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MATRIX METHOD FOR A STABILITY ANALYSIS OF NON-NEWTONIAN FLUID FLOW

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Abstract. *The purpose of this paper is to describe how to solve stability analysis problem for a non-Newtonian fluid flow using the Chebyshev polynomials approximation for flutuation function and their respective derivatives. In spatial analysis, eigenvalues are obtained by solving an equations system for the alpha variable where the highest alpha power is 2nd power. This formulation allows us to obtain all the eigenvalues that are solutions of the stability problem. The domain used is known as Gauss-Lobatto collocation points. Using these collocation points, it is possible to obtain all the Chebyshev polynomials with a recurrence formula facilitating this process. The non-Newtonian constitutive equation used is the Oldroyd-B model. This model admits an analytical solution for the base flow for a channel fluid flow. Using the MATLAB/OCTAVE EIG function to obtain all the eigenvalues of the problem described above, it is obtained all the solutions of the problem for the dimensionless variables which is desired. Therefore, the stability analysis is done looking for the lowest imaginary value in all eigenvalues obtained by solving the eigenproblem. The verification of this solution method is based on results present in the literature, more specifically, the article published in 2019 with the title "DNS and LST stability analysis of Oldroyd-B fluid in a flow between two parallel plates". The verification is performed by comparing values for the amplification rate of disturbances for specified flows and some neutral stability curves. It is worth mentioning that solution obtained by matrix method, in general, are less accurate if we compare to local methods. Global methods of this type are easier to perform, but as we mentioned earlier, this type of solution presents a lower accuracy compared to local methods. The stability analysis problem is not different. Solving this problem for a Newtonian fluid flow, the accuracy is not much less than a local method, but when we are solving this problem for a non-Newtonian fluid flow, that accuracy is, in general, less.*

Keywords: *Linear Stability Analysis, Chebyshev Polynomials, Matrix Method, Oldroyd-B model, Non-Newtonian Fluid Flow*

1. INTRODUCTION

Viscoelastic fluids are present in many cotidian flows such as polymers, petroleum, food, cosmetics and others, therefore, the study of the flows of these types of fluids is important for optimizing production, manufacturing, transportation, and etc.

In the literature could be found many works about constitutive models of viscoelastic fluids, such as (Beris *et al.*, 1987), (Brasseur *et al.*, 1994), (Mompean and Deville, 1997), (Giesekus, 1962), (Giesekus, 1982), (Bird *et al.*, 1980), (Phan-Thien and Tanner, 1977), and their study has been extensively developed in recent decades, but in terms of stability analysis, least has been studied (Bistagnino *et al.*, 2007), (Zhang *et al.*, 2013), (Brandi *et al.*, 2019).

The stability analysis is performed by checking whether a given disturbance is amplified or smoothed in the flow field. In the practical fluid flow, the disturbances appears by many factors, such that, structural vibrations, surface roughness, noise, external turbulence, etc. If these disturbances are not smoothed the laminar flow undergoes transition to another complex state, but not necessarily to turbulence state flow (Souza *et al.*, 2005). The mechanisms and phenomena relationed with the disturbences growth in the laminar flow are called instability.

The hidrodynamic stability theory investigates how these disturbances are amplified or smoothed, how the evolution of these disturbances are relationed with the transition phenomena for turbulent flow and the relation of these transition phenomena with the flow components as velocity, viscosity, elasticity, etc.

In this work, the stability analysis of a non-Newtonian fluid (Oldroyd-B model) incompressible two-dimensional straight channel flow is carried out. The stability analysis is performed by Linear Stability Theory (LST) technique in the governing equations of fluid flow and solved by the matrix method using MATLAB/OCTAVE EIG function.

2. MATHEMATICAL FORMULATION

The flow is assumed to be time-dependent, non-Newtonian, two-dimensional and incompressible. The governing equations are the continuity and Navier-Stokes equations with a constitutive relation for the non-Newtonian extra-stress tensor. The conservation equations non-dimensionalized governing the flow without gravitational force action are

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{u}) = -\nabla p + \frac{\beta}{Re}\nabla^2 \mathbf{u} + \nabla \cdot \mathbf{T}, \tag{2}$$

and the Oldroyd-B constitutive model

$$\mathbf{T} + Wi \overset{\nabla}{\mathbf{T}} = 2\frac{(1-\beta)}{Re}\mathbf{D}, \tag{3}$$

where $\mathbf{u}$ denotes the velocity field, t denotes dimensionless variable of time, p denotes dimensionless variable of the pressure, $\mathbf{T}$ is the non-Newtonian extra-stress tensor, $Re = \frac{\rho U L}{\eta_0}$ and $Wi = \frac{\lambda U}{L}$ denote the associated Reynolds and Weissenberg numbers, respectively. The amount of Newtonian solvent is controlled by the dimensionless solvent viscosity coefficient, parameter $\beta = \frac{\eta_s}{\eta_0}$, where $\eta_0 = \eta_s + \eta_p$ denotes the total shear viscosity; $\mathbf{D} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the rate of deformation tensor. The term $\overset{\nabla}{\mathbf{T}}$ is the upper-convected derivative of $\mathbf{T}$, defined by

$$\overset{\nabla}{\mathbf{T}} = \frac{\partial \mathbf{T}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{T}) - \mathbf{T} \cdot (\nabla \mathbf{u})^T - (\nabla \mathbf{u}) \cdot \mathbf{T}. \tag{4}$$

2.1 Linear Stability Theory

In the current analysis, the instantaneous flow is decomposed into two parts, a baseflow and a disturbance flow. The baseflow is invariant in the streamwise direction and baseflow normal velocity component v is null. Schlichting's hypothesis (Schilichting, 1979) says that if the disturbances are infinitesimal then the non-linear terms can be neglected if compared to the linear terms. The disturbance components are represented by $\tilde{u}, \tilde{v}, \tilde{p}$ and $\mathbf{T}$.

The decomposed flow variables are substituted into the governing equations and non-Newtonian extra-stress tensor equations. Then, subtracting the equations describing the baseflow, the following disturbance equations are obtained

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0, \tag{5}$$

$$\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial U}{\partial y} = -\frac{\partial \tilde{p}}{\partial x} + \frac{\beta}{Re}\left(\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2}\right) + \frac{\partial \tilde{T}^{xx}}{\partial x} + \frac{\partial \tilde{T}^{xy}}{\partial y}, \tag{6}$$

$$\frac{\partial \tilde{v}}{\partial t} + U\frac{\partial \tilde{v}}{\partial x} = -\frac{\partial \tilde{p}}{\partial y} + \frac{\beta}{Re}\left(\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2}\right) + \frac{\partial \tilde{T}^{xy}}{\partial x} + \frac{\partial \tilde{T}^{yy}}{\partial y}, \tag{7}$$

$$\tilde{T}^{xx} + Wi\left(\frac{\partial \tilde{T}^{xx}}{\partial t} + U\frac{\partial \tilde{T}^{xx}}{\partial x} + \tilde{v}\frac{\partial T^{xx}}{\partial y} - 2T^{xx}\frac{\partial \tilde{u}}{\partial x} - 2T^{xy}\frac{\partial \tilde{u}}{\partial y} - 2\tilde{T}^{xy}\frac{\partial U}{\partial y}\right) = 2\frac{(1-\beta)}{Re}\frac{\partial \tilde{u}}{\partial x}, \tag{8}$$

$$\tilde{T}^{xy} + Wi\left(\frac{\partial \tilde{T}^{xy}}{\partial t} + U\frac{\partial \tilde{T}^{xy}}{\partial x} + \tilde{u}\frac{\partial T^{xy}}{\partial x} + \tilde{v}\frac{\partial T^{xy}}{\partial y} - T^{xx}\frac{\partial \tilde{v}}{\partial x} - T^{yy}\frac{\partial \tilde{u}}{\partial y} - \tilde{T}^{yy}\frac{\partial U}{\partial y}\right) = \frac{(1-\beta)}{Re}\left(\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y}\right), \tag{9}$$

$$\tilde{T}^{yy} + Wi\left(\frac{\partial \tilde{T}^{yy}}{\partial t} + U\frac{\partial \tilde{T}^{yy}}{\partial x} + \tilde{v}\frac{\partial T^{yy}}{\partial y} - 2T^{xy}\frac{\partial \tilde{v}}{\partial x} - 2T^{yy}\frac{\partial \tilde{v}}{\partial y}\right) = 2\frac{(1-\beta)}{Re}\frac{\partial \tilde{v}}{\partial y}, \tag{10}$$

where U and T are the components of the baseflow.

The resulting equations are linear and their coefficients do not depend on x and t, we can find solutions using some method for this type of equation, such as the variable separation method, as follows:

$$\tilde{u}(x,y,t) = \frac{1}{2}\left(\overline{u}(y)e^{i(\alpha x - \omega t)} + cc\right), \tag{11}$$

where $i = \sqrt{-1}$ and cc is the conjugate complex. These equations indicate that disturbances propagate as waves with frequency ω, wavelength $\lambda = \frac{2\pi}{\alpha}$, wave velocity $c = \frac{\omega}{\alpha}$, where α is the wave number in the x direction and the amplitude of the disturbances are $\overline{u}, \overline{v}, \overline{p}$ and $\overline{T}$.

Applying the variables (11) in the governing and extra-stress tensor equations, a new system of equations for disturbance is obtained.

$$i\alpha\overline{u}(y) + \frac{d\overline{v}(y)}{dy} = 0, \tag{12}$$

$$-i\omega\overline{u}(y) + i\alpha\overline{u}(y)U + \overline{v}(y)\frac{dU}{dy} = -i\alpha\overline{p}(y) + \frac{\beta}{Re}\left(-\alpha^2\overline{u}(y) + \frac{d^2\overline{u}(y)}{dy^2}\right) + i\alpha\overline{T}^{xx}(y) + \frac{d\overline{T}^{xy}(y)}{dy}, \tag{13}$$

$$-i\omega\overline{v}(y) + i\alpha\overline{v}(y)U = -\frac{d\overline{p}(y)}{dy} + \frac{\beta}{Re}\left(-\alpha^2\overline{v}(y) + \frac{d^2\overline{v}(y)}{dy^2}\right) + i\alpha\overline{T}^{xy}(y) + \frac{d\overline{T}^{yy}(y)}{dy}, \tag{14}$$

$$\overline{T}^{xx} + Wi\left(-i\omega\overline{T}^{xx} + i\alpha U\overline{T}^{xx} - 2i\alpha T^{xx}\overline{u} + \frac{dT^{xx}}{dy}\overline{v} - 2\frac{dU}{dy}\overline{T}^{xy} - 2T^{xy}\frac{d\overline{u}}{dy}\right) = 2i\alpha\frac{(1-\beta)}{Re}\overline{u}, \tag{15}$$

$$\overline{T}^{xy} + Wi\left(-i\omega\overline{T}^{xy} + i\alpha U\overline{T}^{xy} - i\alpha T^{xx}\overline{v} + \frac{dT^{xy}}{dy}\overline{v} - \frac{dU}{dy}\overline{T}^{yy} - T^{yy}\frac{d\overline{u}}{dy}\right) = \frac{(1-\beta)}{Re}\left(\frac{d\overline{u}}{dy} + i\alpha\overline{v}\right), \tag{16}$$

$$\overline{T}^{yy} + Wi\left(-i\omega\overline{T}^{yy} + i\alpha U\overline{T}^{yy} - 2i\alpha T^{xy}\overline{v} + \frac{dT^{yy}}{dy}\overline{v} - 2T^{yy}\frac{d\overline{v}}{dy}\right) = 2\frac{(1-\beta)}{Re}\frac{d\overline{v}}{dy}. \tag{17}$$

2.2 Matrix Method

The matrix method consists of rewriting the system of Eq. (12) - (17) and solving the eigenvalue problem associated with flow stability analysis.

Rewriting the system of equations in a matrix system as follows

$$LV = \alpha FV, \tag{18}$$

the problem of stability analysis becomes the problem of finding the eigenvalue α (in the spatial analysis, if the analysis is temporal, the eigenvalue to be found is ω) for the eigenvector V, where $V = [\overline{u};\ \alpha\overline{u};\ \overline{v};\ \alpha\overline{v};\ \overline{p};\ \overline{T}^{xx};\ \overline{T}^{xy};\ \overline{T}^{yy}]$.

Derivatives are approximated by Chebyshev polynomials using the distribution of domain points according to the Gauss-Lobatto distribution, as proposed by (Orzag, 1971) and (Baltensperger and Trummer, 2003).

The eigenvalues and eigenvectors of the system (18) are calculated using the function EIG in MATLAB/OCTAVE. The EIG function is called as $[V, \alpha] = eig(L, F)$, and it automatically selects an algorithm based on the properties of L and F. For Hermitian L and Hermitian positive definite F, it computes the generalized eigenvalues of L and F using the Cholesky factorization of F. Otherwise, it uses the generalized Schur decomposition, which ignores the symmetry of L and F.

3. NUMERICAL RESULTS AND DISCUSSION

In this section, the results obtained using the proposed formulation are presented. The results verification is performed using the results presented in the article (Brandi *et al.*, 2019) with the results obtained by the formulation presented in this work.

3.1 Results Verification

Table 1 shows the comparison between the results presented in the work (Brandi *et al.*, 2019) comparing the LST and DNS techniques with the results obtained by the formulation proposed in this work. It shows a satisfactory agreement

Table 1. Comparison between literature results and results obtained in this work.

Re	β	Wi	ω	Ref. DNS α_i	Ref. LST α_i	Present work α_i
6500	1.0	0.0	0.250	-2.60×10^{-3}	-2.68×10^{-3}	-2.6828×10^{-3}
6500	0.9	10.0	0.400	8.32×10^{-2}	8.32×10^{-2}	8.3054×10^{-2}
8000	0.9	5.0	0.240	-9.00×10^{-3}	-9.00×10^{-3}	-9.2293×10^{-3}
5200	0.9	1.0	0.200	2.08×10^{-2}	2.02×10^{-2}	2.0635×10^{-2}
6500	0.7	10.0	0.276	-3.50×10^{-3}	-3.60×10^{-3}	-3.6947×10^{-3}

The results were obtained by taking 150 Chebyshev modes.

between the results verified of the reference and the results found by the formulation presented in this work, both for the Newtonian case (first case) and for the non-Newtonian cases (other cases).

Figure 1 shows three comparisons between the neutral curves presented by (Brandi *et al.*, 2019) and the neutral curves obtained using the formulation presented in this work. To verify the formulation presented in this work, the neutral curves were obtained vary the parameters β and Wi in order to compare with the results presented in the cited reference, providing a good guarantee on the efficiency of the formulation and the veracity of the results obtained. Figure 1 shows

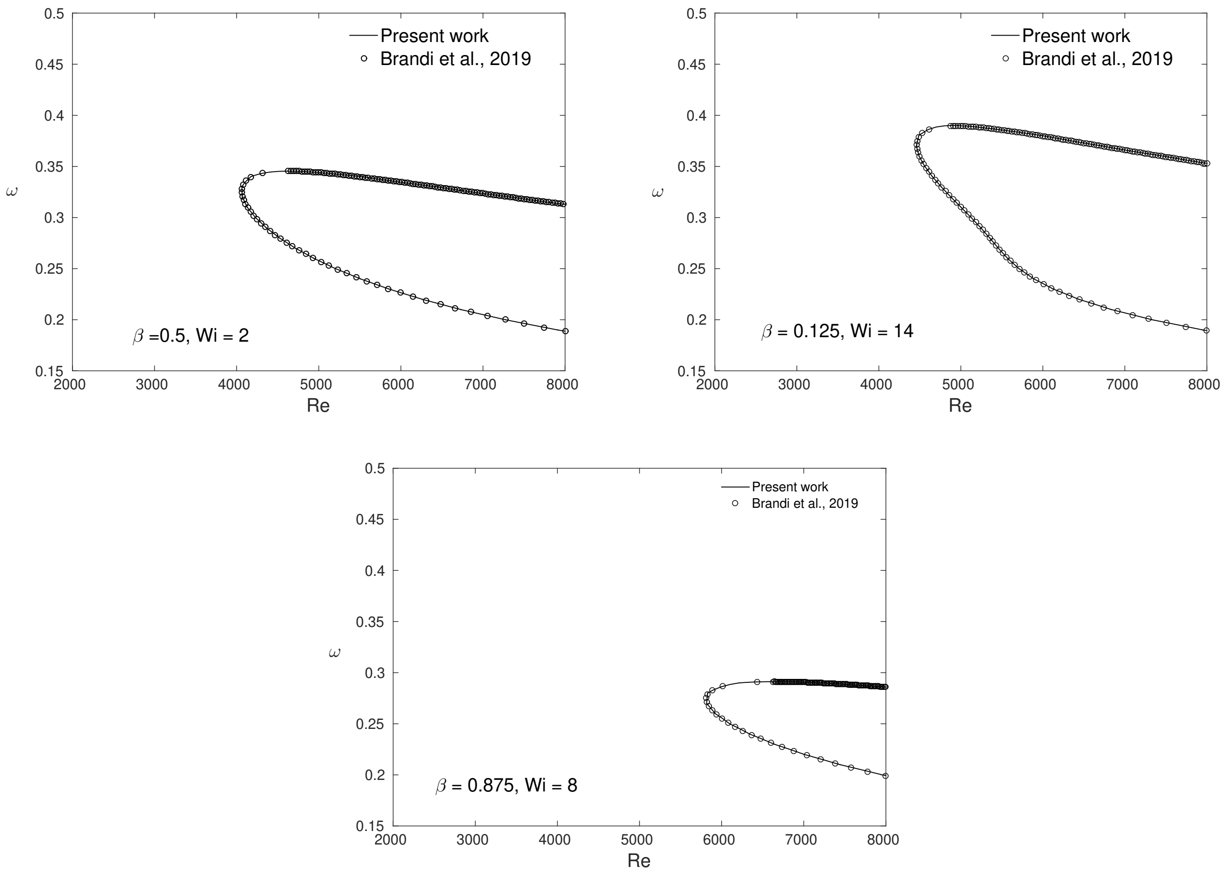

Figure 1. Comparison between the neutral curves presented in Brandi et al., 2019 and the neutral curves obtained through the formulation presented in this work.

that the neutral curves obtained by the formulation presented in this work are in agreement with the curves presented in the work (Brandi *et al.*, 2019), which uses another formulation for the stability analysis and another solution method for the governing equations of the stability problem.

3.2 Neutral Curves Results

Figure 2 presents several neutral curves that were obtained using the formulation presented in this work, varying the Weissenberg number.. It is possible to observe the behavior of the instability region (within the neutral curve $\alpha_i < 0$), which for low values of β, increases as the Weissenberg number increases, and from a given value (in our simulations it was from $Wi = 4$), that this region decreases again, showing that the flow becomes more stable (in the sense that the

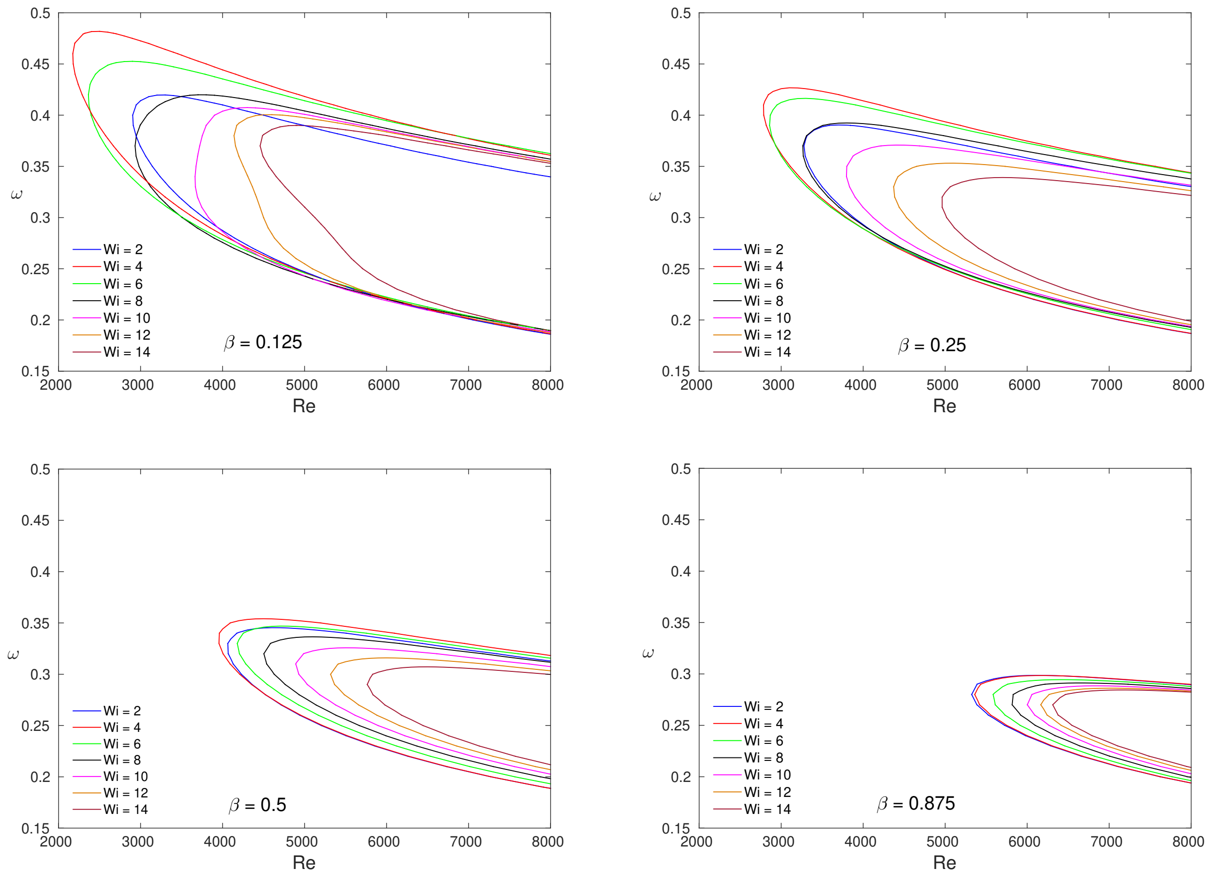

Figure 2. Neutral curves for different values of the constant β, where the Weissenberg number was varied in order to show the influence of this number on the flow stability.

instability region decreases) according to the number of Wi becomes larger than 4. It is possible to notice that the greater the value of the constant β, the influence of the Weissenberg number is smaller on the flow stability. This is consistent, since $\beta = 1$ we have the flow of the Newtonian fluid, where the influence of the Weissenberg number does not exist, since in the Newtonian fluid the Weissenberg number is zero.

Figure 3 shows several neutral curves obtained by the formulation presented in this work by varying the constant β. Analyzing the Fig. 3 it is possible to notice that, as the Weissenberg number increases, the instability region decreases for higher values of the constant β. For values where the constant β is smaller, it can be noted that the instability region increases as the value of the Weissenberg number increases from 2 to 6, but decreases (region) considerably when the Weissenberg value increases for 14.

It is observed that the influence of the Weissenberg number on the stability of the presented flows is greater for smaller values of the constant β and that both Fig. 2 and 3 show this behavior.

4. CONCLUSION

The present work shows results from Linear Stability Theory using the matrix method to investigate the stability analysis of a non-Newtonian fluid flow for the incompressible two-dimensional straight channel flow to unsteady disturbances. The non-Newtonian model performed was the Oldroyd-B model.

The solution of the stability problem of a non-Newtonian fluid flow was obtained through the solution of the matrix method formulation using MATLAB/OCTAVE EIG function for the associated eigenvalue problem.

The results obtained by the formulation presented in this work were compared with the results presented in the article (Brandi *et al.*, 2019), which presents a stability analysis of non-Newtonian flows for the Oldroyd-B model. The comparison showed that the results obtained by the formulation of the matrix method were satisfactory and are in agreement with the results presented in the cited reference. All results presented in this work are in agreement with the results presented in the work cited as reference. This shows that the formulation of the matrix method is efficient for analyzing the stability of viscoelastic flows.

The reference (Brandi *et al.*, 2019) solves the stability problem by obtaining the solution of the Orr-Sommerfeld equation using the Shooting method for the associated eigenvalue problem. The disadvantage of using this method is that

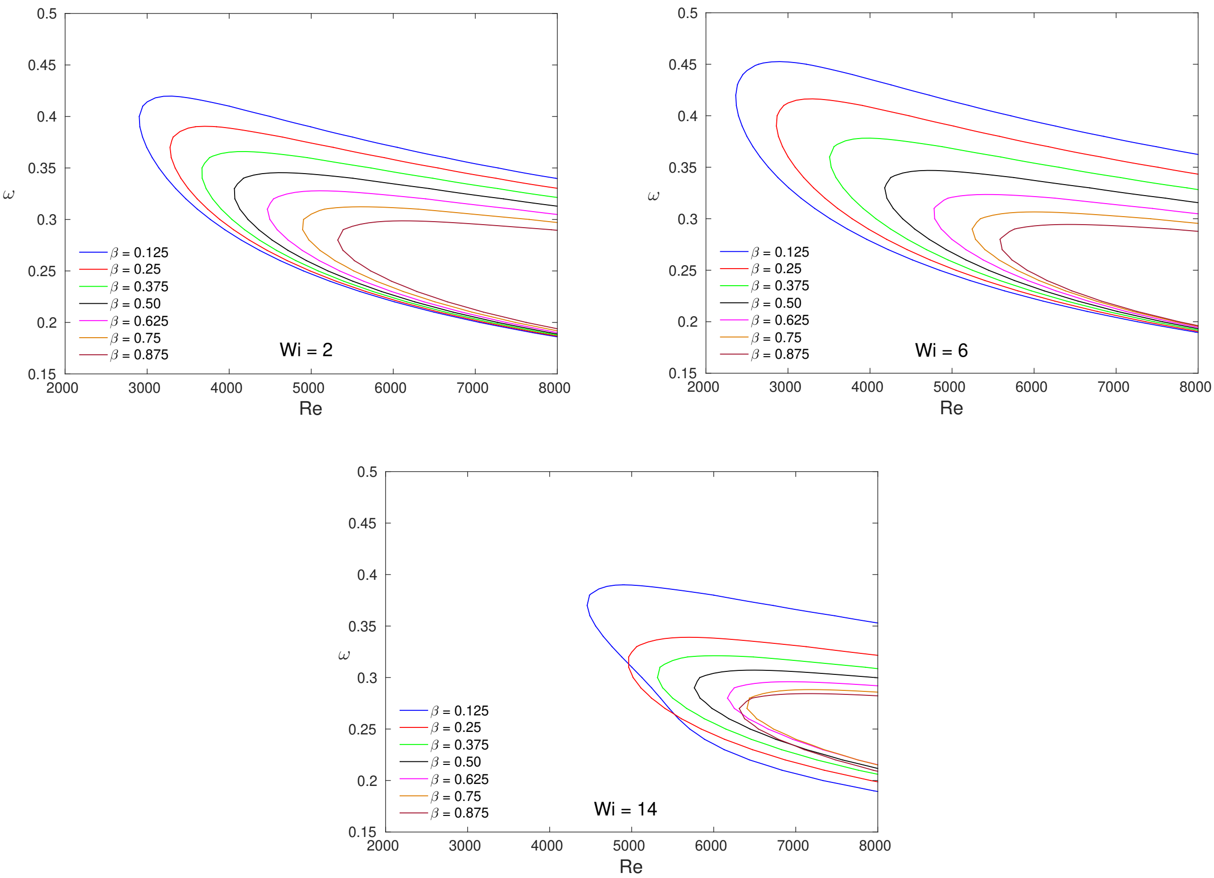

Figure 3. Neutral curves for different values of the Weissenberg number where the constant β was varied in order to show the influence of this parameter on the flow stability.

to get the correct solution to the stability problem, a relatively close initial guess is needed so that the method can converge on the solution, which is not always possible.

The advantage of the formulation proposed in this work is precisely that there is no need an initial guess for the solution, since when solving the stability problem using the matrix method, the entire eigenspectrum of the solution is obtained, it is only necessary to search within this eigenspectrum, the eigenvalue that contains the information regarding the stability of the desired fluid flow.

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