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Globally Robust Sliding Mode Command Tracking Under Input Bounds Using a Time-Optimal Sigmoid Trajectory

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Abstract. *The present paper investigates the smooth-command-tracking control of a continuous-time second-order system with actuator constraints, subject to a matched disturbance input. Particularly, to cope with the disturbance, we propose a novel design for a global first-order sliding-mode control, which ensures disturbance invariance from the very beginning of the motion. Moreover, to address the input constraints and the smooth command tracking requirement, we adopt a novel second-order polynomial S-curve output trajectory connecting the initial point to the specified target with a minimum-time admissible acceleration-cruise-deceleration profile. The proposed method is illustrated by computer simulations, which show its effectiveness to provide global robustness, smooth operation, and avoid input saturation.*

Keywords: *sliding mode control, trajectory generation, input constraint, reference shaping.*

1. INTRODUCTION

Mechatronics systems such as automated machines, robots, and multirotor aerial vehicles usually operate under two crucial antagonistic conditions: disturbances and actuator constraints. On the one hand, the disturbances typically require a high-magnitude control action to overcome their effects, while, on the contrary, the control action must be sufficiently small to avoid saturating the actuators. A typical operation of such systems involves a maneuver to a given setpoint in a smooth fashion to avoid vibration and consequent wearing (Fang *et al.*, 2019). The present work stems from the hypothesis that it is possible to accomplish such a setpoint maneuvering, in theory, with perfect robustness to bounded matched disturbances using global sliding modes (Utkin and Shi, 1996; Bartoszewicz, 1996).

It is well-known that if the input constraints are ignored in the design of a control law, in practice, the actuator commands can be saturated, *e.g.*, due to the windup phenomenon (Kothare *et al.*, 1994), which may lead to degraded performance or even instability. The design of control laws under state and input constraints has been widely investigated in the literature and the available methods can be divided into two classes. In the first one, the methods consist of a control law that guarantees the closed-loop stability in the presence of constraints. The most effective and widespread method in this class is the model predictive control (Rawlings *et al.*, 2017). This class also contains the Lyapunov-based methods such as the one found in (Tyan and Bernstein, 1999), which proposes a linear saturated control law to stabilize a double integrator. In the second class, the methods consist of designing a reference manager (or governor) to modify the original reference input to obtain a feasible one (Garone *et al.*, 2017; Sugie and Yamamoto, 2001).

Input and state constraints have also been addressed in the sliding mode control literature, though with less intensity (Incremona *et al.*, 2017; Choi *et al.*, 2001; Bartoszewicz, 1996). In particular, Incremona *et al.* (2017) have dealt with constrained systems under higher-order sliding mode control by the assumption of a sufficient control margin and the admissibility of the initial conditions. Choi *et al.* (2001) and Bartoszewicz (1996) have investigated the global sliding mode control (GSMC) under input constraints by using time-varying sliding functions designed in an optimal sense while guaranteeing feasibility. Some hierarchical schemes combining MPC with SMC have also been investigated; see, *e.g.*, (Rubagotti *et al.*, 2011). In these schemes, the SMC provides robustness to disturbances and uncertainties and ensures the satisfaction of the input and state constraints. All the above methods fit well in the first of the two above-mentioned classes.

The present paper deals with the control of a single-input-single-output (SISO) disturbed double integrator system with input bounds towards a desired constant setpoint. To face this problem, we propose a novel method that belongs to the second class described above, *i.e.*, it contains a reference manager and a cascaded robust stabilizing control law. In particular, the proposed reference manager provides a non-conservative feasible minimum-time S-shaped trajectory command. The adopted S-shaped command is a second-order polynomial trajectory containing three sequential phases: acceleration, cruise, and deceleration. We argue that this trajectory command allows a smoother operation than the reference governors presented in (Garone *et al.*, 2017; Sugie and Yamamoto, 2001). On the other hand, the adopted robust stabilizing control law is a first-order sliding mode controller that does not induce a reaching phase, thus providing global robustness. Different from (Utkin and Shi, 1996; Bartoszewicz, 1996), the proposed GSMC is formulated on the tracking error dynamics in such a way to eliminate the need for designing a time-varying sliding function. In fact, the global sliding mode is enabled by assuming that the S-shaped command provided by the reference manager starts from the rest at the actual initial system output. Finally, it is worth citing the recent reference (Boukattaya *et al.*, 2020), which has proposed a new time-varying sliding surface that supports a global sliding mode and, different from the classical ones (Utkin and Shi, 1996; Bartoszewicz, 1996), it provides a terminal behavior as well as the capability to specify a desired error convergence time. However, different from our method, the control law presented in (Boukattaya *et al.*, 2020) always run into a singularity. Moreover, this reference has not concerned about the optimization of the arrival time and has not addressed the control constraints.

The remaining text is organized in the following manner. Section 2 formulates the proposed method on a simple disturbed double integrator system. Section 3 illustrates the proposed method using computer simulation. Finally, Section 4 presents the concluding remarks as well as the future works.

2. METHODOLOGY

This section details the proposed method considering a SISO second-order system. Subsection 2.1 formulates the sliding mode controller. Subsection 2.2 designs a feasible minimum-time sigmoid position command. Finally, Subsection 2.3 presents an algorithm to compute the optimal trajectory.

2.1 Global sliding mode control

Consider a SISO second-order system described by

$$\ddot{x} = u + d, \tag{1}$$

where $(x, \dot{x}) \in \mathbb{R}^2$ is the state, $u \in \mathbb{R}$ is the control input, and $d \in \mathbb{R}$ is the disturbance. Denote the initial conditions by $x_0 \triangleq x(0)$ and $\dot{x}_0 \triangleq \dot{x}(0)$.

Consider the objective of tracking a time-varying command $\bar{x} : \mathbb{R}_+ \longrightarrow \mathbb{R}$ of the class C^1, with bounded acceleration, *i.e.*, $|\ddot{\bar{x}}| \leq \ddot{\bar{x}}_{\max}$, for some $\ddot{\bar{x}}_{\max} \in (0, \infty)$.

By defining the tracking error $\tilde{x} \triangleq x - \bar{x}$, we can write the corresponding dynamics as

$$\ddot{\tilde{x}} = u - \ddot{\bar{x}} + d, \tag{2}$$

through which we can solve the tracking problem $x \longrightarrow \bar{x}$ by means of the regulation $\tilde{x} \longrightarrow 0$. For this purpose, define the switching variable

$$\sigma \triangleq \dot{\tilde{x}} \tag{3}$$

and the corresponding sliding surface

$$\mathcal{S} \triangleq \left\{ \left(\tilde{x}, \dot{\tilde{x}}\right) \in \mathbb{R}^2 : \sigma = 0 \right\}. \tag{4}$$

Note that with the above choice of σ, if the adopted command satisfies $\dot{\bar{x}}(0) = \dot{x}_0$, then it holds that $\sigma(0) = 0$, which means that any system trajectory starts from the sliding surface. Therefore, if a control law $u = u\left(\tilde{x}, \dot{\tilde{x}}\right)$ is designed for the error dynamics (2) to satisfy a sliding condition on $\mathcal{S}$ from the very beginning of the trajectory, then the corresponding closed-loop system will undergo a global sliding mode.

Consider the following assumptions:

A1: The disturbance satisfies $|d| < d_{\max}$, where $d_{\max} \in (0, \infty)$ is known.

A2: The command $\bar{x}$ satisfies $\bar{x}(0) = x_0$ and $\dot{\bar{x}}(0) = \dot{x}_0$.

In Assumption A1, the boundedness consideration is quite reasonable in practice, but one can rarely obtain a non-conservative estimate of the bound $d_{\max}$. However, this is not restrictive, since one could avoid the availability of $d_{\max}$ by using some adaptation scheme, such as the one presented in Huang *et al.* (2008) and recently applied to the attitude control of a quadrotor UAV in Ricardo Jr *et al.* (2020). Assumption A2 is not too restrictive neither; essentially, it just requires the knowledge of the plant initial conditions and the use of a command trajectory starting from them.

Let us consider the control law

$$u = -\kappa \operatorname{sign}(\sigma) + \ddot{\bar{x}}, \tag{5}$$

where $\operatorname{sign}(x) = 1$ if $x > 0$ and $\operatorname{sign}(x) = -1$ if $x < 0$, and $\kappa \in (0, \infty) \subset \mathbb{R}$ is a design gain.

Proposition 1. Under Assumptions A1–A2, if $\kappa > d_{\max}$, then the closed-loop system described by (2)–(3), and (5) has a global sliding mode on $\mathcal{S}$.

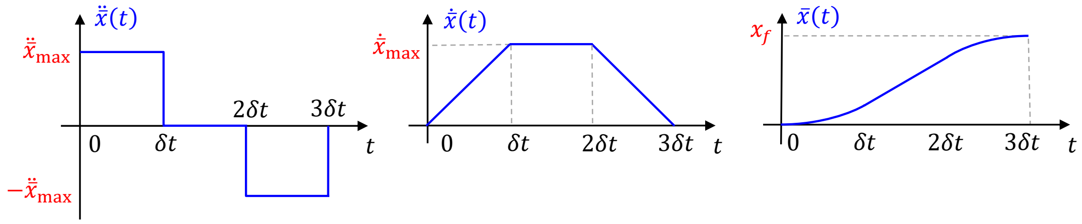

Figure 1. Second-order polynomial S-curve trajectory.

Proof. Define the scalar positive definite function $V(\sigma) \triangleq \sigma^2/2$. Note that if $V(\sigma) \to 0$, then $\sigma \to 0$. Therefore, since $\sigma(t = 0) = 0$, for the system to show a global sliding mode, it suffices to satisfy the finite-time convergence condition $\dot{V} = -\beta V^{1/2}$ [see Bhat and Bernstein (2000)], $\forall t$ and some $\beta > 0$. The latter equation can be rewritten in terms of σ as

$$\sigma\dot{\sigma} = -\beta|\sigma|/\sqrt{2}. \tag{6}$$

By developing the left-hand side of (6) using (2), (3), and (5), we obtain:

$$\sigma\dot{\sigma} = -(\kappa - d\,\text{sign}\,(\sigma))\,|\sigma|. \tag{7}$$

By comparing the right-hand sides of (6) and (7), we obtain:

$$\kappa = d\,\text{sign}\,(\sigma) + \frac{\beta}{\sqrt{2}}. \tag{8}$$

Therefore, κ must satisfy (8), $\forall t$, for guaranteeing the global sliding mode. Since $\beta > 0$ is arbitrary, this condition can be rewritten as $\kappa > d\,\text{sign}\,\sigma, \forall t$, which in turn is always satisfied if $\kappa > d_{\max}, \forall t$, thus concluding the proof. □

Under the conditions of Proposition 1, the closed-loop system described by (2)–(3) and (5) will maintain $\sigma(t) \equiv 0$, $\forall t \geq 0$. Therefore, the global system dynamics correspond to a sliding motion, which can be immediately described from (3) as $\dot{\tilde{x}} = 0$, which implies $\tilde{x} = 0$, since from Assumption A2 $\bar{x}(0) = x_0$. Now that the robust command tracking is guaranteed, we just need to design a sufficiently smooth command trajectory that ensures the control feasibility.

2.2 Feasible reference shaping

Consider now the objective of robustly displacing x from x_0 (at $t = 0$) to a given final output x_f, in a time period t_f, while respecting the input bounds

$$u \in \mathcal{U}, \tag{9}$$

where $\mathcal{U} \triangleq [-u_{\max}, u_{\max}] \subset \mathbb{R}$, for a given $u_{\max} < \infty$.

To accomplish this objective, a time-varying command $\bar{x}(t)$ with bounded second-time derivative $\ddot{\bar{x}}(t)$ will be designed based on a second-order polynomial S-curve, like the one illustrated in Figure 2. The chosen command shape contains three sequential phases, for simplicity, of equal duration $\delta t \equiv t_f/3$: acceleration, cruise, and deceleration. This command trajectory is analytically defined by the following second-order differential equation:

$$\ddot{\bar{x}}(t) = \zeta\left(t; t_f, \ddot{\bar{x}}_{\max}\right) \triangleq \begin{cases} \ddot{\bar{x}}_{\max}, & t \in \left[0, t_f/3\right) \\ 0, & t \in \left[t_f/3, 2t_f/3\right) \\ -\ddot{\bar{x}}_{\max}, & t \in \left[2t_f/3, t_f\right] \end{cases}, \tag{10}$$

where $\ddot{\bar{x}}_{\max}$ is its acceleration bound. Without loss of generality, let us assume that $\ddot{\bar{x}}_{\max} > 0$.

Without loss of generality, let us consider that $\bar{x}_0 = 0$ and $\dot{\bar{x}}_0 = 0$. By integrating equation (12) from $t = 0$ to $t = t_f$, we can obtain the final output $x_f = 2\ddot{\bar{x}}_{\max} t_f^2/9$ as well as the maximum output rate $\dot{\bar{x}}_{\max} = \ddot{\bar{x}}_{\max} t_f/3$. Therefore, by specifying x_f and the final time t_f, one can immediately compute

$$\ddot{\bar{x}}_{\max} = \frac{9}{2}\frac{x_f}{t_f^2}, \tag{11}$$

$$\dot{\bar{x}}_{\max} = \frac{3}{2}\frac{x_f}{t_f}. \tag{12}$$

By defining the state vector $\bar{\mathbf{x}} \triangleq \left(\bar{x}, \dot{\bar{x}}\right)$, we can put the trajectory kinematics (10) into the following linear time-invariant state-space representation:

$$\dot{\bar{\mathbf{x}}} = \mathbf{A}\bar{\mathbf{x}} + \mathbf{B}\bar{\zeta}\left(t; t_f, x_f\right), \tag{13}$$

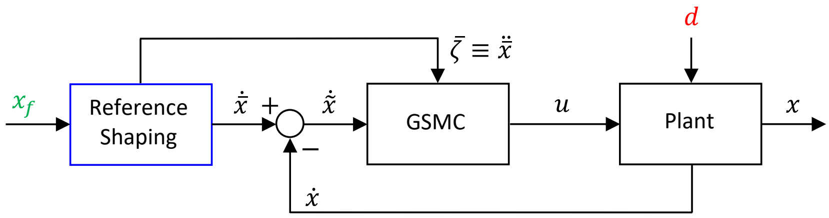

Figure 2. Block diagram describing the implementation of the proposed feasible GSMC.

where $\bar{\zeta}(t; t_f, x_f) \triangleq \zeta(t; t_f, \dddot{x}_{\max})$, and

$$\mathbf{A} \triangleq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \mathbf{B} \triangleq \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let us now detail relation (9) and write it in terms of the state $\bar{\mathbf{x}}$. By using the control law (5), (9) can be equivalently written as

$$-\kappa \operatorname{sign}(\sigma) + \dddot{\bar{x}} \in \mathcal{U}, \tag{14}$$

which is always satisfied under the condition

$$\dddot{x}_{\max} \leq u_{\max} - \kappa. \tag{15}$$

Therefore, for a given final position x_f, in order to generate the S-curve trajectory (13) with the minimum final time t_f which allows the satisfaction of the input constraints (9), we can design the following optimization problem:

$$\min_{t_f \in (0,\infty)} \ t_f$$
$$\text{s.t.}$$
$$\dddot{x}_{\max} \leq u_{\max} - \kappa$$

The analytical solution to the above optimization problem can be immediately obtained from (11) and (15) as

$$t_f^* = \frac{3\sqrt{2}}{2} \sqrt{\frac{x_f}{u_{\max} - \kappa}}. \tag{16}$$

2.3 Implementation

Figure 2 shows a block diagram of the system described by (1) in closed-loop with the feasible GSMC proposed in Subsections 2.1 and 2.2. In particular, the blue block is responsible for generating the command trajectory $\bar{x}$ for a given x_f according to Subsection 2.2. Note, however, that for the particular system considered here (a simple double integrator), the GSMC does not receive feedback of x, but only of $\dot{x}$.

Consider a sampling period T and the set $\mathcal{T} \triangleq \{t_k = kT, k = 1, 2, \ldots, \text{floor}(t_f^*/T)\}$ of discrete instants between $t = 0$ and $t = t_f^*$. To generate the feasible command trajectory from a given reference x_f we can use the following algorithm:

Algorithm 1. Inputs: x_f, $u_{\max}$, $\bar{\mathbf{x}}(0) = \mathbf{0}$.

- Compute t_f^* using (16);
- Compute $\dddot{x}_{\max}$ from t_f^* using (11);
- For all $t_k \in \mathcal{T}$:
  - Compute $\bar{\zeta}\left(t_k; t_f^*, x_f\right)$ using (10);
  - Compute $\bar{\mathbf{x}}(t_k)$ by integrating (13).

We can preview that the above algorithm has a very low computational burden and, for that, can be easily implemented online. This is a remarkable advantage of the proposed method over the reference governors (Garone *et al.*, 2017) and MPC-based schemes (Rubagotti *et al.*, 2011), which usually require the online numerical solution of computationally demanding optimization problems.

3. NUMERICAL EVALUATION

To illustrate the method, the overall system represented by the block diagram of Figure 2, with the reference shaping implemented according to Algorithm 1, is simulated in MATLAB. The integration is carried out by using the Euler method, with a sampling time of 10^{-5}. The disturbance is simulated as a sinusoidal signal with a frequency of 1 Hz and amplitude of 0.5. The maximal input is $u_{\max} = 1$ and the target position is $x_f = 1$.

Figure 3 shows the obtained plots. In particular, Figure 3(a) shows the output trajectory x as well as its command $\bar{x}$ and rate $\dot{x}$. We observe that the sigmoid trajectory command $\bar{x}$ with optimal final time $t_f^* = 3.35$ s is perfectly tracked by the system despite the relevance of the disturbance. In Figure 3(b), we show the system state trajectory in the phase plane, from which we can observe the three trajectory parts (acceleration, constant-rate cruise, and deceleration). In Figure 3(c), we show the control, the equivalent control, and the control bounds. We see that the control bounds are respected. Moreover, we note that the equivalent control is discontinuous at $t_f^*/3$ and $2t_f^*/3$, which is due to the discontinuities in $\ddot{\bar{x}}$ (see equation (5) and Figure 1 (left)). Finally, in Figure 3(d) we show the sliding variable. Note that the system does exhibit a global sliding motion; the very small deviation of σ from zero is due to the sampling time.

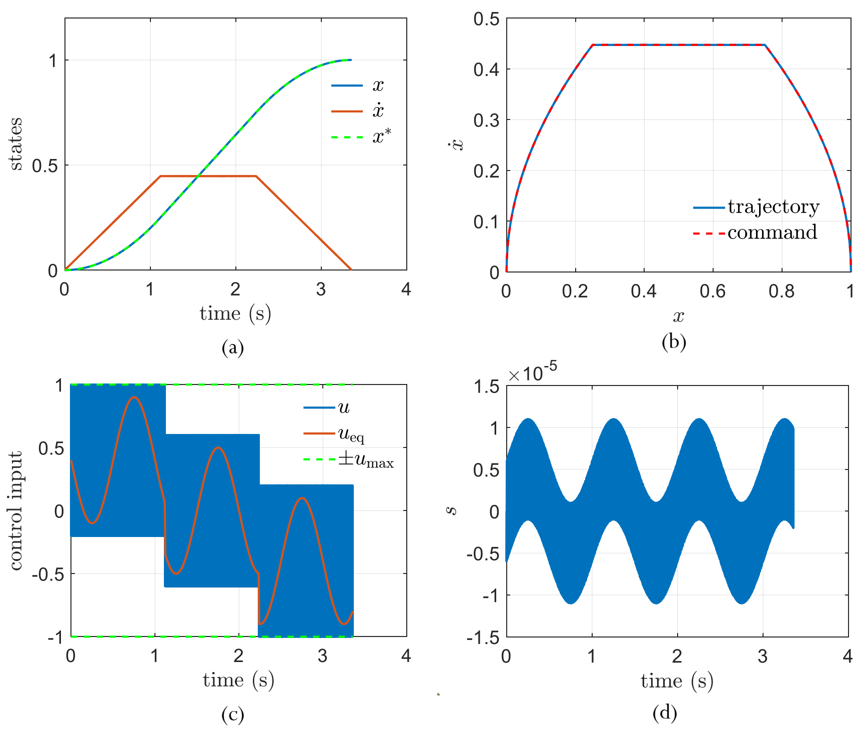

Figure 3. Simulation results illustrating the proposed feasible GSMC. (a) x, $\dot{x}$, and $\bar{x}$ along the time. (b) State trajectory on the phase plane. (c) control u, equivalent control $u_{\rm eq}$, and control bounds $\pm u_{\max}$. (d) sliding variable.

4. CONCLUSION

This paper has presented a simple and effective global sliding mode controller that allows a robust smooth minimum-time maneuver to a given output reference while ensuring the satisfaction of the control input bounds. In this preliminary work, the method has been formulated considering a disturbed scalar double integrator system and a second-order polynomial S-curve output command trajectory. Simulation has been performed, verifying the properties of the proposed method. This method can find applications in any mechatronic system requiring a robust and smooth displacement to a given setpoint. In future works, the proposed method can be extended to nonlinear multi-input systems and experimentally evaluated in some mechatronics systems.

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