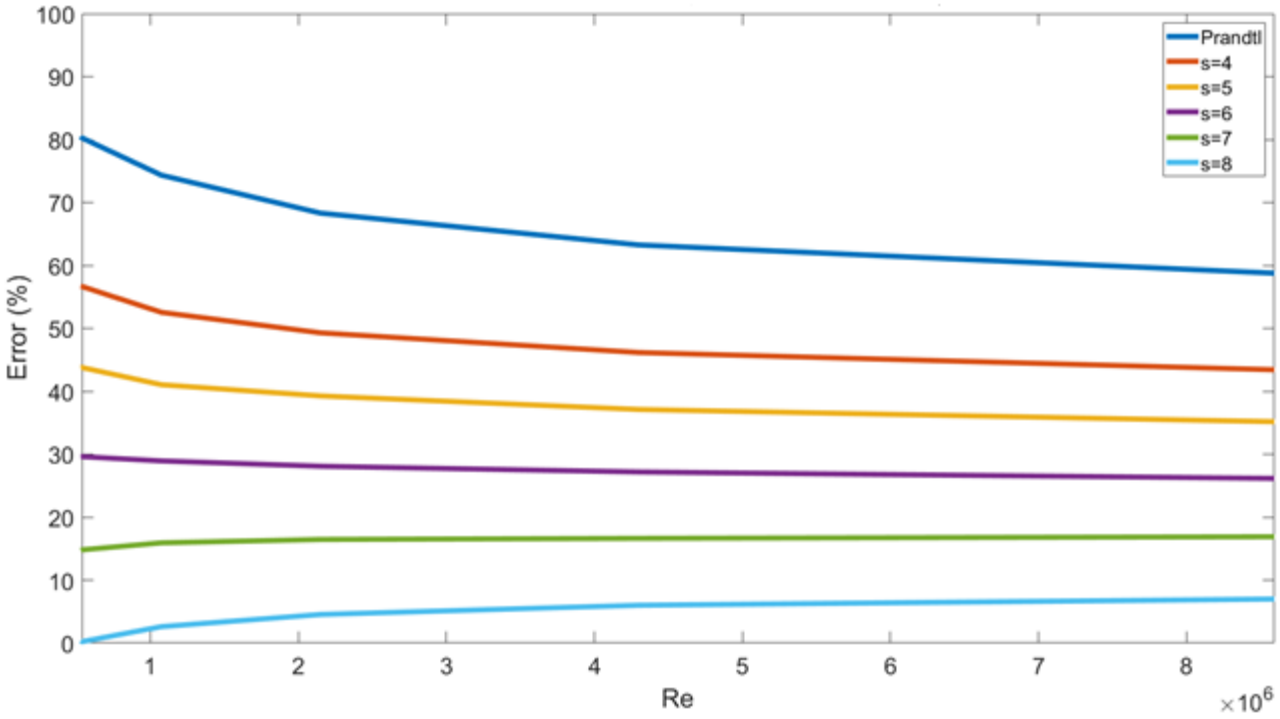

Figure 5: Comparison of the error for different thicknesses of the viscous sublayer using the Prandtl mixing length

It is seen by Figure 4 that the accuracy of the results increases by increasing the viscous sublayer thickness. That's actually expected since the Prandtl mixing length itself doesn't work quite well on the walls. For this case the wall function comes in handy to overcome its deficiency on the walls and very close to it. For example, using the mixing length by itself the error can be up to 80 %, on the other hand, by using a wall function that covers the region from the wall until $s = 8$, the error decreases to up to 8 %.

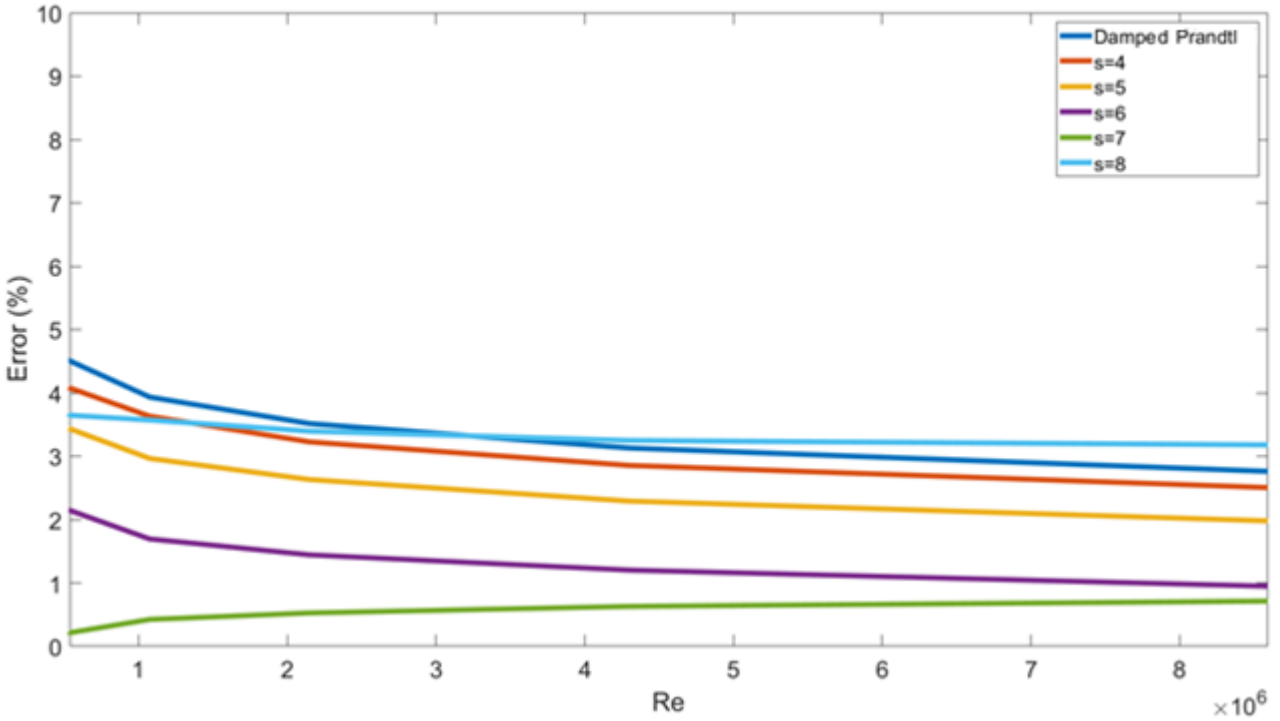

Figure 6: Comparison of the error for different thicknesses of the viscous sublayer using the damped Prandtl mixing length. Notice that the y-axis goes from 0 to 10%.

Figure 5 shows that the insertion of a wall function didn't improve the results significatively when using the damped Prandtl mixing length. This is very likely because the viscous sublayer is more suitably represented by the damping of the mixing length than by the Eq. 14. In any case, its usefulness is proved once again since the error achieves even lower values when a part of the velocity profile is assumed to be linear as in Eq. 14.

5. CONCLUSIONS

The results show that the use of wall functions improved the accuracy of the Prandtl Mixing Length model at predicting the pressure drop in a circular pipe. Errors as low as 1% can be obtained, when compared to the Colebrook Eq. This is true as long as the damped Prandtl model and an appropriate viscous sublayer thickness are utilized. The graphs presented here also show that as the Reynolds number increases the results get even better. The $k-\omega$ model is time consuming and its results are very sensitive to the grid mesh, which might generate wrong results.

The Prandtl Mixing Length, whilst being an algebraic model with crude assumptions in its derivation can be suitable to calculate the pressure drop in pipes. Its proved advantages are low time consuming, good precision in the results and easiness to implement its code. Even though it might not be appropriate for more sophisticated simulations, for the pressure drop and for didactic purposes it is definitely worth it.

6. ACKNOWLEDGEMENTS

The authors would like to acknowledge the financial support provided by the Brazilian Government Funding Agencies, CAPES, CNPq, and FAPERJ.

7. REFERENCES

Balaji, L.S.S. Prakash, 2016. CFD Analysis of a Pressure Drop in a staggered Tube Bundle for a Turbulent Cross Flow.
Çengel, Y. A. and Cimbala, J. M., 2015. Mecânica dos Fluidos, Fundamentos e Aplicações. Capítulo 8: Escoamento interno, pp. 367.
Cowen, Edwin A. Chapter 2: Canonical Turbulent Flows. Pp. 4.
Davidson, P. A. 2014. Turbulence: An Introduction for Engineers and Scientists.
Lars, D., 2020. Chapter 11: Reynolds Stress Models and two-equation Models, pp. 152.
Lim, D. C. Y., Kurnia, J. C. and Al-Kaylem, H. H., 2018. Comparison of different turbulence models in pipe flow of various Reynolds Numbers.
Pope, S. B. Turbulent Flows. Chapter 11: Reynolds Stress and Related Models, pp 433.
Recktenwald, G., 2009. The k-ε Turbulence Model. Chapter 4: The k-ε model, pp. 12.
Sabdenov, K. O. and Erzada, M., 2013. The Equations for Prandtl's Mixing Length. Frontiers in Aerospace Engineering Volume 3 Issue 2, pp. 50.
Sadrehaghighi, I., 2018. Turbulence Modeling, a Review. CFD Open Series.
Souza, J. F. A., Oliveira, L. R., Azevedo, J. L. L., Soares, I. D. and Mata, M. M., 2011. Uma Revisão Sobre a Turbulência e a sua Modelagem. Revista Brasileira de Geofísica, pp 31.
Tennekes, H., Tennekes, H. and Lumley, J. L. A First Course in Turbulence, 1972. Chapter 2: Turbulent Transport of Momentum and Heat, pp. 32.
Luigi A. Antonialli and Aristeu Silveira-Neto. 2018. Theoretical Study of Fully Developed Turbulent Flow in a Channel, Using Prandtl's Mixing Length Model. Scientific Research Publishing.

8. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.