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COBEM2021-1660 A Non-Linear MPFA Based on a Flux Limited Splitting Method Satisfying the Discrete Maximum Principle for One-Phase Fluid Flow Simulation on Heterogeneous and Anisotropic Media

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Abstract. *The locally conservative family of linear Control Volume Distributed Multi-Point Flux Approximation (CVD-MPFA) has been successfully employed over the last two decades to solve the diffusion equation. However, for highly anisotropic and heterogeneous media, these schemes may produce solutions with spurious oscillations that violate the Discrete Maximum Principle (DMP). To overcome this limitation, we present a repair technique based on a new non-linear flux limited splitting approach for MPFA finite volume methods to solve the steady state diffusion problem. In particular, we implement our repair technique using the Multi-Point Flux approximation Method with a Diamond Stencil (MPFA-D). Our technique eliminates spurious oscillations by imposing the DMP without losing mass conservation. This is done similarly to the M-Matrix Flux Splitting method that splits the Two-Point Flux Approximation (TPFA) contribution, which is naturally monotone, from the cross diffusion flux terms. We compute a parameter that calculates, for each control surface flux, the maximum amount of cross diffusion term at every iteration that makes the solution monotonic. Our results are compared with different linear and robust MPFA methods. For all examples solved, the non-linear flux-splitting technique has proven to be robust and accurate in producing solutions free of spurious oscillations honouring the DMP for arbitrary unstructured meshes and heterogeneous and anisotropic diffusion tensors.*

Keywords: *Non-linear Scheme, Flux Splitting, Flux Limited Splitting, Multi-Point Flux Approximation, DMP, Reservoir Simulation*

1. INTRODUCTION

Over the last two decades, the locally conservative flux-continuous family of linear Control Volume Distributed (CVD) (Edwards and Rogers, 1998; Edwards, 2002) finite volumes, also known as Multi-Point Flux Approximation (MPFA) (Aavatsmark *et al.*, 1998; Aavatsmark, 2002), has been successfully employed to solve steady-sate diffusion problems arisen from various engineering and physical phenomena. Unlike its precedent, the Two-Point Flux Approximation (TPFA), they are consistent on non k-orthogonal grids and for full permeability tensors. However, the TPFA is still an industry standard due to its simplicity, computational efficient and the guarantee of solutions free from spurious oscillation as this scheme satisfies the Discrete Maximum Principle (DMP). On the context of subsurface reservoir, the CVD-MPFA methods, are the best option for simulations that require the discretization of more convoluted geometries such as inclined laminated layers, channels, fractures with abrupt variations on the permeability field, just as the ones commonly found on unconventional reservoirs. The advantages of these formulations are clear and they were extended and employed in several different applications (Parramore *et al.*, 2016; de Souza *et al.*, 2020; Cavalcante *et al.*, 2020). Still, the loss of monotonicity on the pressure field may give birth, for instance, to the appearance of spurious gas where the pressure falls incorrectly below the bubble point (Nordbotten *et al.*, 2007) and induce inaccurate non-physical Darcy flux. Despite the best efforts to improve accuracy and mitigate this issue (de Carvalho *et al.*, 2007; Edwards and Zheng, 2008; Chen *et al.*, 2008; Gao and Wu, 2010; Contreras *et al.*, 2016, 2019), any linear scheme that is more than first-order accurate may produce local extrema, according to the Godunov's theorem. This has lead to an increase interest on non-linear techniques not susceptible to this limitation. In the context of diffusion problems, several authors have developed Non-Linear Finite Volumes (NL-FV) formulations (Le Potier, 2005; Lipnikov *et al.*, 2007; Yuan and Sheng, 2008; Cances *et al.*, 2013; Gao and Wu, 2013; Queiroz *et al.*, 2013; Contreras *et al.*, 2021) aiming the creation of monotonic methods. In particular, we are interested on a specific set of strategies capable of repairing non monotonic solutions from second-order or higher schemes. Outside the finite volume context, we highlight the work of Kuzmin *et al.* (2009) who devised a posteriori repair technique for finite element methods that splits diffusive and anti-diffusive fluxes and performs a slope limitation to guarantee DMP. For the Finite Volume family, Edwards (2000) created a technique that splits CVD-MPFA formulations in TPFA and cross diffusion terms (CDT) creating an iterative semi-implicit scheme, conservative at each iteration level, driven by the M-Matrix of TPFA. Pal and Edwards (2006, 2011) extended this scheme by creating pressure limiters that improve the monotonicity of the original CVD formulation. The objective of this article is to create an iterative repair technique for linear finite volume schemes capable of restoring the monotonicity of CVD-MPFA in general, while maintaining the conservation throughout any intermediate step. The presented formulation enhances the Flux Splitting techniques in (Edwards, 2000; Pal and Edwards, 2006, 2011), by calculating at every iteration a pressure dependent relaxation parameter that estimates the right amount of cross diffusion a given flux approximation can have without violating the DMP. The formulation is tested using the Multi-Point Flux Approximation with a Diamond Stencil (MPFA-D) Gao and Wu (2010); Contreras *et al.* (2016), a scheme with full pressure support known to produce accurate solutions even for strongly anisotropic heterogeneous diffusion problems and distorted meshes. The article is organized as follows: in the first three sections, we introduce the mathematical model and numerical schemes that motivated this work. In Section 2, we describe the steady-state diffusion equations arisen from single-phase flow in incompressible porous media, in Section 3, we briefly describe the numeric formulation of TPFA and MPFA-D, and in Section 4, we summarize the procedure used by the original flux splitting technique to obtain an semi-implicit method. The following three sections we detail our formulation. In Section 5, we define a notation and we use it to derive and detail our Flux Limited Splitting (FLS) method, we present the algorithms to compute the cross diffusion relaxation parameter, we discuss stability and initial solution of the scheme, and we present a detailed fluxogram of FLS, in Section 6, we present the numerical examples that compare the FLS with the standard MPFA-D; and finally in Section 7 we present our conclusions.

2. Mathematical Formulation

The steady-state single-phase incompressible and isothermal flow through an incompressible domain Ω is defined:

$$\vec{\nabla} \cdot \vec{v} = Q \quad \text{where} \quad \vec{v} = -K\nabla p \tag{1}$$

where Q stands for the source and sink term, $\vec{v}$ represents a diffusive flux, in this case the Darcian flux, with K representing a 2 x 2 permeability tensor and p the pressure.

We define the appropriate boundary conditions as:

$$\begin{cases} p = g_d & \text{on} \quad \partial\Omega_d \\ \vec{v} \cdot \vec{N} = g_n & \text{in} \quad \partial\Omega_n \end{cases} \tag{2}$$

where $\vec{N}$ denotes the normal area vector, g_d stands for prescribed pressure, g_n prescribed flux, defined respectively on $\partial\Omega_d$, on $\partial\Omega_n$ i.e. Dirichlet and and Neumann boundaries with, $\partial\Omega_n \cap \partial\Omega_d = \varnothing$

3. Finite Volume Formulation

Let Ω also assume the representation of a discrete approximation of the physical domain containing nv control volumes (CV). By integrating Equation 1 and applying Gauss's Divergence Theorem in a CV k, we obtain:

$$\int_{\partial\Omega_k} \vec{v} \cdot \vec{N} dA = \int_{\Omega_k} Q dV \tag{3}$$

By applying mean value theorem, we can write the discrete form of Equation 3 as:

$$\sum_{f \in \partial\Omega_k} \vec{v} \cdot \vec{N}_f = Q_k \quad \forall \Omega_k \in \Omega \tag{4}$$

where Q_v stands for the volumetric source and sink term in k, f represents one of the faces that comprise the boundaries $\partial\Omega_k$ of a control volume k, and $\vec{N}_{IJ}$ the vector area normal to f.

3.1 Two Point Flux Approximation

The Two Point Flux Approximation (TPFA) approximates the flux across a face IJ (see Figure 1) as:

$$(\vec{v} \cdot \vec{N})_{IJ} \simeq -\frac{2K^n_{\hat{L}} K^n_{\hat{R}}}{K^n_{\hat{L}} h^{\hat{R}}_{IJ} + K^n_{\hat{R}} h^{\hat{L}}_{IJ}} (p_{\hat{R}} - p_{\hat{L}}) \tag{5}$$

where the two adjacent volumes to the left and to the right are represented respectively by $\hat{L}$ and $\hat{R}$, K^n_i represents the normal permeability (see Equation 8), and p_i the pressure of the CV i with $i = \hat{L}, \hat{R}$.

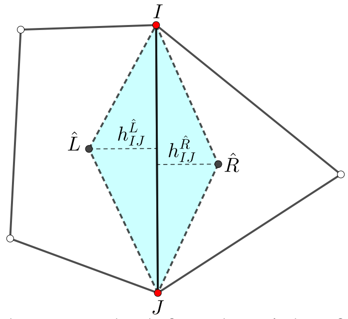


Figure 1: Representation of two adjacent volumes to the left and to right of a shared face IJ with centroids represented by, respectively, $\hat{L}$ and $\hat{R}$, and heights $h_{IJ}^{\hat{L}}$ and $h_{IJ}^{\hat{R}}$. On the MPFA-D, the diamond region consists of two triangles $\triangle I\hat{L}J$ and $\triangle I\hat{R}J$.

3.2 MPFA-D

In this article, we employ the Multi-Point Flux Approximation with a Diamond Stencil (MPFA-D), first introduce by Gao and Wu (2010) and brought to the multiphase flow context by Contreras *et al.* (2016). The MPFA-D unique flux approximation across a face IJ is given by:

$$(\vec{v}\cdot\vec{N})_{IJ} \simeq \tau_{IJ}[p_{\hat{R}} - p_{\hat{L}} - \nu_{IJ}(p_J - p_I)] \tag{6}$$

where p_I and p_J are the pressures projected on the nodes I and J that comprise the analysed face as shown in Figure 1.

The scalar transmissibility τ_{IJ} and the non-dimensional tangential components ν_{IJ} of the flux in Equation 6 are defined as:

$$\tau_{IJ} = -|IJ|\frac{K^n_{IJ(\hat{L})}K^n_{IJ(\hat{R})}}{K^n_{IJ(\hat{L})}h^{\hat{R}}_{IJ} + K^n_{IJ(\hat{R})}h^{\hat{L}}_{IJ}} \qquad \nu_{IJ} = \frac{\overrightarrow{IJ}\cdot\overrightarrow{\hat{L}\hat{R}}}{|IJ|^2} - \frac{1}{|IJ|}\left(\frac{K^t_{IJ(\hat{L})}}{K^n_{IJ(\hat{L})}}h^{\hat{L}}_{IJ} + \frac{K^t_{IJ(\hat{R})}}{K^n_{IJ(\hat{R})}}h^{\hat{R}}_{IJ}\right) \tag{7}$$

We define the normal $K^n_{IJ(i)}$ and tangential $K^t_{IJ(i)}$ projections of the permeability tensor onto IJ as:

$$K^n_{IJ(i)} = \frac{\vec{N}^T_{IJ}K_{(i)}\vec{N}_{IJ}}{|IJ|^2} \quad \text{and} \quad K^t_{IJ(i)} = \frac{\vec{N}^T_{IJ}K_{(i)}\overrightarrow{IJ}}{|IJ|^2} \quad \text{with } i = \hat{L} \text{ or } \hat{R}. \tag{8}$$

Please, see Contreras *et al.* (2016) or Gao and Wu (2010) for more information on boundary conditions and the Linear Preserving Weights type 2 used for the interpolation of p_I and p_J.

3.3 The Discrete system of Equations

By using any flux approximation like those described by Equations 5 and 6, we obtain a discrete system of equation given as:

$$M_i p_i = Q_i \tag{9}$$

where M_i is the transmissiblity matrix, p_i the unknown pressure vector and Q_i a vector containing the source and sink term for each CV, and where i represents the flux approximation scheme used such $i =$ TPFA, MPFA-D, etc.

4. Flux Splitting

Here, we will briefly summarize the Flux Spiting (Edwards, 2000) that transform the solving of a MPFA linear system into a non-linear semi-implicit TPFA system.

Let us start by defining a linear system of equations obtained using any MPFA.

$$M_{\text{MPFA}}p = Q_{\text{MPFA}} \tag{10}$$

This equation can be rewritten by splitting the TPFA terms from the cross diffusion terms (CDT) with no loss of generality as:

$$M_{\text{TPFA}}p + M_{\text{CDT}}p = Q_{\text{TPFA}} + Q_{\text{CDT}} \quad \text{where} \quad M_{\text{CDT}} = M_{\text{MPFA}} - M_{\text{TPFA}}, \quad Q_{\text{CDT}} = Q_{\text{MPFA}} - Q_{\text{TPFA}} \tag{11}$$

Edwards (2000) derived the semi-implicit scheme that converges to the exactly MPFA solution of the form:

$$M_{\text{TPFA}} p^{k+1} = Q_{\text{CDT}} + Q_{\text{TPFA}} - M_{\text{CDT}} p^k \tag{12}$$

According to Edwards (2000), the method is stable, if:

$$||I - M_{\text{TPFA}}^{-1} M_{\text{MPFA}}||_\infty \leq 1 \tag{13}$$

where I stands for the identity matrix and the subscript ∞ represents the infinity norm.

One of the biggest advantages of this technique is that at each level of iteration, flux is conservative (Edwards, 2000; Pal and Edwards, 2011, 2006). This means that, with only a few steps it is possible to obtain an approximate and accurate solution that holds the mass conservation law for all control volumes in the computational domain. Although in this article, we are employing the MPFA-D in a 2-D domain, this method can be easily applicable to 3-D, provided that a proper flux approximation is used.

5. Non Linear Flux Limited Splitting Scheme

Now, we present the new Flux Limited Splitting that can be used to eliminate spurious oscillations from the MPFA technique in general. Before introducing the technique, we will briefly introduce some mathematical tools that will help us deriving the limited flux expression.

By applying the Divergence theorem to a discrete control volume k, we have:

$$\sum_{f \in \partial\Omega_k} \vec{N}_f = \vec{0} \quad \forall \Omega k \in \Omega \tag{14}$$

where f represents a face on the boundaries $\partial\Omega_k$ of the CV k and $\vec{N}_f$ denotes the vector area normal to f.

Let us generalize this idea and create a matrix expression that simultaneously performs the balance of each normal area vector of all CV $\Omega_k \in \Omega$.

We define $\mathbf{N}$ as a vector containing the area vector normal to all faces as:

$$\mathbf{N} = [\vec{N}_i]_{n_f \times 1} = \begin{bmatrix} \vec{N}_1 \\ \vec{N}_2 \\ \dots \\ \vec{N}_{n_f} \end{bmatrix} \tag{15}$$

where n_f stands for the number of discrete faces in Ω and where $\vec{N}_i$ represents the normal area vector of a face i that is calculated with no particular orientation.

The idea is to define a discrete operator $\mathbb{D}$ that performs the divergence theorem in all control volumes in CV of Ω taking into consideration the orientation of area vector in $\mathbf{N}$ such that:

$$\mathbb{D}\mathbf{N} = \vec{0} \tag{16}$$

Thus, the Discrete Divergence Operator $\mathbb{D}$ needs to be defined as:

$$\mathbb{D} = (d_{ij})_{n_v \times n_f} \quad \text{such as} \quad d_{ij} = \begin{cases} 0 & j \notin i \\ \gamma(i,j) & j \in i \end{cases} \quad \text{where} \quad \gamma(i,j) = \begin{cases} 1 & \vec{N}_j \cdot \vec{\hat{i}\bar{j}} > 0 \\ -1 & \text{otherwise} \end{cases} \tag{17}$$

where n_v is the number of CV in Ω, i, j are, respectively, volumes and faces Ω, $\vec{\hat{i}\bar{j}}$ is a vector that connects the centroid $\hat{i}$ of the control volume i to the centre $\bar{j}$ of the face j.

Using a similar notation, we can define an array containing any vector property $\vec{X}$ interpolated on a face and oriented in accordance with the area vectors in $\mathbf{N}$ as:

$$\mathbf{X} = [\vec{X}_i]_{n_f \times 1} = \begin{bmatrix} \vec{X}_1 \\ \vec{X}_2 \\ \dots \\ \vec{X}_{nf} \end{bmatrix} \tag{18}$$

and the operation:

$$\mathbf{X} \odot \mathbf{Y} = [\vec{X}_i \cdot \vec{Y}_i] \tag{19}$$

where $\odot$ represents the element-wise inner product, $\cdot$.

Using this notation let us write an expression for the discrete Darcian flow approximation found in Equations 5 and 6.

$$\mathbf{V} \odot \mathbf{N} = [(\vec{v} \cdot \vec{N})_i] \tag{20}$$

This way, we can interpreted the TPFA in Equation 5, as the following matrix operation:

$$(\mathbf{V} \odot \mathbf{N})_{\text{TPFA}} = T_{\text{TPFA}} p \tag{21}$$

where T_{TPFA} represents the face transmissiblity matrix, a $n_f \times n_v$ sparse matrix that stores TPFA coefficients of all faces.

On some formulations such as the MPFA-D, there are boundary conditions terms that do not depend on the pressure. Thus, the equation is formulated slightly different:

$$(\mathbf{V} \odot \mathbf{N})_{\text{MPFA}} = T_{\text{MPFA}} p - F \tag{22}$$

where represents $T_{\text{MPFA-D}}$ the face transmissiblity matrix, a $n_f \times n_v$ sparse matrix that stores pressure dependent coefficients of the MPFA approximation, and F represents source and sink terms intrinsic of MFPA approximations that are added to the wells source and sink term.

By applying the Discrete Divergence Operator on Equation 20, and expanding, we can write a generalization for the mass conservation equation as defined in Equation 1, given by:

$$\mathbb{D}(\mathbf{V} \odot \mathbf{N}) = \left[\sum_{f \in \partial\Omega_i} \vec{v} \cdot \vec{N}_{f \in \partial\Omega_i} \right]_{n_v} = [Q_i]_{n_v} = Q_{\text{s}} \tag{23}$$

where Q_{s} represents the volumetric source and sink term.

By substituting the definition of Equation 21 in 23, we have:

$$M_{\text{TPFA}} p = Q_{\text{TPFA}} \quad \text{with} \quad M_{\text{TPFA}} = \mathbb{D} T_{\text{TPFA}} \quad Q_{\text{TPFA}} = Q_{\text{s}} \tag{24}$$

Similarly, if we substitute Equation 22 in 23, we have:

$$M_{\text{MPFA}} p = Q_{\text{MPFA}} \quad \text{with} \quad M_{\text{MPFA}} = \mathbb{D} T_{\text{MPFA}} \quad Q_{\text{MPFA}} = Q_{\text{s}} + \mathbb{D} F \tag{25}$$

Now, let us split the MPFA flux in Equation 22 in terms of the TPFA and the CDT components.

$$(\mathbf{V} \odot \mathbf{N})_{\text{MPFA}} = (\mathbf{V} \odot \mathbf{N})_{\text{TPFA}} + (\mathbf{V} \odot \mathbf{N})_{\text{CDT}} \tag{26}$$

By substituting Equations 21 and 22 in 26, we can define the flux of CDT as:

$$(\mathbf{V} \odot \mathbf{N})_{\text{CDT}} = T_{\text{CDT}} p - F \quad \text{with} \quad T_{\text{CDT}} = T_{\text{MPFA}} - T_{\text{TPFA}} \tag{27}$$

As the TPFA is known for respecting the DMP, the idea behind our technique is to create a parameter that limits the cross diffusion terms. Therefore, a Flux Limited Splitting (FLS) expression is written as:

$$(\mathbf{V} \odot \mathbf{N})_{\text{FLS}} = (\mathbf{V} \odot \mathbf{N})_{\text{TPFA}} + B (\mathbf{V} \odot \mathbf{N})_{\text{CDT}} \quad \text{with} \quad B = \begin{bmatrix} \beta_1 & \dots & \beta_{n_f} \end{bmatrix}^T I \quad \text{where} \quad 0 \leq \beta_i \leq 1 \quad \forall i \leq n_f \tag{28}$$

Once again, we apply the Discrete Divergence Operator on Equation 21 and adding the source and sink terms to obtain the discrete mass conservation equation, defined as:

$$\mathbb{D}(\mathbf{V} \odot \mathbf{N})_{\text{FLS}} = \mathbb{D}(\mathbf{V} \odot \mathbf{N})_{\text{TPFA}} + \mathbb{D} B (\mathbf{V} \odot \mathbf{N})_{\text{CDT}} = Q_{\text{s}} \tag{29}$$

Using flux definitions of the TPFA and CDT in Equations 21 and 22, we have:

$$\mathbb{D} T_{\text{FLS}} p = \mathbb{D} T_{\text{TPFA}} p + \mathbb{D} B T_{\text{CDT}} p - \mathbb{D} B F = Q_{\text{s}} \tag{30}$$

We can finally derive the Flux Limited Splitting recurrence law:

$$\mathbb{D} T_{\text{TPFA}} p^{k+1} = Q_{\text{s}} + \mathbb{D} B^k F - \mathbb{D} B^k T_{\text{CDT}} p^k \tag{31}$$

The idea is to calculate at each iteration level $B^k = B(p^k)$, which gives the amount of cross diffusion terms that avoids spurious oscillations.

It is also possible to write Equation 31 as:

$$M_{\text{TPFA}} p^{k+1} = Q_{\text{s}} + Q^k_{\text{LCDT}} - M^k_{\text{LCDT}} p^k \quad \text{with} \quad Q^k_{\text{LCDT}} = \mathbb{D} B^k F \quad \text{and} \quad M^k_{\text{LCDT}} = \mathbb{D} B^k T_{\text{CDT}} \tag{32}$$

where subscript LCDT stands for limited cross diffusion terms.

5.1 Limitation of the Cross Diffusion Terms

To devise an algorithm hat computes a proper relaxation factor B^k for a certain iteration level, let us suppose that instead of limiting the flux, the limitation was imposed on the balance cross diffusion terms. Therefore, we could write Equation 32 as:

$$M_{\text{TPFA}} p^{k+1} = Q_{\text{s}} + A(Q^k_{\text{LCDT}} - M^k_{\text{LCDT}} p^k) \quad A = \begin{bmatrix} \alpha_1 & \dots & \alpha_{n_f} \end{bmatrix}^T I \quad \text{where} \quad 0 \leq \alpha_i \leq 1 \quad \forall i \leq n_v \tag{33}$$

By premultiplying Equation 33 by M^{-1}_{TPFA}, we can isolate p^{k+1}:

$$p^{k+1} = p_{\text{TPFA}} + A M^{-1}_{\text{TPFA}} (Q^k_{\text{CDT}} - M^k_{\text{LCDT}} p^k) \quad \text{where} \quad p_{\text{TPFA}} = M^{-1}_{\text{TPFA}} Q_{\text{s}} \tag{34}$$

The Discrete Maximum Principle states between two distinct iteration levels k and $k+1$, the following inequality must also hold:

$$\min(p^k_i) \leq p^{k+1}_i \leq \max(p^k_i) \tag{35}$$

where $\min(p^k_i)$ and $\max(p^k_i)$ represent, respectively, the minimum and the maximum pressure values of any volume that shares a node with i.

We can expand p^{k+1} using the definition in Equation 34:

$$\min(p^k_i) \leq p_{\text{TPFA}\, i} + \alpha_i W_i \leq \max(p^k_i) \quad \forall i \in \Omega \quad \text{with} \quad W_i = [M^{-1}_{\text{TPFA}} (Q^k_{\text{LCDT}} - M^k_{\text{LCDT}} p^k)]|_i \tag{36}$$

After some algebraic manipulation, we obtain the following interval in which α_i satisfy the DMP:

$$L_i : \min(p^k_i) - p_{\text{TPFA}\, i} \leq +\alpha_i W_i \leq \max(p^k_i) - p_{\text{TPFA}\, i} \quad \forall i \in \Omega \tag{37}$$

To avoid extrapolation, we limit each interval with:

$$N_i : L_i \cap [0, 1] \quad \forall n \in n_f \tag{38}$$

We define the relaxation parameter associated with the balance of the cross diffusion terms as:

$$\alpha_i = \begin{cases} \max(N_i) & N_i \neq \varnothing \\ 1 & N_i = \varnothing \end{cases} \quad \forall i \leq n_v \tag{39}$$

At this point, the limitation factor is computed for each control volume. To obtain a relaxation parameter at each face of the domain we use the following relation:

$$\beta_i = \min(\alpha_{\hat{L}}, \alpha_{\hat{R}}) \quad \forall i \leq n_f \tag{40}$$

where $\hat{L}$ and $\hat{R}$ represent respectively, the control volumes neighbours to the left and right of the face i.

The algorithm presented on this section was devised in collaboration with Cavalcante (2021).

5.2 Stability

With the introduction of this relaxation factor for the cross-diffusion term, have devised a non-liner scheme where $p = p(B)$ and $B = B(p)$. Thus, unlike the work of Edwards (2000) that computes the solution of the MPFA system of equations using the semi-implicit relation in Equation 12, the FLS system of equation is modified at every iteration according to Equation 30. Therefore, it is natural to study the stability of each step.

Let p_H be the exact solution of the implicit FLS system of equations, such that:

$$\mathbb{D} T^k_{\text{FLS}} p_H = \mathbb{D} T_{\text{TPFA}} p_H + \mathbb{D} B^k T_{\text{CDT}} p_H - \mathbb{D} B^k F = Q_{\text{s}} \tag{41}$$

If we subtract Equation 41 from the semi-implicit law of recurrence of the FLS in Equation31, we have:

$$\mathbb{D} T_{\text{TPFA}} e^{k+1} + \mathbb{D} B^k T_{\text{CDT}} e^k = 0 \tag{42}$$

with the relative discrete solution error $e^{k+1} = p^{k+1} - p_H$ and $e^k = p^k - p_H$.

After some algebraic manipulation, we have:

$$\frac{e^{k+1}}{e^k} = M^{-1}_{\text{TPFA}} M^k_{\text{CDT}} \quad \text{with} \quad M^k_{\text{CDT}} = \mathbb{D} B^k T_{\text{CDT}} \tag{43}$$

A method is known to be stable if the spectral radius is bounded by unity (Edwards, 2000), which follows:

$$||M^{-1}_{\text{TPFA}} M^k_{\text{CDT}}||_\gamma = ||I - M^{-1}_{\text{TPFA}} M^k_{\text{FLS}}||_\gamma \leq 1 \quad \text{with} \quad M^k_{\text{CDT}} = M^k_{\text{FLS}} - M_{\text{TPFA}} \tag{44}$$

calculated using a γ norm.

The constrains imposed on β ensure that M^k_{FLS} is always bounded by M^k_{MPFA} and M^k_{TPFA}. This way, as we update B, the restrictions on the cross diffusion terms increase the overall stability of FLS. In our experience, the FLS has converged in all tested cases, even when the original M-Matrix Flux Splitting did not.

5.3 Flux Limited Spiting Initial Solution

As a non-linear scheme, the FSL can be interpreted as an iterative technique that simultaneously optimize the relaxation parameters and pressure solution. Therefore, a proper choice of an initial pressure p^0 can improve the convergence and quality of the FSL solution. Edwards (2000) and Pal and Edwards (2006, 2011) use the $p^0 = p_{\text{TPFA}}$ but in our experience, this choice introduces more artificial diffusion making the FLS solution more diffusive. Another possible initial solution could be the p_{MPFA}, however this would require solving an implicit scheme. A possible compromise is to allow the method to run with no flux limitation so the pressure solution can freely approximate p_{MPFA} and after a few iterations we start the FSL. This alternative has proven to present the best results, however during these few iterations while the FLS has not started, the scheme is subjected to stability criteria in Equation 13 more restrictive that the one in Equation 44. This may lead to an unnecessary loss of stability at the beginning of the iterative process. To overcome this issue, we employ the technique designed by Pal and Edwards (2006, 2011) that computes a limited pressure $\tilde{p}$ described as follows:

$$\tilde{p}_i^{k+1} = \begin{cases} p_i^{k+1} & \min(p_i^k) \leq p_i^{k+1} \leq \max(p_i^k) \\ p_i^k & \text{otherwise} \end{cases} \tag{45}$$

The main idea behind strategy is that, if the method loses stability, it will also lose monotonicity. In this cases, we do not update the pressure value. In all other parts of the domain, the solution continues to be regularly updated. This way, when the process of flux limitation is started, the pressure solution is as close as possible to the p_{MPFA}.

5.4 Flux Limited Splitting Algorithm

All the ideas described on the previous sections are summarized in the fluxogram presented on Figure 2. In our experience, it is enough to set (lim = 5) the number of iterations before the beginning of the flux limiting algorithm.

6. Fluid flow in a Heterogenous domain with a square hole in a extremely anisotropic media

In order to evaluate our method, we have adapted the problem from (Queiroz *et al.*, 2013; Contreras *et al.*, 2019) originally devised to test loss of monotonicity of FV schemes. The problem consists in a square domain $\Omega = [0, 1]^2$ with a concentric square hole $\Omega = [4/9, 5/9]^2$ as described on Figure 3. The boundaries of the domain are subjected to Dirichlet boundary conditions, $g_D^1 = 0$ at $\partial\Omega_1$, and $g_D^2 = 2$ at $\partial\Omega_2$.

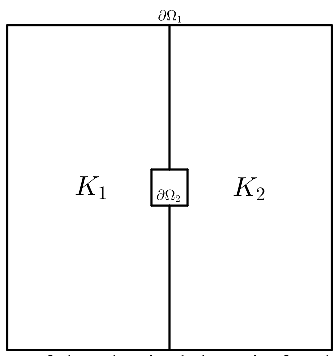


Figure 3: Representation of the physical domain for the example in Section 6.

The domain is split in half, with each side described by extremely anisotropic permeability tensors. This creates a discontinuity in the middle of the domain. The permeability tensor field is given by:

$$K(x,y) = \begin{cases} \begin{bmatrix} \cos(0) & -\sin(0) \\ sin(0) & cos(0) \end{bmatrix} \begin{bmatrix} 100 & 0 \\ 0 & 0.01 \end{bmatrix} \begin{bmatrix} \cos(0) & \sin(0) \\ -sin(0) & cos(0) \end{bmatrix} & K_1 : x \leq 0.5 \\ \begin{bmatrix} (y+\epsilon)^2 + \delta(x+\epsilon)^2 & -(1-\delta)(y+\epsilon)(x+\epsilon) \\ -(1-\delta)(y+\epsilon)(x+\epsilon) & (x+\epsilon)^2 + \delta(y+\epsilon)^2 \end{bmatrix} & K_2 : x > 0.5 \end{cases} \quad \text{with} \quad \epsilon = 10^{-3}, \delta = 10^3 \tag{46}$$

The domain is discretized creating a unstructured mesh with 2848 quadrilateral control volumes. For the sake of comparison, we have run a simulation using three different schemes, the original M-Matrix Flux Splitting, a direct simulation using the MPFA-D, and our method, the Flux Limited Splitting.

First of all, for the original Flux Splitting, the method has not converged. This happened because of the the extremely heterogeneous, anisotropic and discontinuous permeability field that gave birth to a linear system of equations that violates

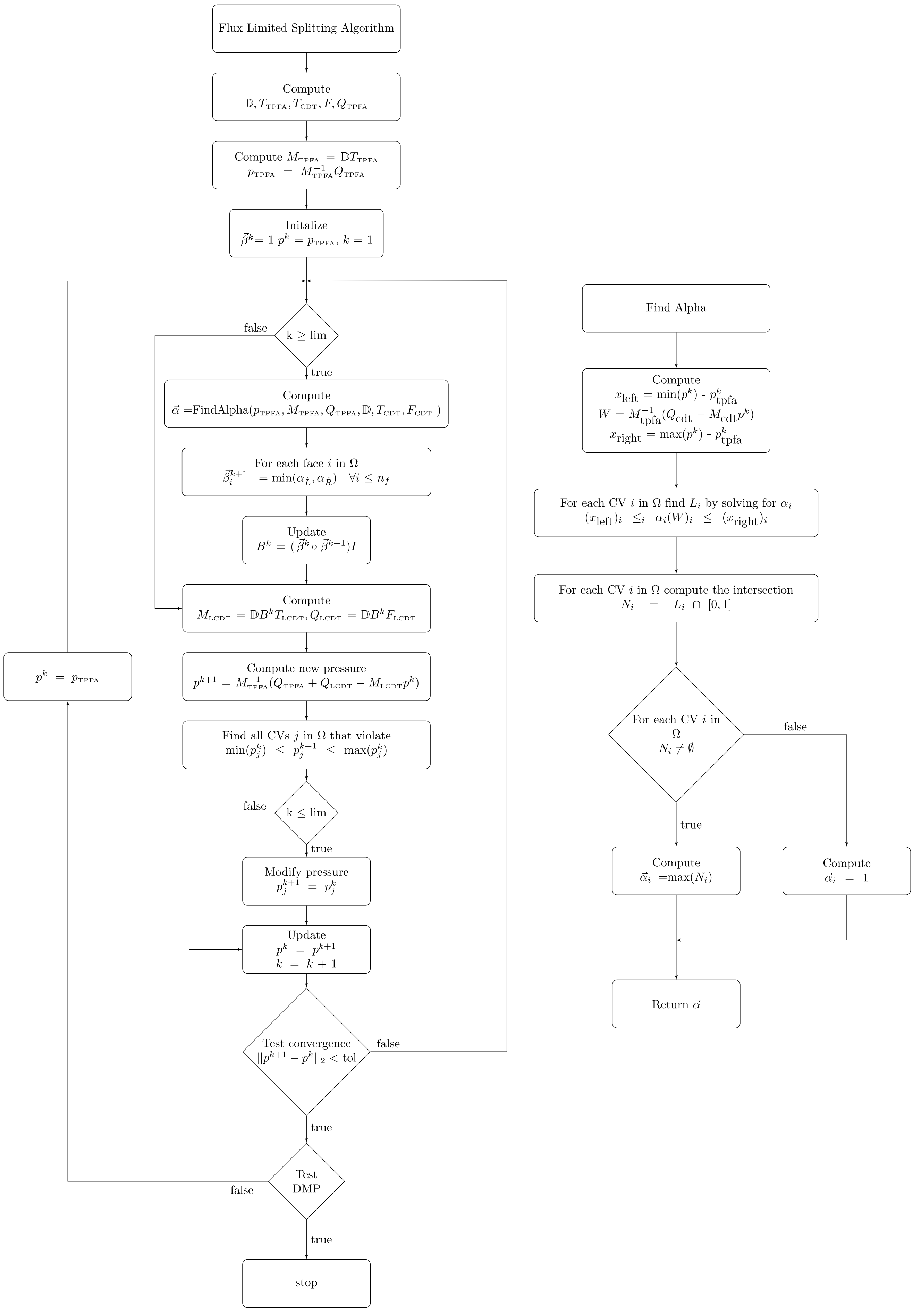
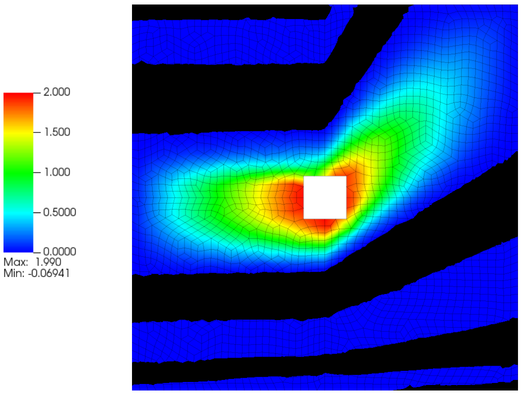
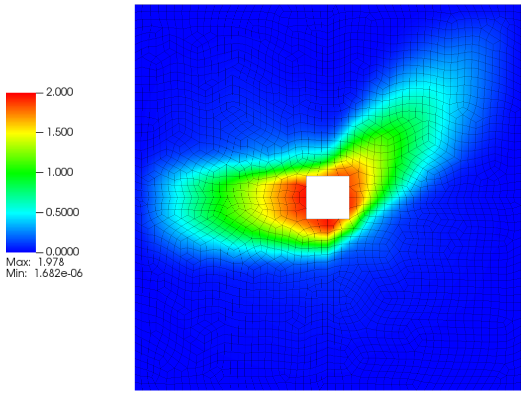


Figure 2: Flux Limited Splitting Fluxogram.

the stability condition of Equation 13. The MPFA-D, in turn, has produced, qualitatively, a rather good pressure solution. The most important aspect of the features of fluid flow were accurately represented. However, the method has clearly violated the DMP, producing a pressure field ranging from -0.06941 to 1.990. The Flux Limited Splitting, in turn, has produced a solution a little more diffusive than the MPFA-D, however, the method was capable of repairing the MPFA-D solution avoiding overshooting and undershooting. With 42 iterations, including the 5 initial iterations to capture the MPFA influence, plus 37 iterations of the FLS, the method has converged, producing a DMP respecting solution that ranges between $1.682e^{-6}$ to 1.978.



(a) MPFA-D pressure solution.



(b) MPFA-D with FLS pressure solution.

Figure 4: Comparison between the pressure solution results for a fluid flow in a heterogenous domain with a square hole in a extremely anisotropic media. The undershooting region is represented by black.

7. Conclusions

In this article, we have introduced the Flux Limited Splitting, an extensions to the M-Matrix Flux Splitting (Edwards, 2000) , that calculates and limits the amount of cross diffusion terms that allow the solution to remain monotonic. The formulation, we have developed was coupled with a MPFA-D, and tested in an extremely heterogeneous, anisotropic media. In presented example, the method was capable of repairing the solution of the MPFA-D solution, retrieving the loss monotonicity. The advantages of our formulation include:

- Our method is capable of repairing the loss of monotonicity even for challenging problems.
- The Flux Limited Splitting increases the stability of the original M-Matrix Flux Splitting.
- The FLS is conservative at any iteration level.
- The FLS can be easily applied to general structured or unstructured meshes in 2-D or 3-D, as long as a consistent flux approximation is provided.

8. ACKNOWLEDGEMENTS

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