26th International Congress of Mechanical Engineering
COBEM 2021
ABCM Associação Brasileira de Engenharia e Ciências Mecânicas
November 22-26, 2021, Virtual Congress, Brazil

COB-2021-0588
ANALYSIS OF EQUILIBRIUM IN ELASTICITY USING A REGULAR PERTURBATION TECHNIQUE

Adair Roberto Aguiar
Department of Structural Engineering, São Carlos School of Engineering, University of São Paulo, Av. Trabalhador Sãocarlense, 400, Cx. P. 359, 13560-590, São Carlos - SP – Brazil
aguiarar@sc.usp.br

Julián Bravo Castillero
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, 01000 CDMX, México
julian@mym.iimas.unam.mx

Lucas Almeida Rocha
Department of Structural Engineering, São Carlos School of Engineering, University of São Paulo, Av. Trabalhador Sãocarlense, 400, Cx. P. 359, 13560-590, São Carlos - SP – Brazil
lucas.almeida.rocha@usp.br

Abstract. *Numerical methods, such as the finite element method, are often used to obtain approximate solutions of problems whose exact solutions are not known. In this work, we adopt a different approach; we employ an analytical method, namely, the regular perturbation method, to construct approximate solutions in the form of truncated power series in a small parameter ε. We apply the method to obtain approximate solutions for the problem of a cylindrically orthotropic disk in equilibrium subjected to a compressive prescribed displacement along its boundary. This problem has an exact solution that has a singular derivative at the center of the disk and predicts inversion of material. The approximate solutions reproduce those behaviors and tend to the exact solution as either ε decreases or the number of terms in the truncated series increases. In addition, the approximate solutions can be cast in a very concise form. The approach presented in this work can also be used to validate numerical solutions and to obtain insight on the solutions of complex problems that are not known in closed form.*

Keywords: *Linear elasticity, Orthotropy, Asymptotic analysis, Regular perturbation method*

1. INTRODUCTION

Nonlinear elasticity is more adequate to model physical phenomena involving large deformations than its linear counterpart. Large deformations may occur due to external loads or to the presence of singular points in the solid, such as corners on the boundary of the solid, crack tips, and certain points inside anisotropic solids. The governing equations are quasilinear and, in general, have no closed-form solutions. Numerical methods, such as the finite element method together with a Newton-Raphson type of iterative scheme, have been used by Aguiar and Fosdick (2000) to search for approximate solutions to these equations.

In this work, we adopt a different approach and employ a regular perturbation technique to gain insight on the singular nature of solutions near singular points. In particular, we consider the problem of equilibrium without body force of a cylindrically orthotropic disk subjected to a prescribed radial displacement along its boundary. We also consider that the solid is linearly elastic, so that the solution of the corresponding linear problem can be used for comparison purposes. The equation of equilibrium depends on a parameter $\varepsilon = 1 - c_{22}/c_{11}$, where c_{11} and c_{22} are the elastic moduli in the radial and the tangential directions, respectively.

Lekhnitskii (1968) has solved the disk problem in the context of the classical linear elasticity and showed that the radial and hoop stresses are singular at the center of the disk when $c_{11} > c_{22}$, which is a material property found in carbon fibers with radial microstructure (Christensen, 1994; Avery and Herakovich, 1986) and certain types of wood (Forest Products Laboratory, 2010; Johnson *et al.*, 2005). Later, Fosdick and Royer-Carfagni (2001) have shown that, in addition to the singular stress, the solution of linear elasticity predicts the existence of regions where the determinant of the deformation gradient is not positive. This characterizes inversion of material and is not physically admissible. The authors have proposed a constrained minimization theory in the context of which the solution does not present this anomalous behavior.

Here, we construct approximate solutions of the linear disk problem in the form of a truncated power series in ε, substitute these approximate solutions in the governing differential equations, set the coefficients of the resulting series to zero, and obtain a system of differential equations, which together with appropriate boundary conditions, allow the formulation of a sequence of boundary value problems that are simpler to solve than the original problem. This procedure is the basis of the regular perturbation method, which can be used to solve different types of problems. See, for instance, (Givoli, 2021) and the references cited therein for examples of 3D elasticity problems reduced in terms of a sequence of 2D problems. Next, we construct a sequence of solutions of these problems parameterized by the number of terms in the series. An interesting aspect of this investigation is that all the solutions can be cast in a very concise form. For a given ε, the sequence of approximate solutions converges to the solution of the original problem as the number of terms in the series tends to infinity. Even though the technique is conceived to yield convergent sequences for small ε, we also obtain convergent sequences for values of ε that are not small.

In Section 2 we present the governing equation of the disk problem and its exact solution, with a particular interest on its singular behavior. In Section 3 we use the regular perturbation method to obtain approximate solutions, which can be written in a very concise form, of the disk problem. In Section 4, we consider a numerical example and obtain a sequence of approximate solutions that converges to the exact solution of the disk problem as either the number of terms in the series increases and ε is fixed or ε decreases and the number of terms in the series is fixed. In Section 5, we present some concluding remarks.

2. THE DISK PROBLEM

We consider the problem of a homogeneous and cylindrically orthotropic circular disk of radius R_e compressed along its boundary by an imposed radial displacement $\bar{u} = \beta\, R_e$, $\beta < 0$. There is no body force and the center of the disk is placed at the origin of the polar coordinate system (R, Θ), which has an associated orthonormal basis $\{\mathbf{e}_r, \mathbf{e}_\theta\}$. Relative to this basis, the components of the stress and infinitesimal strain tensors are related to each other by the relations

$$\sigma_{rr} = c_{11}\,\epsilon_{rr} + c_{12}\,\epsilon_{\theta\theta}\,, \qquad \sigma_{\theta\theta} = c_{12}\,\epsilon_{rr} + c_{22}\,\epsilon_{\theta\theta}\,, \qquad \sigma_{r\theta} = 2\,c_{66}\,\epsilon_{r\theta}\,, \tag{1}$$

where the constants c_{11}, c_{22}, c_{12} and c_{66} are the elastic moduli. In the context of the classical linear elasticity, the displacement field has the form $\mathbf{u}(R,\Theta) = u(R)\,\mathbf{e}_r$, yielding the strain components

$$\epsilon_{rr} = \frac{du}{dR}\,, \qquad \epsilon_{\theta\theta} = \frac{u}{R}\,, \qquad \epsilon_{r\theta} = 0\,. \tag{2}$$

There is only one nontrivial equation of equilibrium, given by

$$\frac{d\sigma_{rr}}{dR} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{R} = 0\,, \tag{3}$$

which, because of Eq. (1) and Eq. (2), takes the form

$$\frac{d^2u}{dR^2} + \frac{1}{R}\frac{du}{dR} - \frac{c_{22}}{c_{11}}\frac{u}{R^2} = 0\,, \qquad R \in (0, R_e)\,. \tag{4}$$

By letting $\rho = R/R_e$ and $\hat{u}(\rho) = u(R)/R_e$, we can write Eq. (4) in terms of the dimensionless variables ρ and $\hat{u}(\rho)$ as

$$\frac{d^2\hat{u}}{d\rho^2} + \frac{1}{\rho}\frac{d\hat{u}}{d\rho} - \frac{c_{22}}{c_{11}}\frac{\hat{u}}{\rho^2} = 0\,, \qquad \rho \in (0,1)\,. \tag{5}$$

By imposing the boundary condition $u(R_e) = \bar{u}$ and the compatibility condition $u(0) = 0$, that is, $\hat{u}(1) = \beta$ and $\hat{u}(0) = 0$, respectively, the solution of Eq. (5) is given by

$$\hat{u}(\rho) = \beta\,\rho^{\alpha}, \qquad \alpha := \sqrt{\frac{c_{22}}{c_{11}}} > 0\,. \tag{6}$$

It follows from Eq. (1), (2), and (6) that the radial and hoop stresses are given by

$$\sigma_{rr} = (c_{11}\,\alpha + c_{12})\,\beta\,\rho^{\alpha-1}\,, \qquad \sigma_{\theta\theta} = (c_{12}\,\alpha + c_{22})\,\beta\rho^{\alpha-1}\,. \tag{7}$$

Note that both stresses are singular when $\alpha < 1$. In addition, Eq. (6) yields a determinant of the deformation gradient, given by

$$J := \det(\mathbf{I} + \nabla\mathbf{u}) = \left[1 + \alpha\,\beta\rho^{\alpha-1}\right]\left[1 + \beta\rho^{\alpha-1}\right], \tag{8}$$

where **I** is the identity matrix, that is negative in the interval

$$-\alpha\,\beta < \rho^{1-\alpha} < -\beta\,, \tag{9}$$

for $\alpha < 1$ and any value of $\beta < 0$. This represents inversion of material, which, of course, is not physically admissible. A discussion on such an anomalous behavior is beyond the scope of this work. Here, we use the regular perturbation method to obtain sequences of approximations that converge to the exact solution, given by Eq. (6).

3. ASYMPTOTIC RESULTS

The basis of the regular perturbation method consists in identifying a small parameter ε in the equation to be solved, which here is Eq. (5). We begin by considering that $c_{22} \approx c_{11}$. We then define the small parameter

$$\varepsilon := 1 - \frac{c_{22}}{c_{11}} \tag{10}$$

and rewrite Eq. (5) as

$$L^{\varepsilon}\hat{u} = 0\,, \qquad L^{\varepsilon} := \frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2}(1-\varepsilon)\,. \tag{11}$$

In the particular case $\varepsilon = 0$, the expressions in Eq. (11) become

$$L^{0}\hat{u} = 0\,, \qquad L^{0} := \frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2}\,. \tag{12}$$

In perturbation theory, equations Eq. (11) and Eq. (12) are called perturbed and unperturbed equations, respectively (Nayfeh, 1993). The regular perturbation method is particularly appropriate when the unperturbed equation corresponds to a relatively simple problem with known solution. This is the case here, because $\varepsilon = 0$ in Eq. (10) corresponds to the isotropic case, $c_{11} = c_{22}$, which is simpler than the anisotropic case represented by the perturbed equation, given by Eq. (11), with $\varepsilon \neq 0$.

Next, we propose an approximate solution of Eq. (11) given by the power series

$$\hat{u}^{(m)}(\rho,\varepsilon) = \sum_{i=0}^{m} \varepsilon^{i}\,\hat{u}_i(\rho)\,, \tag{13}$$

where $\hat{u}_i$ are functions to be determined below and $m \in \mathbb{N}$. Substituting Eq. (13) into Eq. (11), we obtain

$$L^{\varepsilon}\,\hat{u}^{(m)} = \varepsilon^{m+1}\,\frac{\hat{u}_m}{\rho^2} + \varepsilon^{0}\left(\frac{d^2}{d\rho^2}\,\hat{u}_0 + \frac{1}{\rho}\,\frac{d}{d\rho}\,\hat{u}_0 - \frac{\hat{u}_0}{\rho^2}\right) + \sum_{i=1}^{m}\varepsilon^{i}\left(\frac{d^2}{d\rho^2}\,\hat{u}_i + \frac{1}{\rho}\,\frac{d}{d\rho}\,\hat{u}_i - \frac{\hat{u}_i}{\rho^2} + \frac{\hat{u}_{i-1}}{\rho^2}\right)\,. \tag{14}$$

To determine $\hat{u}_i(\rho)$, $i = 0, 1, 2, ..., m$, we solve a sequence of differential equations given by

$$\begin{aligned} &\frac{d^2}{d\rho^2}\,\hat{u}_0 + \frac{1}{\rho}\,\frac{d}{d\rho}\,\hat{u}_0 - \frac{\hat{u}_0}{\rho^2} = 0\,,\\ &\frac{d^2}{d\rho^2}\,\hat{u}_i + \frac{1}{\rho}\,\frac{d}{d\rho}\,\hat{u}_i - \frac{\hat{u}_i}{\rho^2} = -\frac{\hat{u}_{i-1}}{\rho^2}\,, \qquad \text{for } 1 \le i \le m\,, \quad \rho \in (0,1)\,. \end{aligned} \tag{15}$$

We then have that the substitution of $\hat{u}^{(m)}$, given by Eq. (13) together with the solutions $\hat{u}_i(\rho)$, $i = 0, 1, \ldots, m$, into the differential equation Eq. (14) yields a residue of order $O(\varepsilon^{m+1})$, which is given by $\varepsilon^{m+1}\,\hat{u}_m/\rho^2$.

The approximation $\hat{u}^{(m)}(\rho,\varepsilon)$ must also satisfy the boundary condition $\hat{u}^{(m)}(1,\varepsilon) = \beta$ and the compatibility condition $\hat{u}^{(m)}(0,\varepsilon) = 0$. This is achieved by requiring that $\hat{u}_i(\rho)$ satisfies

$$\hat{u}_i(1) = \begin{cases} \beta & \text{for } i = 0\,,\\ 0 & \text{for } 1 \le i \le m\ , \end{cases} \qquad \hat{u}_i(0) = 0 \quad \text{for } 0 \le i \le m\,. \tag{16}$$

It follows from both Eq. (15.a) and Eq. (16) that $\hat{u}_0(\rho)$ is given by

$$\hat{u}_0(\rho) = \beta\,\rho\,. \tag{17}$$

Clearly, Eq. (17) is the solution of the isotropic disk problem. Similarly, for $m = 5$, we obtain from Eq. (15.b) together with both Eq. (16) and Eq. (17) that $\hat{u}_i(\rho)$, for $i = 1, 2, 3, 4, 5$, is given by

$$\begin{aligned}
\hat{u}_1(\rho) &= \frac{1}{2}\,\beta\,\rho\,\left[-\log(\rho)\right] , \\
\hat{u}_2(\rho) &= \frac{1}{8}\,\beta\,\rho\,\left[\log(\rho)^2 - \log(\rho)\right] , \\
\hat{u}_3(\rho) &= \frac{1}{48}\,\beta\,\rho\,\left[-\log(\rho)^3 + 3\,\log(\rho)^2 - 3\,\log(\rho)\right] , \\
\hat{u}_4(\rho) &= \frac{1}{384}\,\beta\,\rho\,\left[\log(\rho)^4 - 6\,\log(\rho)^3 + 15\,\log(\rho)^2 - 15\,\log(\rho)\right] , \\
\hat{u}_5(\rho) &= \frac{1}{3840}\,\beta\,\rho\,\left[-\log(\rho)^5 + 10\,\log(\rho)^4 - 45\,\log(\rho)^3 + 105\,\log(\rho)^2 - 105\,\log(\rho)\right] .
\end{aligned} \tag{18}$$

The expressions in Eq. (18) can be written concisely as

$$\hat{u}_i(\rho) = \frac{1}{2^i\,i!}\,\beta\,\rho\,\sum_{j=1}^{i}(-1)^j\,\log(\rho)^j\,A_{(i,j)} , \qquad \text{for } i \geq 1 , \tag{19}$$

where $A_{(i,j)}$ is given by

$$A_{(i,j)} = \begin{cases} 1 , & \text{for } 1 \leq i \leq 2 , \\ \dfrac{(2\,a+1)!}{2^a\,a!} , \qquad a := i-2 , & \text{for } i \geq 3 \quad \text{and} \quad 1 \leq j \leq 2 , \\ \dfrac{2}{j\,(j-1)}\left[(j-1)\,A_{(i,\,j-1)} - i\,A_{(i-1,\,j-2)}\right] , & \text{for } i \geq 3 \quad \text{and} \quad 3 \leq j \leq i . \end{cases} \tag{20}$$

Remark: In general terms, to obtain the expressions in Eq. (20), we replace Eq. (19) into Eq. (15.b) and set the coefficients that multiply each power of $\log(\rho)$ to zero. This approach is explained in detail in the Appendix.

4. NUMERICAL RESULTS

We consider $\beta = -0.05$ and $\varepsilon = 0.1, 0.5, 0.9$, which, because of Eq. (6.b) and Eq. (10), correspond to $\alpha = \sqrt{0.9}, \sqrt{0.5}, \sqrt{0.1}$, respectively. In Fig. 1, we show curves for the radial displacement, $\hat{u}$, on the left-hand column and for the determinant of the deformation gradient, J, on the right-hand column versus the radius, ρ, in a neighborhood of $\rho = 0$. Here, we are interested in showing how the approximate solutions behave near the singular point at $\rho = 0$ and in the region where the determinant is negative, which is given by Eq. (9). In fact, if the graphs were plotted for $\rho \in (0, 1)$ the curves would be indistinguishable. We consider three cases with an increasing degree of orthotropy, that is $\varepsilon = 0.1, 0.5, 0.9$ represented in Fig. 1a, 1b, and 1c, respectively. In all three figures, the black lines represent the exact solution, given by Eq. (6) for the displacement and by Eq. (8) for the determinant of the deformation gradient, and the colored lines represent the approximations obtained with different values of m, as indicated by the legend of each figure.

From all three figures 1a, 1b, and 1c, we see that the approximate solutions of both the radial displacement and the determinant of the deformation gradient tend to their exact values, given by Eq. (6) and Eq. (8), respectively. In addition, we see from Fig. 1a that, for a low degree of orthotropy, represented by $\varepsilon = 0.1$, the differences between the approximate values (of $\hat{u}$ and J) and their respective exact values are very small; they are only noticeable for ρ in the order of 10^{-25}. On the other hand, we see from Fig. 1c that for a high degree of orthotropy, represented by $\varepsilon = 0.9$, the differences are significant and we need $m = 35$ to make them barely noticeable. We then formalize this discussion by defining the relative error between an approximate solution $\hat{u}^{(m)}$ and the exact solution $\hat{u}$, given by Eq. (6), as

$$e := \frac{|\hat{u} - \hat{u}^{(m)}|_{L^2(0,1)}}{|\hat{u}|_{L^2(0,1)}} , \tag{21}$$

where $|\cdot|_{L^2(0,1)}$ represents the $L^2(0,1)$ norm.

In Tab. 1, we show the relative error e of the approximate solution $\hat{u}^{(m)}$, $m = 1, 2, 3, 4, 5, 11$, for the cases $\varepsilon = 0.1$ and $\varepsilon = 0.5$. In Tab. 2, we show the relative error e for $m = 1, 5, 10, 15, 25, 35, 50$ in the case $\varepsilon = 0.9$. We see from both tables that the value of e decreases as either, ε decreases and m is fixed, or m increases and ε is fixed. In addition, in order to e be within a given a tolerance, for instance $e < 0.01$, we see that we must have $m \geq 1$, $m \geq 3$, and $m \geq 15$ in the cases $\varepsilon = 0.1$, $\varepsilon = 0.5$, and $\varepsilon = 0.9$, respectively. The necessity of larger values of m as ε increases is expected since, in Section 3, we have seen that the approximate solution $\hat{u}^{(m)}$ satisfy Eq. (14) to within a error $\varepsilon^{m+1}\,\hat{u}_m/\rho^2$.

Table 1: Relative error e of the approximate solution $\hat{u}^{(m)}$, $m = 1, 2, 3, 4, 5, 11$, for $\varepsilon = 0.1$ and $\varepsilon = 0.5$.

	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 11$
$\varepsilon = 0.1$	0.0013	8.4×10^{-5}	6.0×10^{-6}	4.6×10^{-7}	4.2×10^{-8}	2.5×10^{-8}
$\varepsilon = 0.5$	0.0414	0.0140	0.0052	0.0020	0.0008	5.5×10^{-6}

Table 2: Relative error e of the approximate solution $\hat{u}^{(m)}$, $m = 1, 5, 10, 15, 25, 35, 50$, for $\varepsilon = 0.9$.

	$m = 1$	$m = 5$	$m = 10$	$m = 15$	$m = 25$	$m = 35$	$m = 50$
$\varepsilon = 0.9$	0.2201	0.0617	0.0204	0.0081	0.0016	0.0004	5.0×10^{-5}

5. CONCLUSIONS

We have adopted a regular perturbation technique to gain insight on singular points in a solid. In particular, we have investigated the problem of equilibrium of a cylindrically orthotropic disk subjected to a prescribed displacement along its boundary. We have considered that the solid is linearly elastic, so that the solution of the corresponding problem can be used for comparison purposes. By using the regular perturbation method, we have constructed approximate solutions $\hat{u}^{(m)}(\rho, \varepsilon)$ in the form of truncated power series with the form given by Eq. (13), where $\hat{u}_0(\rho)$ is given by Eq. (17) and $\hat{u}_i(\rho)$, $i = 1, 2, ..., m$, is given by Eq. (19) and Eq. (20). Then, we have verified that the approximate solutions $\hat{u}^{(m)}(\rho, \varepsilon)$ tends to the exact solution $\hat{u}$, given by Eq. (6), as either m increases and ε is fixed or ε decreases and m is fixed. The regular perturbation method employed in this work could be used in more complex problems, where an exact solution is not known, to provide information about the solution, such as the existence of singular points. In addition, the method could also be used in the validation of numerical solutions.

6. ACKNOWLEDGEMENTS

This work was initiated during the visit of the second author at the São Carlos School of Engineering, University of São Paulo, Brazil. Support of the PRInt USP/CAPES program, grant n° 88887.372694/2019-00, is gratefully acknowledged. The first author acknowledges the support of National Council for Scientific and Technological Development (CNPq), grant n° 420099/2018-2, and the third author acknowledges the financial support provided by Coordination for the Improvement of Higher Education Personnel (CAPES) - Finance Code 001.

7. REFERENCES

Aguiar, A. and Fosdick, R., 2000. "A singular problem in incompressible nonlinear elastostatics". *Mathematical Models and Methods in Applied Sciences*, Vol. 10, No. 08, pp. 1181–1207.

Avery, W.B. and Herakovich, C.T., 1986. "Effect of fiber anisotropy on thermal stresses in fibrous composites". *Journal of Applied Mechanics*, Vol. 53, No. 4, pp. 751–756.

Christensen, R.M., 1994. "Properties of carbon fibers". *Journal of the Mechanics and Physics of Solids*, Vol. 42, No. 4, pp. 681–695.

Forest Products Laboratory, 2010. "Wood handbook - Wood as an engineering material". Technical report, U.S. Department of Agriculture, Forest Service, Forest Products Laboratory, Madison.

Fosdick, R. and Royer-Carfagni, G., 2001. "The constraint of local injectivity in linear elasticity theory". *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, Vol. 457, No. 2013, pp. 2167–2187.

Givoli, D., 2021. "Asymptotic analysis for plane stress problems". *Journal of Elasticity*, Vol. 144, No. 1, pp. 1–14.

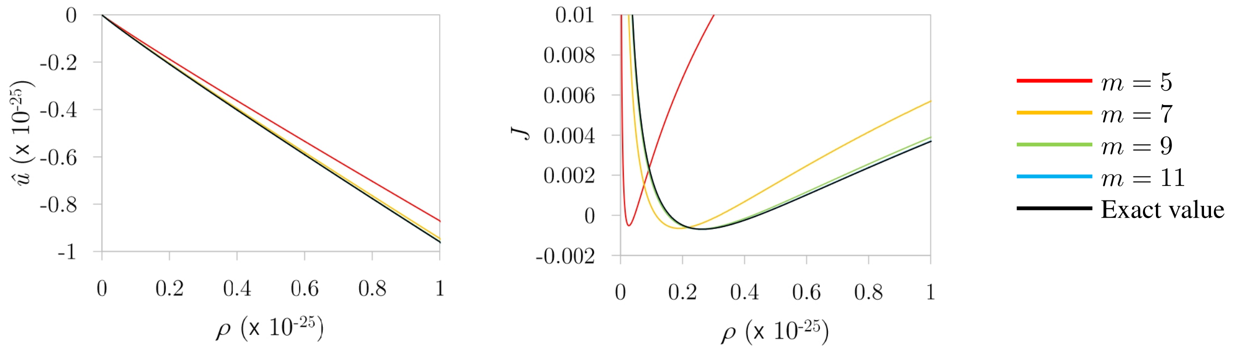
Johnson, J.A., Hermanson, J.C., Cramer, S.M. and Amundson, C., 2005. "Stress singularities in a model of a wood disk under sinusoidal pressure". *Journal of Engineering Mechanics*, Vol. 131, No. 2, pp. 153–160.

Lekhnitskii, S.G., 1968. *Anisotropic plates*. Gordon & Breach, New York.

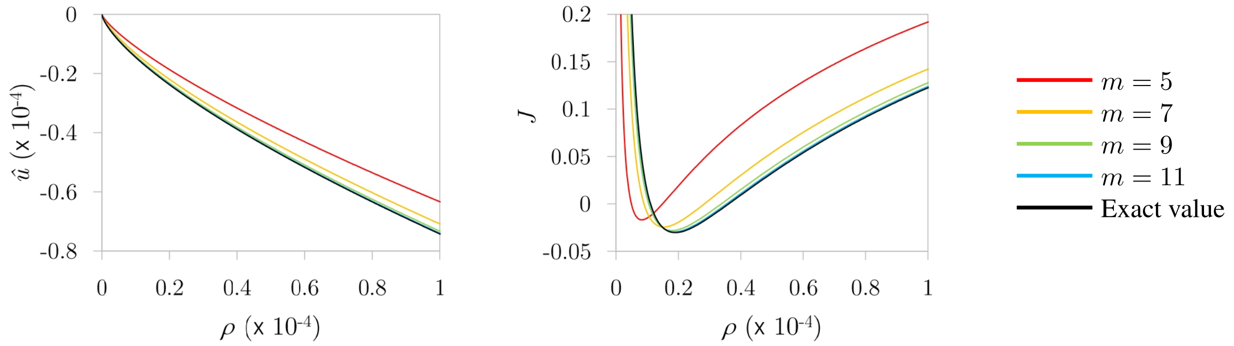
Nayfeh, A.H., 1993. *Introduction to perturbation techniques*. Wiley, New York.

8. RESPONSIBILITY NOTICE

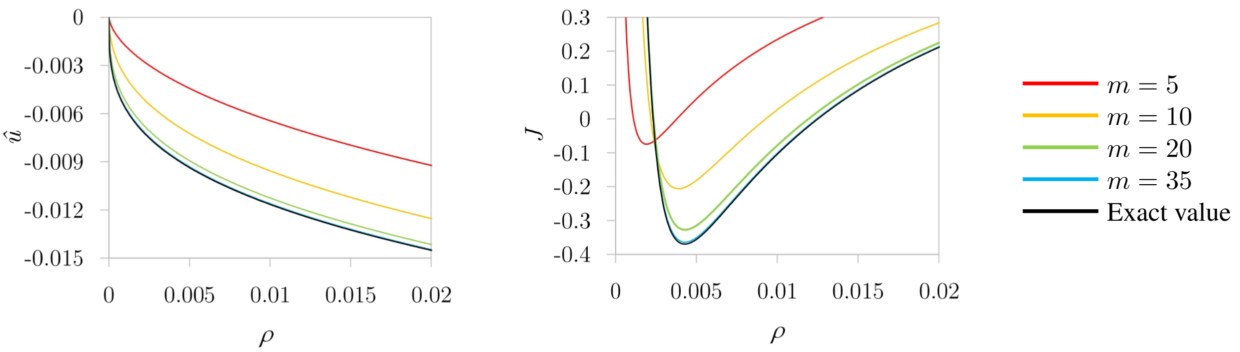
The authors are solely responsible for the printed material included in this paper.



(a) $\varepsilon = 0.1$



(b) $\varepsilon = 0.5$



(c) $\varepsilon = 0.9$

Figure 1: Fields $\hat{u}$ (on the left-hand column) and J (on the right-hand column) versus ρ for different approximate solutions $\hat{u}^{(m)}$ and their respective exact values, given by Eq. (6) for $\hat{u}$ and by Eq. (8) for J.

APPENDIX

Here, we explain the approach used to obtain the expressions of $A_{(i,j)}$ in Eq. (20). By substituting Eq. (19) into Eq. (15.b), we obtain

$$(-1)^i\, i \left(\frac{i-1}{\log\rho} + 2\right) \log(\rho)^{i-1} A_{(i,i)} + \sum_{j=1}^{i-1} (-1)^j \Big[2\, i\, \log(\rho)^j A_{(i-1,\,j)} + 2\, j\, \log(\rho)^{j-1} A_{(i,j)} - j\, \log(\rho)^{j-2} A_{(i,j)} + j^2 \log(\rho)^{j-2} A_{(i,j)} \Big] = 0\,. \tag{22}$$

For Eq. (22) to be satisfied, the coefficients that multiply each power of $\log(\rho)$ must be null. Next, we use these relations to determine the constants $A_{(i,j)}$. We begin with the coefficient that multiplies the power $\log(\rho)^{-1}$. This coefficient has only the contribution from $j=1$ in the summation of Eq. (22). Therefore, we have

$$A_{(i,1)} - A_{(i,1)} = 0\,, \tag{23}$$

which is trivially satisfied and does not provide any information on the value of $A_{(i,1)}$. However, from Eq. (18), we see that

$$\begin{cases} A_{(1,1)} = A_{(2,1)} = A_{(2,2)} = 1\,, \\ A_{(i,1)} = 1 \times 3 \times 5 \times ... \times (2\,a+1) = \dfrac{(2\,a+1)!}{2^a\,a!}\,, \quad a := i-2\,, \quad \text{for } i \geq 3\,. \end{cases} \tag{24}$$

Then, we assume that $i \geq 3$, for otherwise all $A_{(i,j)}$ would be given by Eq. (24.a). The coefficient that multiplies the power $\log(\rho)^0$ has contributions from both $j=1$ and $j=2$ in the summation of Eq. (22). To make this coefficient vanish, we must have

$$-2\,A_{(i,1)} - 2\,A_{(i,2)} + 4\,A_{(i,2)} = 0 \qquad \Rightarrow \qquad A_{(i,2)} = A_{(i,1)}\,, \tag{25}$$

which can also be easily verified in Eq. (18).

Then, for the power $\log(\rho)^1$, we have contributions from $j=1$, $j=2$, and $j=3$. More generally, for the powers $\log(\rho)^{k-2}$, $k = 3, 4, ..., i-1$ and $i \geq 3$, we have contributions from $j = k-2$, $j = k-1$, and $j = k$, which allow us to write

$$(-1)^{k-2}\,2\,i\,A_{(i-1,\,k-2)} + (-1)^{k-1}\,2\,(k-1)\,A_{(i,\,k-1)} + (-1)^k\left(-k+k^2\right)A_{(i,k)} = 0\,, \tag{26}$$

and, therefore, for $3 \leq k \leq i-1$ and $i \geq 3$, we have that $A_{(i,k)}$ is given by

$$A_{(i,k)} = \frac{1}{k\,(k-1)}\left[2\,(k-1)\,A_{(i,\,k-1)} - 2\,i\,A_{(i-1,\,k-2)}\right]. \tag{27}$$

Next, for the power $\log(\rho)^{i-2}$, in Eq. (22), we have contributions from the term outside the summation and from $j = i-2$ and $j = i-1$ inside the summation. Therefore, we have that

$$(-1)^{i-2}\,2\,i\,A_{(i-1,\,i-2)} + (-1)^{i-1}\,2\,(i-1)\,A_{(i,\,i-1)} + (-1)^i\,i\,(i-1)\,A_{(i,i)} = 0\,, \tag{28}$$

which yields

$$A_{(i,i)} = \frac{1}{i\,(i-1)}\left[2\,(i-1)\,A_{(i,\,i-1)} - 2\,i\,A_{(i-1,\,i-2)}\right]. \tag{29}$$

By comparing Eq. (27) and Eq. (29), we conclude that Eq. (27) is also valid for $k = i$.

For the last power, which is $\log(\rho)^{i-1}$, we have the contribution from the term outside the summation and from $j = i-1$ in the summation. Hence, we have that

$$(-1)^{i-1}\,2\,i\,A_{(i-1,\,i-1)} + (-1)^i\,2\,i\,A_{(i,i)} = 0 \qquad \Rightarrow \qquad A_{(i,i)} = A_{(i-1,\,i-1)}\,, \tag{30}$$

which is easily verified in Eq. (18). Thus it follows from Eq. (24), (27), and (29) that $A_{(i,j)}$ is given by Eq. (20).