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Semi-Analytic First-order Solution for Optimal Low-Thrust Trajectories around Moon

Sandro da Silva Fernandes
Luiz Arthur Gagg Filho
Instituto Tecnológico de Aeronáutica, Praça Marechal Eduardo Gomes 50, Vila das Acácias, 12228-900, São José dos Campos-SP, Brazil
arthurga@ita.br, sandro@ita

Abstract. *This work considers the development of a semi-analytical first-order solution for computing optimal low-thrust limited power trajectories around Moon. Based on the study about the main terms of Kaula's development for the disturbing potential of the Moon as function of the orbital elements for a lunar orbiter, it is assumed that for a preliminary mission analysis of a space vehicle propelled by a low-thrust engine this perturbing potential is described by the main three zonal harmonics J_2, J_3 and J_4. The second zonal harmonic is not as dominant with respect to the higher terms as it occurs for Earth; indeed, the difference in the order of magnitude of J_2 with respect to J_3 and J_4 is not so high. The semi-analytical solution is derived by using a canonical transformation approach and it is expressed in closed-form with no development in powers of eccentricity. Some numerical results show the influence of the zonal harmonics on the optimal trajectories.*

Keywords: Low-thrust trajectories, zonal harmonics of gravitational field, semi-analytical solution.

1. INTRODUCTION

In the last decades, the use of low-thrust propulsion systems has played an important role in space missions and the problem of determining optimal low-thrust trajectories is essential in mission planning (Morante *et al.*, 2021). Such propulsion systems are characterized by high specific impulse and low-thrust capability - the ratio between the maximum thrust acceleration and the gravity acceleration on the ground is small, between 10^{-4} and 10^{-2} (Marec, 1979). So, they operate continuously for several hours, or even, days.

Analytical solutions for specific transfers problems have been determined since the beginning of space exploration with the pioneer work by Tsien (1953). In his work, Tsien derived analytical approximated planar solutions in case of radial and circumferential thrust for initially circular orbits. From this pioneer work, several analytical works have been developed for specific maneuvers by applying averaging techniques Edelbaum (1965, 1966); Marec and Vinh (1980); Haissig *et al.* (1993); Geffroy and Epenoy (1997); Bonnard *et al.* (2006). It is remarkable to mention that analytical solutions are convenient for rapidly computation of low-thrust trajectories, or, they can be combined with powerful numerical techniques as initial guess.

In da Silva Fernandes and Carvalho (2019), a numerical-analytical procedure has been developed for computing optimal time-fixed low-thrust limited power transfers between arbitrary orbits in Earth's gravitational field considering the effects of the main zonal harmonics J_2, J_3 and J_4. The proposed procedure involves the development of a two-stage algorithm: in the first stage, a neighboring extremals algorithm is applied to solve the two-point boundary-value problem of going from an initial orbit to a final orbit at the prescribed final time, which is governed by a mean canonical system describing the secular behavior of the optimal trajectories. In the second stage, a Newton-Raphson algorithm is applied to adjust the initial values of the adjoint variables when the first order periodic terms associated with the second zonal harmonic J_2 and the optimal thrust acceleration are included.

The present work derives a semi-analytical solution to determine optimal low-thrust limited-power trajectories around Moon. Differently from Earth, the second zonal harmonic is not as dominant with respect to the higher terms; indeed, the difference in order of magnitude of J_2 with respect to J_3 and J_4 is not so high. Thus, a semi-analytical first order solution is derived including the periodic terms associated with J_3 and J_4. This solution is expressed in closed-form with no development in powers of eccentricity. It should be noted that study presented in this work does not include orbits with small eccentricities and/or inclinations, such that classical orbital elements are introduced. Some numerical results show the influence of the zonal harmonics on the optimal trajectories. This semi-analytical solution can be used in the previous proposed algorithm for computing optimal time-fixed low-thrust transfers between prescribed terminal orbits. This procedure considers the solution of boundary-value problem of going from the initial orbit to the final orbit.

The paper is organized as follows. In Section 2, a discussion about the perturbing gravitational potential of the Moon is presented. In Section 3, the optimization problem of time-fixed low-thrust limited-power trajectories is formulated and the maximum Hamiltonian function is derived. In Section 4, an approximated first order solution is obtained by applying a perturbation technique. Numerical results are presented for specified maneuvers in Section 5. Concluding remarks are presented in Section 6.

2. ANALYSIS OF PERTURBING GRAVITATIONAL POTENTIAL OF THE MOON

To build a semi-analytical first-order solution for low-thrust limited-power trajectories around Moon, the first step is to define which perturbations must be considered. According to Knezevic and Milani (1998) for low lunar orbit with altitude varying around 100 km, the main perturbations acting on the spacecraft are due to the lunar gravity field. Moreover, for a low-thrust propulsion system the time of flight of the maneuver is of the order of hours or even few days such that the other perturbations (gravitational attraction of Earth and Sun, and, radiation pressure from the Sun) can be ignored in a preliminary mission analysis.

The perturbations due to the lunar gravity field can be described by the well-known expansion of the lunar gravitational potential in spherical harmonics,

$$\Phi = -\frac{Gm_M}{r}\left[1 - \sum_{l=2}^{\infty} J_l\left(\frac{R_M}{r}\right)^l P_l\left(\sin\phi\right) + \sum_{l=2}^{\infty}\sum_{m=1}^{l} J_{lm}\left(\frac{R_M}{r}\right)^l P_{lm}\left(\sin\phi\right)\cos m\left(\lambda - \lambda_{lm}\right)\right] \tag{1}$$

where m_M is the mass of the Moon, G is the universal gravitational constant, R_M denotes the mean equatorial radius of the Moon ($R_M = 1738$ km), P_l is Legendre polynomial, P_{lm} is associated Legendre polynomial, ϕ is the latitude, λ is the longitude, λ_{lm} is a constant associated to the tesseral or sectorial harmonic, J_l is the coefficient for the zonal harmonic, J_{lm} is the coefficient for tesseral ($l = m$) or sectorial ($l \neq m$) harmonic. The main coefficients for the Moon are presented in Tab. 1 (Konopliv *et al.*, 2001) in which $C_{22} = J_{22}\cos m\lambda_{lm}$. In this paper, the gravity acceleration is given by $\mathbf{g} = -\nabla\Phi$.

Table 1. Coefficients for zonal and tesseral harmonics of the lunar gravitational potential

Coefficients	Value	Coefficients	Value
J_2	$+2.032337 \times 10^{-4}$	J_5	$+7.154090 \times 10^{-7}$
J_3	$+8.475900 \times 10^{-6}$	J_6	-1.357770×10^{-5}
J_4	-9.591930×10^{-6}	C_{22}	$+2.235700 \times 10^{-5}$

The gravitational potential can be represented as function of classical orbital elements $(a, e, I, \Omega, \omega, M)$ as developed by Kaula (2013) and Osório (1973), where a is the semimajor axis, e is the eccentricity, I is the inclination of the orbital plane, Ω is the longitude of ascending node, ω is the argument of periapsis (perilune) and M is the mean anomaly. The general term of the development proposed by Osório is given by

$$\Phi_{lm} = -\frac{Gm_M}{a}\left(\frac{R_M}{a}\right)^l J_{lm} \sum_{j=0}^{l} A_{lmj}(I) \sum_{p=-\infty}^{\infty} H_p^{-(l+1),(l-2j)}(e)\cos\Psi_{lmjp} \tag{2}$$

where $A_{lmj}(I)$ is Kaula's inclination function, $H_p^{-(l+1),(l-2j)}(e)$ is Hansen's coefficients and the angle Ψ_{lmjp} is defined by

$$\Psi_{lmjp} = pM + (l-2j)\omega + m(\Omega - \theta - \lambda_{lm}) + (l-m)\frac{\pi}{2} \tag{3}$$

where θ is the phase of the lunar rotation. For each selection of the integers j, l, m, p we can define the short periodic terms as those with $p \neq 0$ (terms containing the mean anomaly M), medium period terms as those with $p = 0$ but $m \neq 0$ (terms containing $m\theta$) and long periodic terms as those with $p = 0$ and $m = 0$. Besides these terms, resonant terms arise for satellite with mean montion $\dot{M}$ commensurable with $\dot{\theta}$. Such resonant terms are related to tesseral harmonics ($m \neq 0$). Finally, it should be noted that the contribution of a specific harmonic decreases with the term $(R_M/r)^l$ (see Eq. 1).

Taking into account the preceding discussion about the lunar potential and the values of harmonic coefficients in Tab. 1, it is assumed that a simplified model of the lunar gravitational potential to be used in the analysis of optimal low-thrust limited-power trajectories involves only the three main zonal harmonics J_2, J_3 and J_4. Although C_{22} is almost the

same order of magnitude of the selected zonal harmonics, its contribution will be ignored in this analysis, since it is also assumed that the powered trajectory is not a resonant orbit. In this way, the simplified model adopted to describe the lunar gravitational potential is quite similar to one used to study the orbit maintenance of a lunar polar orbiter (Knezevic and Milani, 1998).

3. THE MAXIMUM HAMILTONIAN FOR OPTIMAL LOW-THRUST LIMITED-POWER TRAJECTORIES

The optimization problem related to low-thrust limited-power transfers is stated as follows: "Transfer the space vehicle $\mathbf{M}$ from the initial state $(\mathbf{r_0}, \mathbf{v_0}, 0)$ at the initial time $t_0 = 0$ to the final state $(\mathbf{r_f}, \mathbf{v_f}, J_f)$ at the specified final time $t_f = T$, such that the fuel consumption is a minimum".

The state equations are;

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \qquad\qquad \frac{d\mathbf{v}}{dt} = -\frac{\mu}{r^3}\mathbf{r} + \nabla U + \mathbf{\Gamma} \qquad\qquad \frac{dJ}{dt} = \frac{1}{2}\Gamma^2 \qquad\qquad (4)$$

where μ is the gravitational parameter of Moon ($\mu = Gm_M$), $\mathbf{\Gamma}$ denotes the thrust acceleration, J is the consumption variable (it is a decreasing function of the space vehicle mass), U is the disturbing force function ($U = -\Phi$) related to the three main zonal harmonics J_2, J_3 and J_4, $U = U_2 + U_3 + U_4$, where

$$U_n = -\frac{\mu}{r} J_n \left(\frac{R_M}{r}\right)^n P_n (\sin\phi), n = 2, 3, 4 \qquad\qquad (5)$$

It is assumed that the thrust acceleration (control vector) $\mathbf{\Gamma}$ is unconstrained; that is, the thrust direction is free and the thrust magnitude is unbounded (Marec, 1979).

By applying Pontryagin Maximum Principle (Pontryagin, 2018), the optimal thrust acceleration must be selected from the admissible controls such that the Hamiltonian function H reaches its maximum. The Hamiltonian function is formed using Eq. 4, as it follows

$$H = \mathbf{p_r} \cdot \mathbf{v} + \mathbf{p_v} \cdot \left(-\frac{\mu}{r^3}\mathbf{r} + \nabla U + \mathbf{\Gamma}\right) + \frac{1}{2}p_J\Gamma^2 \qquad\qquad (6)$$

where $\mathbf{p_r}$, $\mathbf{p_v}$ p_J are the adjoint variables. Accordingly, the optimal thrust acceleration and the maximum Hamiltonian function are given, respectively, by

$$\mathbf{\Gamma}^* = \mathbf{p_v} \qquad\qquad (7)$$

$$H = \mathbf{p_r} \cdot \mathbf{v} + \mathbf{p_v} \cdot \left(-\frac{\mu}{r^3}\mathbf{r}\right) + \mathbf{p_v} \cdot \nabla U + \frac{1}{2}p_v^2 \qquad\qquad (8)$$

J is an ignorable variable (obtained by quadrature) and its adjoint assumes the value -1, computed from the transversality conditions.

The undisturbed part of the maximum Hamiltonian function describes an integrable canonical system, whose general solution defines a canonical transformation $(\mathbf{r}, \mathbf{v}, \mathbf{p_r}, \mathbf{p_v}) \to (a, e, I, \Omega, \omega, M, p_a, p_e, p_I, p_\Omega, p_\omega, p_M)$, where a, e, I, Ω, ω and M are the classical orbital elements, previously introduced, and $p_a, p_e, p_I, p_\Omega, p_\omega$ and p_M are the adjoint variables.

The maximum Hamiltonian function is invariant with respect to the canonical transformation and it is written in terms of orbital elements as it follows (da Silva Fernandes and Carvalho, 2019)

$$H^* = H_0 + H_U + H_\Gamma \qquad\qquad (9)$$

where

$$H_0 = np_M \qquad\qquad (10)$$

$$
\begin{aligned}
H_U = {} & \frac{2}{na}\frac{\partial U}{\partial M}p_a + \frac{\sqrt{1-e^2}}{na^2 e}\left[-\frac{\partial U}{\partial \omega} + \sqrt{1-e^2}\frac{\partial U}{\partial M}\right]p_e + \frac{1}{\sqrt{1-e^2}na^2\sin I}\left[-\frac{\partial U}{\partial \Omega} + \cos I\frac{\partial U}{\partial \omega}\right]p_I \\
& + \frac{1}{\sqrt{1-e^2}na^2\sin I}\frac{\partial U}{\partial I}p_\Omega + \frac{\sqrt{1-e^2}}{na^2 e}\left[\frac{\partial U}{\partial e} - \frac{e\cot I}{1-e^2}\frac{\partial U}{\partial I}\right]p_\omega + \frac{1}{na}\left[-2\frac{\partial U}{\partial a} - \frac{1-e^2}{ae}\frac{\partial U}{\partial e}\right]p_M \quad (11)
\end{aligned}
$$

$$
\begin{aligned}
H_\Gamma &= \frac{1}{2n^2a^2(1-e^2)} \left\{ \frac{1}{2}(1-\cos 2f)\left[2aep_a + (1-e^2)p_e\right]^2 + 2(1-e^2)\sin 2f\left[-ap_ap_\omega - \frac{(1-e^2)}{2e}p_ep_\omega\right] \right. \\
&+ 4(1-e^2)^{\frac{3}{2}}\sin f\left(\frac{-2e}{1+e\cos f}+\cos f\right)\left[ap_ap_M + \frac{(1-e^2)}{2e}p_ep_M\right] + \frac{(1-e^2)^2}{2e^2}(1+\cos 2f)p_\omega^2 \\
&- \frac{2(1-e^2)^{\frac{5}{2}}}{e^2}\cos f\left(\frac{-2e}{1+e\cos f}+\cos f\right)p_\omega p_M + \frac{(1-e^2)^3}{e^2}\left(\frac{-2e}{1+e\cos f}+\cos f\right)^2 p_M^2 \\
&+ 4a^2(1-e^2)^2\left(\frac{a}{r}\right)^2 p_a^2 + 4a(1-e^2)^2\left(\frac{a}{r}\right)(\cos E + \cos f)\,p_ap_e + (1-e^2)^2(\cos E + \cos f)^2\,p_e^2 \\
&+ \frac{4a(1-e^2)^2}{e}\left(\frac{a}{r}\right)\sin f\left(1+\frac{1}{1+e\cos f}\right)\left[p_ap_\omega - \sqrt{1-e^2}p_ap_M\right] \\
&+ \frac{2(1-e^2)^2}{e}(\cos E + \cos f)\sin f\left(1+\frac{1}{1+e\cos f}\right)\left[p_ep_\omega - \sqrt{1-e^2}p_ep_M\right] \\
&+ \left[\frac{(1-e^2)}{e}\sin f\left(1+\frac{1}{1+e\cos f}\right)\left[p_\omega - \sqrt{1-e^2}p_M\right]\right]^2 + \frac{1}{2}\left(\frac{r}{a}\right)^2\left[p_I^2 + \left(\frac{p_\Omega}{\sin I}-p_\omega\cot I\right)^2\right] \\
&\left. + \frac{1}{2}\left(\frac{r}{a}\right)^2\cos 2(\omega+f)\left[p_I^2 - \left(\frac{p_\Omega}{\sin I}-p_\omega\cot I\right)^2\right] + \left(\frac{r}{a}\right)^2\sin 2(\omega+f)p_I\left(\frac{p_\Omega}{\sin I}-p_\omega\cot I\right) \right\} \quad (12)
\end{aligned}
$$

with the disturbing force functions related to the main zonal harmonics written in orbital elements as

$$
U_2 = \frac{\mu}{a}J_2\left(\frac{R_M}{a}\right)^2\left[\left(\frac{1}{2}-\frac{3}{4}\sin^2 I\right)\left(\frac{a}{r}\right)^3 + \frac{3}{4}\sin^2 I\left(\frac{a}{r}\right)^3\cos 2(\omega+f)\right] \tag{13}
$$

$$
U_3 = -\frac{\mu}{a}J_3\left(\frac{a_e}{a}\right)^3\sin I\left\{\left(\frac{15}{8}\sin^2 I - \frac{3}{2}\right)\left(\frac{a}{r}\right)^4\sin(\omega+f) - \frac{5}{8}\sin^2 I\left(\frac{a}{r}\right)^4\sin(3(\omega+f))\right\} \tag{14}
$$

$$
\begin{aligned}
U_4 &= -\frac{\mu}{a}J_4\left(\frac{R_M}{a}\right)^4\left\{\left[\left(\frac{3}{8}-\frac{15}{8}\sin^2 I + \frac{105}{64}\sin^4 I\right)\left(\frac{a}{r}\right)^5\right] \right. \\
&+ \left. \left[\left(\frac{15}{8}\sin^2 I - \frac{35}{16}\sin^4 I\right)\left(\frac{a}{r}\right)^5\cos 2(\omega+f) + \frac{35}{64}\sin^4 I\left(\frac{a}{r}\right)^5\cos 4(\omega+f)\right]\right\}. \tag{15}
\end{aligned}
$$

Variable f denotes the true anomaly.

4. SEMI-ANALYTICAL FIRST ORDER SOLUTION

In this section, a semi-analytical first order solution which includes periodic terms is derived by applying Hori method (Hori, 1966), a perturbation technique based on Lie series. An infinitesimal canonical transformation,

$$
(a, e, I, \Omega, \omega, M, p_a, p_e, p_I, p_\Omega, p_\omega, p_M) \to (a', e', I', \Omega', \omega', M', p'_a, p'_e, p'_I, p'_\Omega, p'_\omega, p'_M)
$$

is built by a generating function S which involves a small parameter. It is assumed that the disturbing Hamiltonians related to the gravitational potential and to the optimal thrust acceleration, H_U and H_Γ, have the same order in the small parameter.

According to the algorithm of Hori method, the mean value of the maximum Hamiltonian function must be computed as follows

$$
\langle H^*\rangle = \frac{1}{2\pi}\int_0^{2\pi} H^* dM \tag{16}
$$

So, the new Hamiltonian function F resulting from the canonical transformation is given by

$$
F = \langle H^*\rangle = \langle H_0\rangle + \langle H_U\rangle + \langle H_\Gamma\rangle \tag{17}
$$

where $\langle H_0 \rangle$ is given by Eq. 10 with no modification, $\langle H_U \rangle$ is computed from Eq. 11 and is written with $\langle U \rangle$ replacing U. The mean values of the terms of disturbing function are

$$\langle U_2 \rangle = \frac{\mu}{a'} J_2 \left(\frac{R_M}{a'} \right)^2 \left(\frac{1}{2} - \frac{3}{4} \sin^2 I' \right) \left(1 - e'^2 \right)^{-\frac{3}{2}} \tag{18}$$

$$\langle U_3 \rangle = -\frac{3}{8} \frac{\mu}{a'} J_3 \left(\frac{R_M}{a'} \right)^3 e' \left(1 - e'^2 \right)^{-\frac{5}{2}} \left(1 - 5\cos^2 I' \right) \sin I' \sin \omega' \tag{19}$$

$$\begin{aligned} \langle U_4 \rangle &= -\frac{\mu}{a'} J_4 \left(\frac{R_M}{a'} \right)^4 \left(1 - e'^2 \right)^{-\frac{7}{2}} \left(1 + \frac{3}{2} e'^2 \right) \left(\frac{3}{8} - \frac{15}{8} \sin^2 I' + \frac{105}{64} \sin^4 I' \right) \\ &\quad - \frac{3}{4} \frac{\mu}{a'} J_4 \left(\frac{R_M}{a'} \right)^4 \left(1 - e'^2 \right)^{-\frac{7}{2}} e'^2 \left(\frac{15}{8} \sin^2 I' - \frac{35}{16} \sin^4 I' \right) \cos 2\omega' \end{aligned} \tag{20}$$

And, the mean value of the maximum Hamiltonian related to the optimal control is given by

$$\begin{aligned} \langle H_\Gamma \rangle &= \frac{a'}{2\mu} \left\{ 4a'^2 p_a'^2 + \frac{5}{2} \left(1 - e'^2 \right) p_e'^2 + \frac{p_I'^2}{2\left(1 - e'^2 \right)} \left[\left(1 + \frac{3}{2} e'^2 \right) + \frac{5}{2} e'^2 \cos 2\omega' \right] + \frac{5 - 4e'^2}{2e'^2} p_\omega'^2 \right. \\ &\quad + \frac{5e'^2 \sin 2\omega'}{2\left(1 - e'^2 \right)} p_I' \left(\frac{p_\Omega'}{\sin I'} - \cot I' p_\omega' \right) + \frac{1}{2e'^2} \left(5 + 11e'^2 + 4e'^4 \right) p_M^2 - \frac{\sqrt{1 - e'^2}}{e'^2} \left(5 + 2e'^2 \right) p_\omega' p_M' \\ &\quad \left. + \frac{1}{2\left(1 - e'^2 \right)} \left(\frac{p_\Omega'}{\sin I'} - \cot I' p_\omega' \right)^2 \left[\left(1 + \frac{3}{2} e'^2 \right) - \frac{5}{2} e'^2 \cos 2\omega' \right] \right\} \end{aligned} \tag{21}$$

The generating function of the infinitesimal canonical transformation built by means of Hori method is computed, at the first order, from the following equation:

$$\frac{\partial S}{\partial M} = \frac{1}{n} \left[H^* \right]_p \tag{22}$$

where $\left[H^* \right]_p$ denotes the periodic part of the Hamiltonian; that is, $H^* - \langle H^* \rangle$. Thus,

$$S = S_U + S_\Gamma \tag{23}$$

where S_U denotes the part of the generating function related to the gravitational potential, which is computed straightforwardly by integrating the periodic part of U, and, it can be written as

$$\begin{aligned} S_U &= \frac{2}{n'^2 a'} \frac{\partial W}{\partial M'} p_a' + \frac{\sqrt{1 - e'^2}}{n'^2 a'^2 e'} \left[-\frac{\partial W}{\partial \omega'} + \sqrt{1 - e'^2} \frac{\partial W}{\partial M'} \right] p_e' \\ &\quad + \frac{1}{\sqrt{1 - e'^2} n'^2 a'^2 \sin I'} \left[-\frac{\partial W}{\partial \Omega'} + \cos I' \frac{\partial W}{\partial \omega'} \right] p_I' + \frac{1}{\sqrt{1 - e'^2} n'^2 a'^2 \sin I'} \frac{\partial W}{\partial I'} p_\Omega' \\ &\quad + \frac{\sqrt{1 - e'^2}}{n'^2 a'^2 e'} \left[\frac{\partial W}{\partial e'} - \frac{e' \cot I'}{1 - e'^2} \frac{\partial W}{\partial I'} \right] p_\omega' + \frac{1}{n'^2 a'} \left[-2 \frac{\partial W}{\partial a'} - \frac{1 - e'^2}{a' e'} \frac{\partial W}{\partial e'} \right] p_M' \end{aligned} \tag{24}$$

with

$$W = W_2 + W_3 + W_4 \tag{25}$$

$$\begin{aligned} W_2 &= \int \left[U_2 \right]_p dM = \frac{\mu}{a'} J_2 \left(\frac{a_e}{a'} \right)^2 \left(1 - e'^2 \right)^{-\frac{3}{2}} \left\{ \left(\frac{1}{2} - \frac{3}{4} \sin^2 I' \right) \left(f' - M' + e' \sin f' \right) \right. \\ &\quad \left. + \frac{3}{4} \sin^2 I' \left[\frac{1}{2} \sin 2\left(\omega' + f' \right) + \frac{1}{6} e' \left(\sin \left(2\omega' + 3f' \right) + 3 \sin \left(2\omega' + f' \right) \right) \right] \right\} \end{aligned} \tag{26}$$

$$
\begin{aligned}
W_3 \;=\; \int [U_3]_p \, dM =\;& -\frac{\mu}{a'} J_3 \left(\frac{a_e}{a'}\right)^3 \sin I' \left(1-e'^2\right)^{-\frac{5}{2}} \left\{ \frac{3}{8} \left(1 - 5\cos^2 I'\right) \left[e'\left(f' - M' + \frac{1}{2}e'\sin f'\right) \sin\omega' \right. \right. \\
& -\; \left(1 + \frac{1}{4}e'^2\right)\cos\left(\omega' + f'\right) - \frac{1}{2}e'\cos\left(\omega' + 2f'\right) - \frac{1}{12}e'^2 \cos\left(\omega' + 3f'\right) \Big\} \\
& +\; \frac{5}{8}\sin^2 I' \left[\frac{1}{4}e'^2 \cos\left(3\omega' + f'\right) + \frac{1}{2}e'\cos\left(3\omega' + 2f'\right) + \left(\frac{1}{3} + \frac{1}{6}e'^2\right)\cos\left(3\omega' + 3f'\right) \right. \\
& +\; \frac{1}{4}e'\cos\left(3\omega' + 4f'\right) + \frac{1}{20}e'^2\cos\left(3\omega' + 5f'\right) \Big] \bigg\}
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
W_4 \;=\; \int [U_4]_p \, dM =\;& -\frac{\mu}{a'} J_4 \left(\frac{a_e}{a'}\right)^4 \left(1-e'^2\right)^{-\frac{7}{2}} \left\{ \left(\frac{3}{8} - \frac{15}{8}\sin^2 I' + \frac{105}{64}\sin^4 I'\right)\left[\left(1 + \frac{3}{2}e'^2\right)(f' - M')\right.\right. \\
& +\; \left(3e' + \frac{3}{4}e'^3\right)\sin f' + \frac{3}{4}e'^2\sin 2f' + \frac{1}{12}e'^3\sin 3f' \Big] \\
& +\; \left(\frac{15}{8}\sin^2 I' - \frac{35}{16}\sin^4 I'\right)\left[\frac{3}{4}e'^2\,(f'-M')\cos 2\omega' + \frac{1}{4}e'^3\sin f'\cos 2\omega' + \left(\frac{3}{2}e' + \frac{1}{4}e'^3\right)\sin\left(2\omega' + f'\right)\right. \\
& +\; \left(\frac{1}{2} + \frac{3}{4}e'^2\right)\sin\left(2\omega' + 2f'\right) + \left(\frac{1}{2}e' + \frac{1}{8}e'^3\right)\sin\left(2\omega' + 3f'\right) + \frac{3}{16}e'^2\sin\left(2\omega' + 4f'\right) \\
& +\; \frac{1}{40}e'^3\sin\left(2\omega' + 5f'\right)\Big] + \frac{35}{64}\sin^4 I'\left[\frac{1}{8}e'^3\sin\left(4\omega' + f'\right) + \frac{3}{8}e'^2\sin\left(4\omega' + 2f'\right)\right. \\
& +\; \left(\frac{1}{2}e' + \frac{1}{8}e'^3\right)\sin\left(4\omega' + 3f'\right) + \left(\frac{1}{4} + \frac{3}{8}e'^2\right)\sin\left(4\omega' + 4f'\right) + \frac{3}{5}\left(\frac{1}{2}e' + \frac{1}{8}e'^3\right)\sin\left(4\omega' + 5f'\right) \\
& +\; \frac{1}{8}e'^2\sin\left(4\omega' + 6f'\right) + \frac{1}{56}e'^3\sin\left(4\omega' + 7f'\right)\Big] \bigg\}
\end{aligned}
\tag{28}
$$

S_Γ denotes the part of the generating function related to the optimal thrust acceleration, and, it is computed similarly to S_U. Thus, taking into account that terms factored by p_M are ignored for simple transfers (no rendezvous), one finds

$$
\begin{aligned}
S_\Gamma \;=\;& \frac{1}{2}\sqrt{\frac{a'^5}{\mu^3}} \left\{ \left(8e'a'^2\sin E'\right) p_a'^{\,2} + \left(8a'\left(1-e'^2\right)\sin E'\right)p_a'p_e' - \left(\frac{8\sqrt{1-e'^2}}{e'}\cos E'\right)p_a'p_\omega' \right. \\
& +\; \left(1-e'^2\right)\left[-\frac{5}{4}e'\sin E' + \frac{3}{4}\sin 2E' - \frac{1}{12}e'\sin 3E'\right]p_e'^2 + \frac{\sqrt{1-e'^2}}{e'}\left[\frac{5}{2}e'\cos E'\right. \\
& -\; \frac{1}{2}\left(3 - e'^2\right)\cos 2E' + \frac{1}{6}e'\cos 3E'\right]p_e'p_\omega' + \frac{1}{\left(1-e'^2\right)}\left[\left(-e' + \frac{3}{8}e'^3\right)\sin E' + \frac{3}{8}e'^2\sin 2E'\right. \\
& -\; \frac{1}{24}e'^3\sin 3E'\right]\left[p_I'^{\,2} + \left(\frac{p_\Omega'}{\sin I'} - \cot I'p_\omega'\right)^2\right] + \frac{1}{\left(1-e'^2\right)}\left[p_I'^{\,2}\cos 2\omega' + 2p_I'\left(\frac{p_\Omega'}{\sin I'} - \cot I'p_\omega'\right)\sin 2\omega'\right. \\
& -\; \left(\frac{p_\Omega'}{\sin I'} - \cot I'p_\omega'\right)^2\cos 2\omega'\right]\left[\frac{5}{8}\left(-2e' + e'^3\right)\sin E' + \frac{1}{4}\left(1 + \frac{1}{2}e'^2\right)\sin 2E' + \frac{e'^3 - 2e'}{24}\sin 3E'\right] \\
& +\; \frac{1}{\sqrt{1-e'^2}}\left[\frac{5}{4}e'\cos E' - \frac{1 + e'^2}{4}\cos 2E' + \frac{1}{12}e'\cos 3E'\right]\left[-p_I'^{\,2}\sin 2\omega' + 2p_I'\left(\frac{p_\Omega'}{\sin I'} - \cot I'p_\omega'\right)\cos 2\omega'\right. \\
& +\; \left(\frac{p_\Omega'}{\sin I'} - \cot I'p_\omega'\right)^2\sin 2\omega'\right] + \frac{p_\omega'^{\,2}}{e'^2}\left[\left(\frac{5}{4}e' - e'^3\right)\sin E' + \left(-\frac{3}{4} - \frac{1}{2}e'^2\right)\sin 2E' + \frac{1}{12}e'\sin 3E'\right] \bigg\}
\end{aligned}
\tag{29}
$$

Variable E' denotes the eccentric anomaly.

The semi-analytical first order solution for simple transfers is derived from the following transformation equations determined by Hori method,

$$
x = x' + \frac{\partial S}{\partial p_x'}
\tag{30}
$$

where x denotes the orbital elements. The new set of canonical variables $(a', e', I', \Omega', \omega', M', p'_a, p'_e, p'_I, p'_\Omega, p'_\omega, p'_M)$ is obtained by integrating numerically the system of differential equations governed by the new Hamiltonian F, defined by Eqs. 17 to 21.

So, from the transformation equations described above, one finds that a semi-analytical solution can be written as

$$x(t) = x'(t) + \delta x(t) - \delta x(0) \tag{31}$$

where $\delta x(t)$ stands for $\left(\frac{\partial S}{\partial p'_x}\right)$ computed at time t. It is assumed that the original variables (orbital elements) and the new ones satisfy the same specified initial conditions, $a(0) = a'(0) = a_0$, $e(0) = e'(0) = e_0$, $I(0) = I'(0) = I_0$, $\Omega(0) = \Omega'(0) = \Omega_0$ and $\omega(0) = \omega'(0) = \omega_0$.

5. NUMERICAL RESULTS

In this section, the semi-analytical solution derived in the previous sections is applied to generate optimal low thrust limited-power trajectories around the Moon, in order to analyze the effects of the zonal harmonics. This procedure is called direct problem by Marec and Vinh (1980). Tables 2 and 3 show the initial values of the orbital elements and the adjoint variables for two numerical trajectory propagation. Note that semi-major axis is defined in kilometers and the angular variables are defined in degrees. The set of differential equations governed by the mean Hamiltonian is solved numerically by using a Runge-Kutta-Fehlberg algorithm of 4-5 order. To avoid different orders of magnitude between orbital elements, dimensionless variables are used. The canonical units used to make variables dimensionless are: the distance unit corresponds to $R_M = 1738$ km, the time unit corresponds to $1/2\pi$ of the period of the circular orbit with radius equal to R_M, and, the unit mass is defined such that $\mu = 1$. For simplicity, adjoint variables are dimensionless in Tabs. 2 and 3. The time of flight is 5 days for both simulations.

Table 2. First set of initial conditions

Orbital element	Initial value	Adjoint variable	Initial value
a (km)	2000.0	p_a	1.5532×10^{-5}
e	0.10	p_e	-3.6285×10^{-5}
I (°)	20.0	p_I	-8.4079×10^{-5}
Ω (°)	60.0	p_Ω	-7.9349×10^{-5}
ω (°)	30.0	p_ω	1.0035×10^{-7}
M (°)	0.0	p_M	0.0

Table 3. Second set of initial conditions

Orbital element	Initial value	Adjoint variable	Initial value
a (km)	1950.0	p_a	2.6837×10^{-5}
e	0.05	p_e	-1.2698×10^{-5}
I (°)	10.0	p_I	1.1418×10^{-4}
Ω (°)	60.0	p_Ω	-1.2610×10^{-4}
ω (°)	90.0	p_ω	-4.0950×10^{-6}
M (°)	0.0	p_M	0.0

Figures 1 and 2 depict the time evolution of the orbital elements considering the different gravitational models: the black line represents the solution computed by the central field model, the blue line represents the solution computed by the harmonics model that includes the disturbing effects due to the second zonal harmonic J_2, and, the red line represents the solution computed by the harmonics model that includes the disturbing effects of the three main zonal harmonics J_2, J_3 and J_4. From the results presented in these figures, the major comments are:

1. The disturbing effects of the main zonal harmonics can be significant for a optimal low thrust maneuver. The main contribution is due to the second zonal harmonic, but the contributions due to the third and the fourth zonal harmonic can be also relevant.
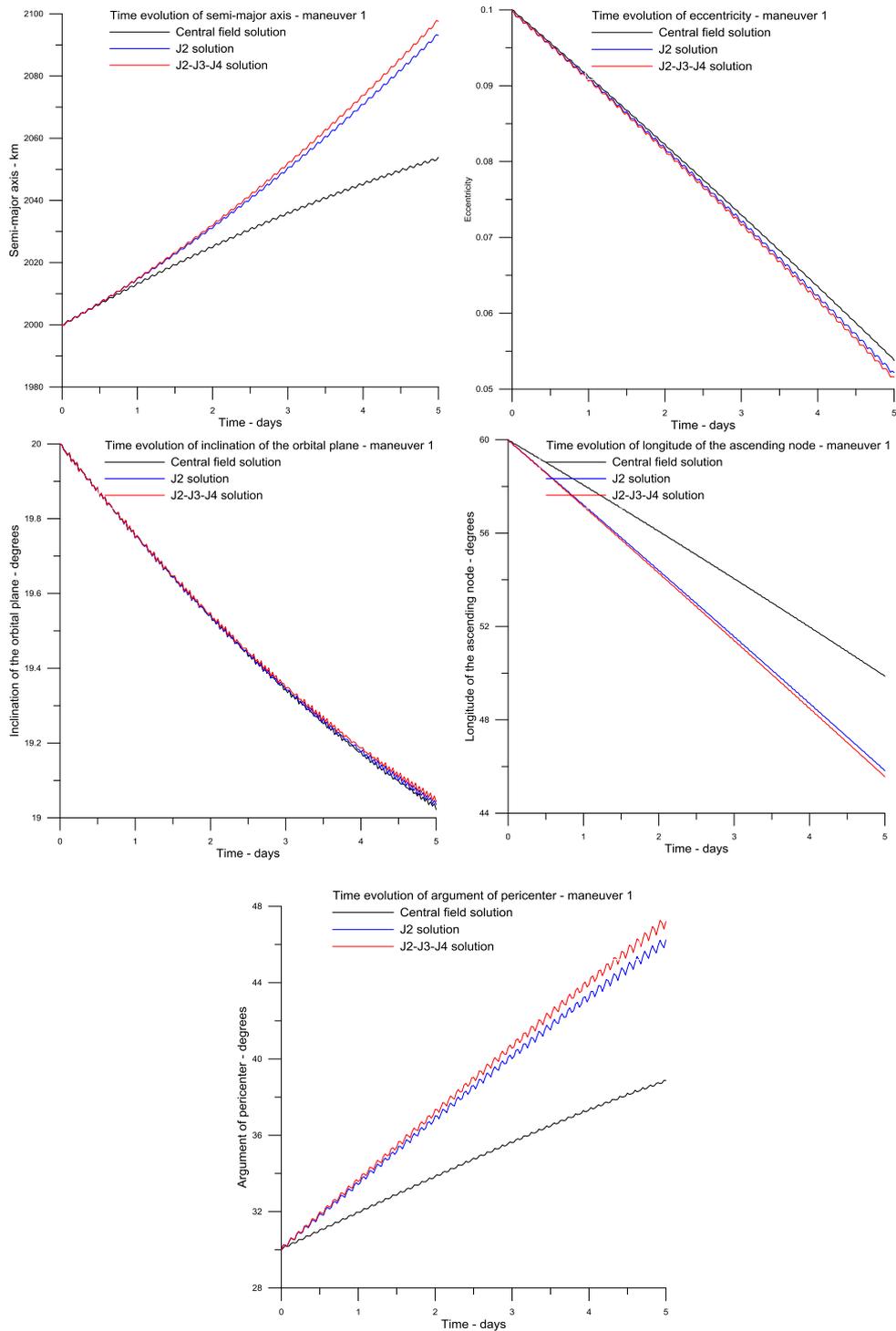
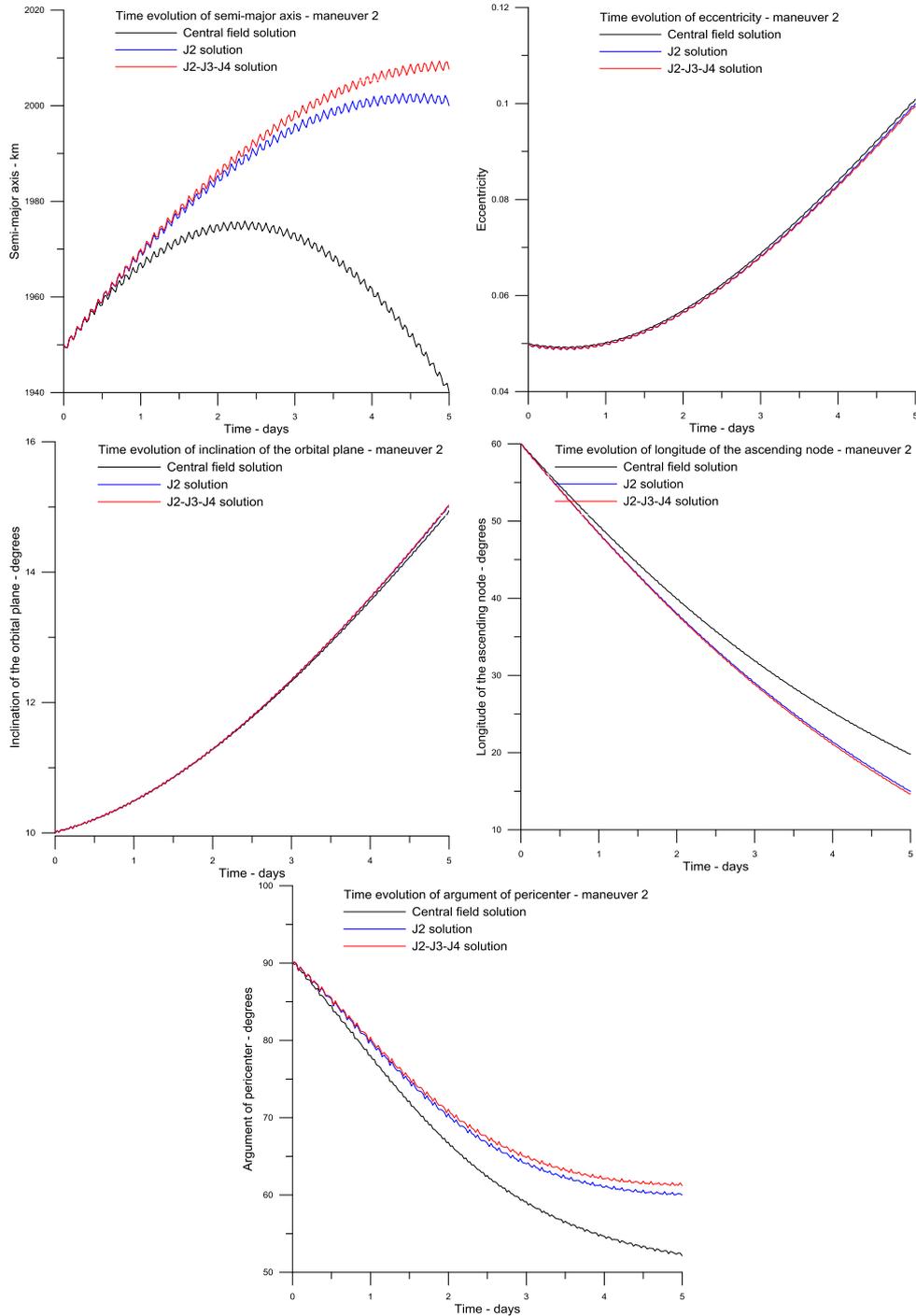Figure 1. Time evolution of orbital elements for maneuver 1

Figure 2. Time evolution of orbital elements for maneuver 2

2. For both maneuvers, semi-major axis, argument of pericenter and longitude of the ascending node seem to be more affected by the disturbing effects due to the zonal harmonics.

3. The amplitude of periodic terms can not be ignored in the solution of the two-point boundary value problem of going from an initial orbit to a prescribed final orbit at a specified final time. It should be noted that the time of flight of both maneuvers is 5 days. For maneuvers with very large time of flight, the amplitude of periodic terms can become less significant.

6. CONCLUSIONS

In this paper, a semi-analytical solution has been derived for computing optimal low thrust limited-power trajectories around the Moon, considering that the gravitational field includes the three main zonal harmonics in the development of the Moon's potential. This solution is obtained by means of canonical transformation theory and it includes, besides secular terms, periodic terms associated with the optimal thrust acceleration and the zonal harmonics. Numerical results show that the disturbing effects due to the zonal harmonics can be relevant to describe the time evolution of the orbital elements in a transfer problem. Moreover, the periodic terms cannot be ignored in the solution of the two-point boundary value problem of going from an initial orbit to a prescribed final orbit if the time of flight is not too large. So, the semi-analytical solution can be used in the solution of such boundary-value problem by using the algorithm described in previous work (da Silva Fernandes and Carvalho (2019)).

7. ACKNOWLEDGEMENTS

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