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COB-2021-0313 ANALYSIS OF PRESTRESSED CABLE STRUCTURE THROUGH AN ENERGY MINIMIZATION TECHNIQUE

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Abstract. *In this paper, the principle of minimum total potential energy was applied to determine the static configuration of an elastic prestressed cable structure, such as those widely used in light roof structures, often called tenso structures. As large displacements are expected, a geometric nonlinear approach must be observed in the solution. In nonlinear formulations, the equilibrium configuration can only be computed at the final, initially unknown, displaced position. To find the forces acting on the bars and the deformed configuration of the structure, instead of a traditional nonlinear Finite Element Analysis via Newton's Method, this work proposes to apply a numerical optimization technique to minimize the total potential energy. In the end, the deformed configuration of the structure was obtained.*

Keywords: *Total potential energy minimization, prestressed structure, optimization, geometric nonlinearity.*

1. INTRODUCTION

Prestressed structures can be defined as a spatial system, formed by tensile cables and membranes, which are usually associated to light roof applications, thus classified due to the low self-weight of the elements compared to the supported loads. The Architect Frei Otto was responsible for the first studies of this constructive system (Graefe, 2018), which is currently, widely, used in covering soccer stadiums, cultural centers, commercial buildings etc. Studies that evaluated the efficiency of the prestressed structures can be observed in the works carried out by Haber and Abel (1982), Berger (1999), Gil and Bonet (2006) and Zhang et al. (2021), being the evolutionary process of the theory of these structures, formed by cables, presented by Irvine (1981).

Notably, these kinds of structures are usually designed to overcome large spans, therefore, susceptible to large displacements, which, consequently, lead to geometric nonlinearity as an important aspect to be considered to verify the structural performance of the elements (Yang and Tsay, 2007).

With the current technological advancement, provided by the development of high-powered computers, there are a range of efficient nonlinear numerical methods, which can be applied to determine the equilibrium of the prestressed structural elements. The varieties of mathematical methods, applicable to these problems, can be found in Brasil et al. (2015), being the nonlinear Finite Element Method (FEM), with resolution by the Newton-Raphson method, one of the more well known in the scientific community. An alternative and efficient way to solve this problem is through energy methods, applying the principle of stationary total potential energy, where the aim is to find the minimum total potential energy of the system, which guarantee de equilibrium state. The total potential energy is calculated by the sum of two parts: the internal potential energy, also characterized by the strain energy stored internally in the system, and the external potential energy, which corresponds to the work of external forces, determined through the multiplication of external forces by the displacements in the corresponding directions.

Different mathematical techniques can be used to solve the problem, defining the total potential energy as an objective function, for example, the technique called dynamic relaxation (Lewis (1989)); finite difference method, incorporated in the fast Lagrangian analysis of continuous three-dimensional (Halvordson (2007)) and optimization of the Total Potential Optimization using Meta-heuristic Algorithms (TPO/MA) (Toklu et. al (2017)). One other way to find the minimum total potential energy of the system is through mathematical techniques to minimize functions. An important theory in the application of such mathematical techniques is called structural optimization, where an objective function is defined, which drives the choice of variable values in order to optimize a certain parameter. Arora (2016) and Brasil and Silva (2019) present the tools and necessary mathematical conditions for applying the optimization concepts into many problems in structural engineering, whether involving static or dynamic restrictions.

In this paper, optimization techniques were applied, without constraints, to minimize the total potential energy of a spatial prestressed structure, considering geometric nonlinearity of the system, the initial stress of the cables and the concentrated loads applied in x, y and z directions. A generalized mathematical formulation was developed to determine the function of the total potential energy of the system in the deformed position. The function, defined as the objective function of the problem, is dependent on the displacements u, v and w that occurs at the directions x, y and z respectively. The formulation was applied for two practical cases: a benchmark example, solved by different methods and the other formed by a spatial system containing four cables, joined by a single free node, where the deformed position of the structure was obtained by minimizing the total potential energy of the system through the implementation, in the *Matlab* software.

2. GENERAL FORMULATION

To generalize the problem, consider the cable in Figure 1, in three dimensions, with initial length L_i, cross-sectional area A_i that starts at a fixed node i with coordinates:

$$(x_i, y_i, z_i), \tag{1}$$

not time-dependent, and connected to a free node A (x_a, y_a, z_a), which also belongs to the other cables, where the loads are applied, and whose displacements are:

$$\{\Delta\} = \{u, v, w\}^T, \tag{2}$$

where u, v e w correspond to the displacements in the directions x, y e z, respectively.

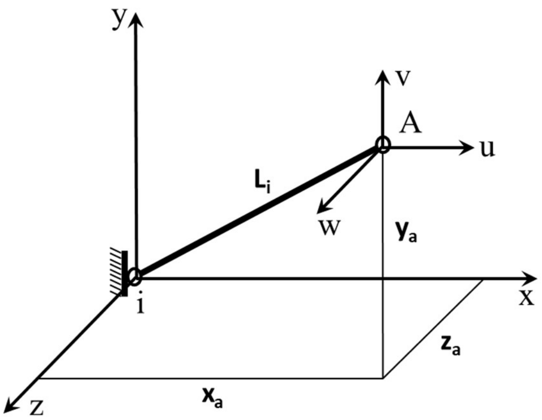


Figure 1. Schematic representation of the cable

Thus, the final coordinates of the point A are given by:

$$(x_a + u, y_a + v, z_a + w). \tag{3}$$

The initial (L_i) and final (L_f) length of the cable are represented by Eq. (4) and (5), respectively:

$$L_i = \sqrt{(x_a - x_i)^2 + (y_a - y_i)^2 + (z_a - z_i)^2}, \tag{4}$$

$$L_f = \sqrt{(x_a + u - x_i)^2 + (y_a + v - y_i)^2 + (z_a + w - z_i)^2}. \tag{5}$$

It is noteworthy that x_a-x_i; y_a-y_i e z_a-z_i are constant for each cable. It is possible to determine the normal elastic force in cable i (variable over time) by multiplying the axial stiffness (K_i) by the length change (ΔL_i):

$$K_i = \frac{E_i A_i}{L_i}, \tag{6}$$

$$N_i = K_i \Delta L_i, \tag{7}$$

where E_i is the Young's module (of the cable i) and ΔL_i the length change of the element, given by the difference of the final and initial lengths:

$$\Delta L_i = L_f - L_i. \tag{8}$$

It should be noted that the normal elastic force in the cable can be added to an initial normal force, constant in time (prestress force) N_0. Thus, the internal strain energy of cable i will be:

$$U_{\text{int}} = \sum_i^n \left(N_0 + \frac{1}{2} N_i \right) \Delta L_i \text{ , if } \Delta L_i > 0 \text{ or } U_{\text{int}} = 0 \text{ , if } \Delta L_i < 0. \tag{9}$$

The potential external energy of the system (work carried out by external loads) can be calculated by:

$$W_{ext} = \sum_{i=1}^n F_i \Delta_i, \tag{10}$$

where n is the number of actions, F_i the external loads and Δ_i the generic displacements. Thus, the total potential energy of the n cable system is given by:

$$\Pi = U_{\text{int}} - W_{ext}. \tag{11}$$

3. UNCONSTRAINED NONLINEAR OPTIMIZATION

The main objective of optimization is to maximize or minimize functions subjected to restrictions, the problem can be presented as follows:

minimize f(x)

subjected to g,

where f is the objective function, if the interest is to maximize, simply minimize $-f$; x is the variable vector of the objective function and g is the vector corresponding to the constraints.

Nonlinear optimization refers to the mathematical techniques for optimization of a nonlinear objective function or nonlinear constraints. In the particular case of a nonlinear unconstrained optimization, the function is maximized, or minimized, without considering the constraints, making:

minimize f(x)

To solve the unconstrained problem, it is necessary to define the convexity properties of the objective function that ensure that the point found is a global minimum point, i.e, the gradient (∇) of the function $f(x)$ is equal to zero:

$$\nabla f(x) = 0, \tag{12}$$

and that the Hessian matrix, $\nabla^2 f(x^*)$, is positive defined:

$$\left\langle d, \nabla^2 f(x^*) d \right\rangle > 0, \ \forall d \neq 0, \tag{13}$$

where x^* is a global minimum point, ∇^2 represents the second derivate of the function $f(x^*)$ and d is the difference given by $d = x\text{-}x^*$.

4. CASE STUDY

To apply the concepts in the previous section, two problems were modeled, a benchmark example for validating the results given by the present formulation and other more complex case. The benchmark example was proposed by Lewis (1989) and solved by Halvordson (2007), as presented in the work by Toklu et. al (2017) with different solution techniques. The first example is a one free and four fully constrained system that consists of four cables (Figure 2). The cross-sectional area of each cable is 0.785 mm^2 and the modulus of elasticity is 124800 N/mm^2. The pretension force at each cable is 200 N, there is a downward load on the node number 3 with intensity of -15 N.

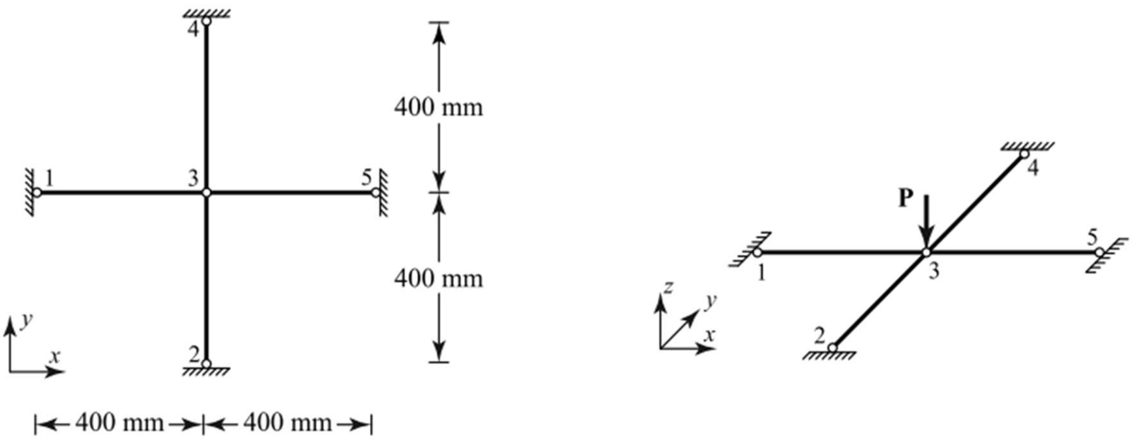
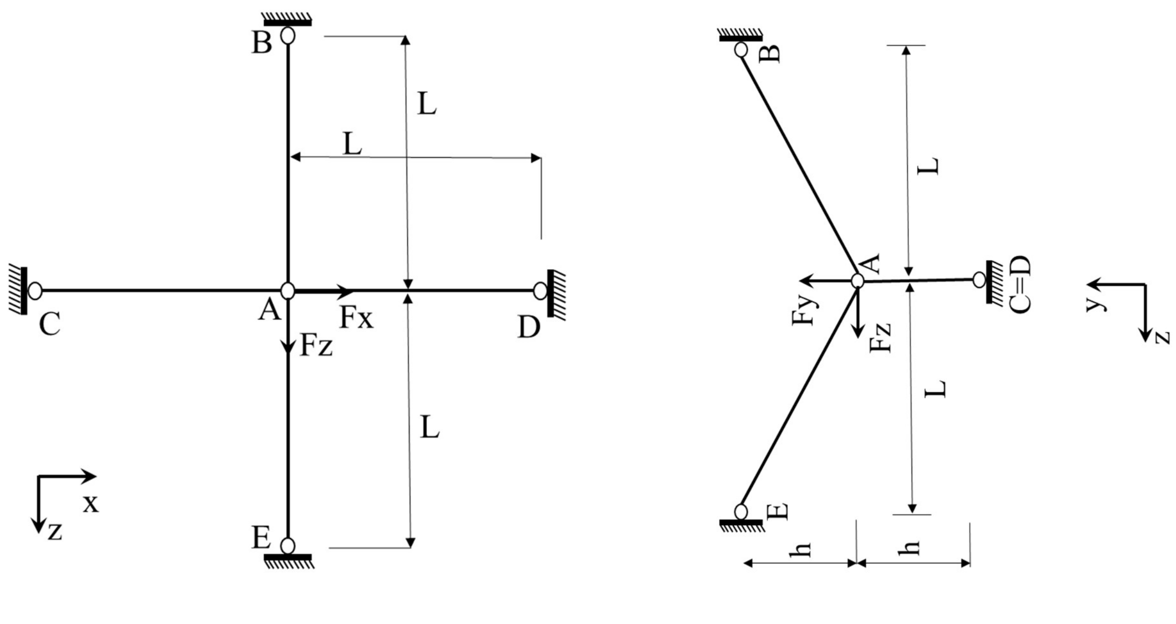


Figure 2. Schematic of flat cable net (Lewis (1989))

The second system consists of four cables (*AB*, *AC*, *AD* e *AE*) with circular cross-section of diameter $\phi = 16$ *mm*; distance $L = 10$ *m* and $h = 2$ *m*; Young's modulus $E = 21000$ *kN/cm*^{2}. Three concentrated forces are applied to node *A,* in the orthogonal directions *x, y, z*, $F_x = 30$ *kN*, $F_y = 20$ *kN*, and $F_z = 10$ *kN* and an initial prestress of 50 *MPa* (Figure 3). Note that node *A* is the only one with unconstrained translations, i.e., it is free to move in space, *x, y* and *z*, identified by the notation *u, v* and *w* in the respective directions, whose final position is the objective analysis.



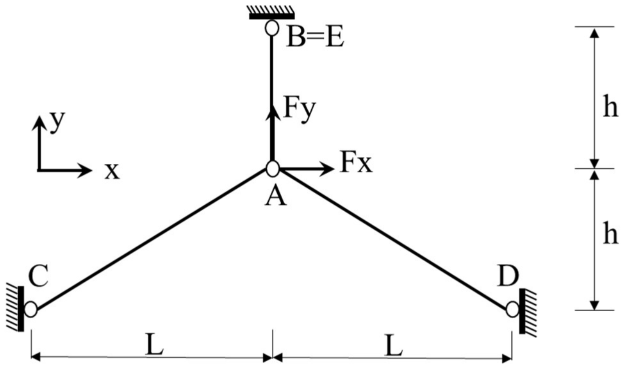


Figure 3. Schematic of the spatial structure.

The coordinates of the points, displayed in Figure 2 are: *A* (0, 0, 0); *B* (0, 200, -1000); C (-1000, -200, 0); *D* (1000, -200, 0) and *E* (0, 200, 1000). Thus, the initial lengths of each bar *AB*, *AC*, *AD* and *AE* are established, being equal, due to the symmetry of the system, and equal to 1019.80 cm. The final length of each bar is described as a function of the relative displacements in the *x, y* and *z* directions, given by *u, v* and *w*:

$$L_{fab} = \sqrt{(u)^2 + (v-200)^2 + (w+1000)^2}\ , \tag{14}$$

$$L_{fac} = \sqrt{(u+1000)^2 + (v+200)^2 + (w)^2}\ , \tag{15}$$

$$L_{fad} = \sqrt{(u-1000)^2 + (v+200)^2 + (w)^2}\ , \tag{16}$$

$$L_{fae} = \sqrt{(u)^2 + (v-200)^2 + (w-1000)^2}\ . \tag{17}$$

The axial stiffness can be calculated by applying Eq. (6), where $K_{ab} = K_{ac} = K_{ad} = K_{ae} = 41.39$ *kN/cm* for all cables, since they have the same cross section, material, and equivalent initial length. Thus, the total strain energy of the system will depend on the displacements *u, v* and *w*, is:

$$U_{int} = U_{ab} + U_{ac} + U_{ad} + U_{ae}\ , \tag{18}$$

where:

$$U_{ab} = \left(10.05 + \frac{41.39}{2}\left(\sqrt{(u)^2 + (v-200)^2 + (w+1000)^2} - 1019.8\right)\right)\left(\sqrt{(u)^2 + (v-200)^2 + (w+1000)^2} - 1019.8\right), \tag{19}$$

$$U_{ac} = \left(10.05 + \frac{41.39}{2}\left(\sqrt{(u+1000)^2 + (v+200)^2 + (w)^2} - 1019.8\right)\right)\left(\sqrt{(u+1000)^2 + (v+200)^2 + (w)^2} - 1019.8\right), \tag{20}$$

$$U_{ad} = \left(10.05 + \frac{41.39}{2}\left(\sqrt{(u-1000)^2 + (v+200)^2 + (w)^2} - 1019.8\right)\right)\left(\sqrt{(u-1000)^2 + (v+200)^2 + (w)^2} - 1019.8\right), \tag{21}$$

$$U_{ae} = \left(10.05 + \frac{41.39}{2}\left(\sqrt{(u)^2 + (v-200)^2 + (w+1000)^2} - 1019.8\right)\right)\left(\sqrt{(u)^2 + (v-200)^2 + (w+1000)^2} - 1019.8\right). \tag{22}$$

The work of external forces is given by:

$$W_{ext} = F_x u + F_y v + F_z w = 30u + 20v + 10w\ . \tag{23}$$

Therefore, the total potential energy of the system is calculated by:

$$\Pi = U_{ab} + U_{ac} + U_{ad} + U_{ae} - (F_x u + F_y v + F_z w) \, , \tag{24}$$

and the displacements *u*, *v* e *w* are determined by:

Minimize Π

An alternative to solve the minimization problem is through the implementation, in the *Matlab* software, which has pre-established routines (toolbox) for nonlinear unconstrained optimization. The routine used in the present analysis was *fminunc* which solves unconstrained multivariable optimization problems. The vector *x* represents the final displacements and is given by:

x = fminunc(fun,x0,options)

that minimizes the function (*fun*). In the studied case, the function to be minimized is the total potential energy, *x0*, which is the initial vector, i.e., the starting point for the search of the minimum point of the function, and *options* are used if it is wanted to create or modify the routine optimization options. In choosing that, the *optimset* command must be set.

5. RESULTS AND DISCUSSIONS

The formulation presented previously was implemented in *Matlab* to determine the displaced position of both structures, as described in section 4. The displacement of the node 3 (benchmark example) of the system are presented in Table 1, obtained by three studies in the literature and the proposed in this study, using the techniques of unconstrained optimization of the *fminunc* function.

Table 1.Benchmark example results

Node	Lewis and Jones (1984))			Halvordson (2007)			Toklu et. al (2017)			Present Method		
	u	*v*	*w*	*u*	*v*	*w*	*u*	*v*	*w*	*u*	*v*	*w*
3	0	0	-6.97 *mm*	0	0	-6.98 *mm*	0	0	-6.97 *mm*	0	0	-6.97 *mm*

The displacement vector, in the second example, was found to be:

$$\{\Delta\} = \{0.378,\ 5.401,\ 1.065\}^T \text{, in cm.} \tag{25}$$

Therefore, the minimum total potential energy of the system is:

$$\Pi = -55.06 J \, , \tag{26}$$

and the final axial forces on *AB*, *AC*, *AD*, and *AE* bars are:

$$N_{ab} = 10.05 kN \, , \tag{27}$$

$$N_{ac} = 39.92 kN \, , \tag{28}$$

$$N_{ad} = 24.61 kN \, , \tag{29}$$

$$N_{ae} = 10.05 kN \, . \tag{30}$$

Figure 3 shows a schematic representation of the deformed position (not to scale) of the structure, after applying the concentrated loads in node *A*.

$$u = 0.378 cm \, , \tag{31}$$

$$v = 5.401 cm \, , \tag{32}$$

$$w = 1.065 cm \, , \tag{33}$$

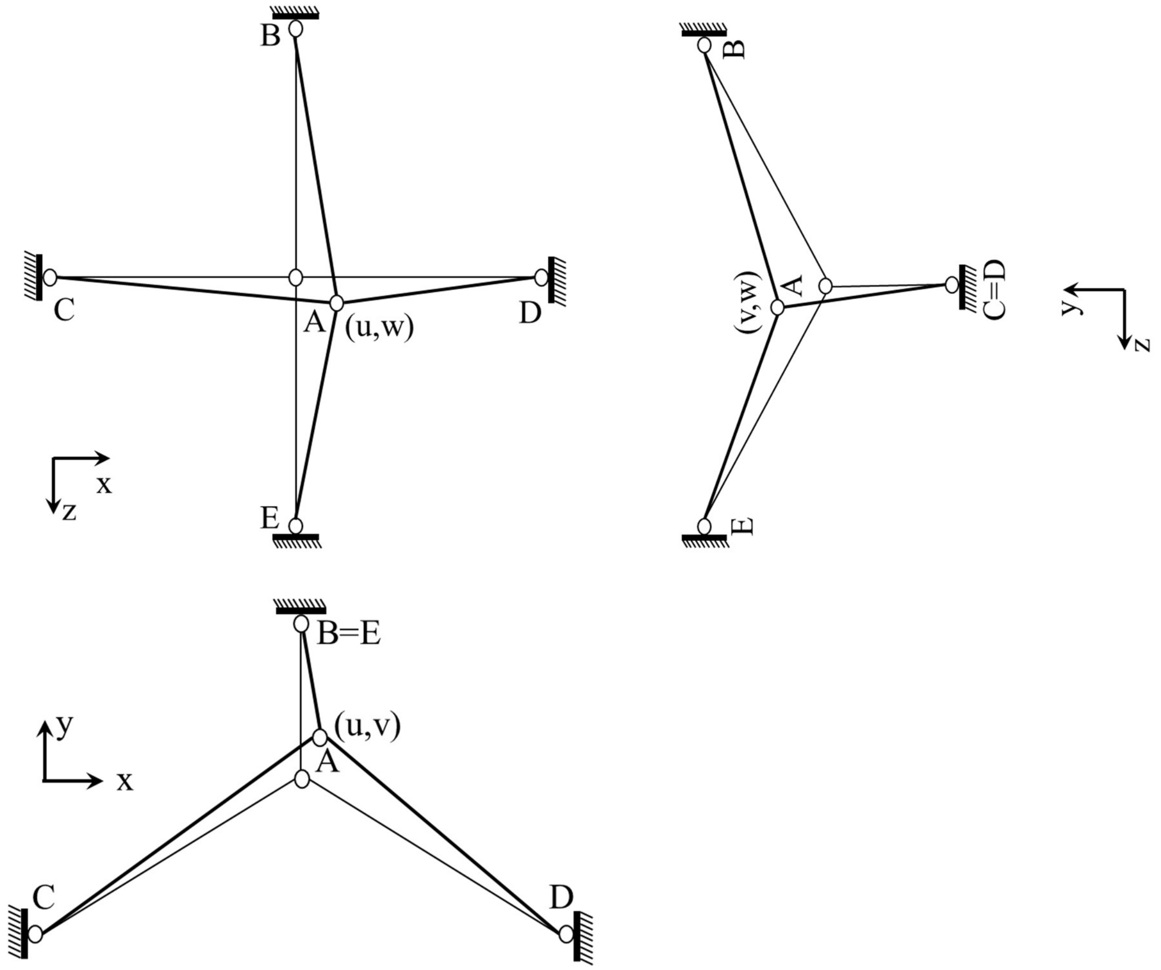


Figure 4. Schematic of the deformed position of the structure (not to scale).

6. CONCLUSION

In the present work, a static analysis through unconstrained nonlinear optimization techniques of a spatial prestressed structure was developed. In the analysis, the geometric nonlinearity that characterizes the system was considered in the solution together with the initial stress of the cables, and the concentrated loads in *x*, *y* and *z* directions. The mathematical solution of the problem was implemented in the *Matlab* program by using the *fminunc* routine, which allowed determining the displacements that minimized the total potential energy of the system.

A benchmark example available in the literature, solved by different authors and mathematical techniques, was used to validate the present formulation. At the end of the processing, it was possible to know the deformed position of the structure, the final lengths, and axial forces of all the bars. In conclusion, the mathematical modeling of the problem was presented with the resolution based on the deformed position of the structure and with the determination of its final displacements. For doing that, an unconstrained nonlinear optimization by concepts of potential stationary energy was employed. This technique proved to be an efficient and simplified alternative for analyzing structures that show similar behaviors.

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