COB-2021-0665
LINEAR STATIC ANALYSIS OF CABLES BY THE GENERALIZED FINITE ELEMENT METHOD

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Abstract. Cables have been extensively used as structural elements in suspension bridges, transmission lines, mooring lines, guyed towers, marine and off-shore constructions, cable trusses and roof structures. This is due to light weight of the cables, their cost-effective construction and the possibility of pre-tensioning. The analysis of cable structures by the Finite Element Method (FEM) using straight elements usually requires a high number of degrees of freedom in order to obtain acceptable results for the cable profile and its properties, such as cable tension and length. Therefore, in this paper the Generalized Finite Element Method (GFEM) is studied with the use of several different enrichment functions, considering a linear and inextensible cable analysis. The results obtained by the GFEM using the proposed polynomial, trigonometric and hyperbolic enrichment functions are compared to analytical solutions found in literature and to linear and nonlinear solutions provided by other researchers. The computational cost is analyzed in terms of the total number of degrees of freedom and the program execution time. The condition number of the stiffness matrix in each analysis is also discussed.

Keywords: Generalized Finite Element Method, Linear Analysis, Cables

1. INTRODUCTION

Cables can be used as structural elements in several engineering branches. Among its constructive advantages, one can mention its low weight and ease of assembly. One of the most notable uses of cables in structures is in suspension and cable-stayed bridges. In addition to the high architectural appeal, these bridge typologies have enabled increasingly larger spans, reaching up to 1 km (Karoumi, 1999).

According to Ren and Peng (2005), cable-stayed bridges still have advantages over other types of bridges, including suspension bridges, and these are: better efficiency in the use of materials, greater stiffness and smaller size of structural elements, facilitating their manufacture/construction. Another essential use of cables is in transmission lines. Costa (2014) highlights that, due to Brazil being a large country, the energy generated in plants needs to travel large distances in transmission lines to reach the final consumer. From an economic point of view, it is important that cables span great distances without compromising their structural behaviour, because, as in bridges, such cables are subject to static, thermal and dynamic loads.

Another use of cables consists in suspension roofs. In 1950, the State Fair Arena project in Raleigh, North Carolina, promoted further study and construction of suspended roofs, as an advantage of this type of coverage is the ability to cover large areas with little material (Tibert, 1999). However, all of the cited uses for cables require a rigorous structural analysis. An initial difficulty in the study of cables is that its geometry is load dependent, that is, its shape varies according to the applied forces (Irvine, 1981). Loads distributed along the arc length of the cable, such as its own weight, make it assume the form of a catenary. On the other hand, loads distributed along the span produce the shape of a parabolic curve.

In this context, the Finite Element Method (FEM) is widely used in the study of Computational Mechanics. In the case of cable analysis, the use of straight elements requires a high number of degrees of freedom in order to obtain satisfactory results for the cable profile and their properties, such as their length and the stresses developed in the cable itself. Possibilities to overcome this problem, using the MEF itself, are the *h*, *p* and *hp* refinements. The *h* refinement consists of refining the mesh by increasing the number of elements, which can demand a high computational cost. The *p* refinement, on the other hand, consists in increasing the degree of the shape functions, however, the formulation of elements in the *p* refinement is not simple, making its computational implementation difficult (Proença and Torres, 2007). In this context, enriched methods were developed, including the Generalized Finite Element Method (GFEM), which

includes known information (a priori) of the problem in order to improve the solution with lower computational cost than the FEM. The Generalized Finite Element Method (GFEM) has been successfully used in several problems of computational mechanics, such as in linear dynamic analysis (Arndt *et al.*, 2016), nonlinear dynamic analysis (Piedade Neto and Proença, 2015) and three-dimensional nonlinear analysis (Proença and Torres, 2007). Therefore, this paper intends to apply the GFEM in the analysis of cable structures.

2. CABLES

In Figure 1 a generic cable and an infinitesimal segment of the cable are shown.
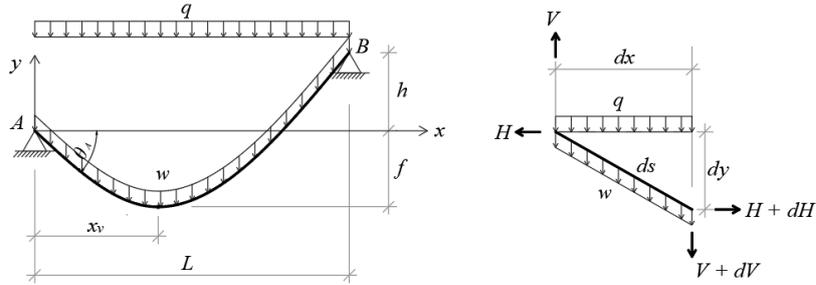

Figure 1. General cable and infinitesimal cable segment.

In Figure 1, L is the span of the cable, f is its sag, the vertical distance between supports A and B is h and θ_A is the angle measured between the cable and the horizontal at support A. The loads q and w are distributed along the span and the cable length, respectively, and forces V and H are the vertical and horizontal components of the axial force in the cable, respectively. Equilibrium of the infinitesimal segment yields the following governing differential equation

$$H\frac{d^2y}{dx^2} = q + w\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{1}$$

In the following, the simplification

$$\frac{dy}{dx} = \sinh\left(\frac{wx}{H} - \frac{wL}{2H}\right) \tag{2}$$

is used. Substituting Eq. (2) in Eq. (1), one obtains

$$H\frac{d^2y}{dx^2} = q + w\sqrt{1 + \sinh^2\left(\frac{wx}{H} - \frac{wL}{2H}\right)} = q + w\cosh\left(\frac{wx}{H} - \frac{wL}{2H}\right) \tag{3}$$

which is the strong form that can be used with Galerkin Method.

2.1 Linear Finite Element Method

The cable element hereby used for the linear analysis, presented by Przybysz *et al.* (2019), is shown in Figure 2, where ξ is the local coordinate, L_e is the element length and y_1 and y_2 are the vertical displacements of nodes 1 and 2, respectively.
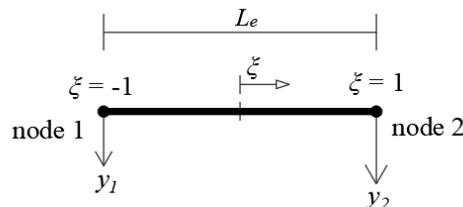

Figure 2. Linear cable element.

According to the element presented in Figure 2, the displacement field y is

$$y(\xi) = \boldsymbol{N}^T \boldsymbol{y} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = N_1 y_1 + N_2 y_2, \qquad N_1 = \frac{1-\xi}{2}, \qquad N_2 = \frac{1+\xi}{2} \tag{4}$$

where $\boldsymbol{N}^T$ is the transpose of the vector containing the linear shape functions N_i. Replacing Eq. (4) in Eq. (3) and applying the Galerkin Method yields

$$\int_{-1}^{1} H \frac{d\boldsymbol{N}}{d\xi} \frac{d\boldsymbol{N}^T}{d\xi} \frac{2}{L_e} \boldsymbol{y} \, d\xi - \int_{-1}^{1} \boldsymbol{N}^T \left[q + w \cosh\left(\frac{wx}{H} - \frac{wL}{2H}\right) \right] \frac{2}{L_e} \, d\xi = \boldsymbol{0} \tag{5}$$

where it has been considered that

$$\frac{d\boldsymbol{N}}{d\xi} = \frac{d\boldsymbol{N}}{d\xi} \frac{d\xi}{dx} = \frac{d\boldsymbol{N}}{d\xi} \frac{2}{L_e} \tag{6}$$

Considering $\boldsymbol{B} = d\boldsymbol{N}/d\xi$ and replacing it into Eq. (5), one obtains

$$\int_{-1}^{1} H \boldsymbol{B} \boldsymbol{B}^T \frac{2}{L_e} \boldsymbol{y} \, d\xi - \int_{-1}^{1} \boldsymbol{N}^T \left[q + w \cosh\left(\frac{wx}{H} - \frac{wL}{2H}\right) \right] \frac{2}{L_e} \, d\xi = \boldsymbol{0} \tag{7}$$

From Eq. (7), it is possible to define

$$\boldsymbol{K}_e = \int_{-1}^{1} H \boldsymbol{B} \boldsymbol{B}^T \frac{2}{L_e} \, d\xi, \qquad \boldsymbol{F}_e = \int_{-1}^{1} \boldsymbol{N}^T \left[q + w \cosh\left(\frac{wx}{H} - \frac{wL}{2H}\right) \right] \frac{2}{L_e} \, d\xi \tag{8}$$

where $\boldsymbol{K}_e$ is the element stiffness matrix and $\boldsymbol{F}_e$ is the element external load vector.

2.2 Iterative scheme

The element stiffness matrix presented in Eq. (8) depends on the horizontal thrust of the cable H. By using an estimate for H, called H_i, it is possible to evaluate the stiffness matrix of the elements, obtain the displacement vector and, therefore, the sag f_i, based on the estimate H_i. Since the value of the sag f is known, it is possible to use an iterative process so that the difference $|f_i\text{-}f|$ is smaller than the given tolerance when approaching better values for H_i. Using the Secant Method, as used by Przybysz *et al*. (2019), one obtains

$$H_{i+1} = \frac{H_{i-1}(f_i - f) - H_i(f_{i-1} - f)}{(f_i - f) - (f_{i-1} - f)}, \quad i = 1, 2, \dots \tag{9}$$

The iterative process is halted when the difference $|f_i\text{-}f|$ is less than a given tolerance, that is, $|f_i\text{-}f| \leq \epsilon_{tol}$. Especially in the case of enriched analyses, to find the value of the sag f_n it is not necessary that a nodal point of an element coincides with the abscissa where the sag f occurs once f_n can be found through interpolation. For every enriched analysis presented in this work, H_0 was considered

$$H_0 = \frac{pl^2}{8f} \tag{10}$$

where p is the distributed load acting on the cable. The value for H_1 is taken as $H_1 = 1.01 H_0$.

2.3 Generalized Finite Element Method

The Generalized Finite Element Method (GFEM), proposed by Melenk and Babuška (1996), and also by Duarte and Oden (1996), originated from the Partition of Unity Method. In the GFEM, the approximations do not necessarily use polynomial functions.

The shape functions of the GFEM are built by the product between the partition of unity function and an enrichment function, not necessarily polynomial (Kim *et al*., 2009). These functions are then assigned to the element nodes, expressed as a function of the system global coordinates (Schwebke and Holzer, 2002).

A key feature of the GFEM is the use of previously known information about the solution of the differential equation that describes the system for the construction of the enrichment functions, presenting good local and global results. GFEM has been successfully used in various fields such as crack analysis (Sukumar *et al.*, 1999; O'Hara *et al.*, 2016) and structural dynamics (Arndt, 2009; Torii, 2012; Piedade Neto and Proença, 2015).

For the two-node element showed in Figure 2, the solution provided by the GFEM for the element displacements y^e can be written as

$$y^e(\xi) = y^e_{FEM} + y^e_{ENRICH} = \sum_{i=1}^{2} \eta_i\, y_i + \sum_{i=1}^{2} \eta_i \left(\sum_{j=1}^{n_l} \gamma_j\, a_{ij} \right) \tag{11}$$

where y^e_{FEM} is the displacement obtained by the FEM, y^e_{ENRICH} is obtained through the enrichment functions, η_i are the linear partition of unity functions, y_i are the nodal displacements, n_l is the level of enrichment, γ_j are the enrichment functions and a_{ij} are the degrees of freedom related to those functions.

In this paper, the set of polynomial functions used for enrichment are the following Lobatto polynomials

$$L_2 = \frac{1}{2}\sqrt{\frac{3}{2}}(\xi^2 - 1), \qquad L_3 = \frac{1}{2}\sqrt{\frac{5}{2}}(\xi^2 - 1)\xi, \qquad L_4 = \frac{1}{8}\sqrt{\frac{7}{2}}(\xi^2 - 1)(5\xi^2 - 1),$$

$$L_5 = \frac{1}{8}\sqrt{\frac{9}{2}}(\xi^2 - 1)(7\xi^2 - 3)\xi, \qquad L_6 = \frac{1}{16}\sqrt{\frac{11}{2}}(\xi^2 - 1)(21\xi^4 - 14\xi^2 + 1) \tag{12}$$

The hyperbolic enrichment functions, proposed by Przybysz *et al.* (2019), are also used. They are given by

$$\begin{cases} HY1_1 = -\cosh(\xi + 1) - \xi \\ HY1_2 = -\cosh(\xi - 1) + \xi \end{cases} \tag{13}$$

$$\begin{cases} HY2_1 = -\cosh\left(\dfrac{\xi w}{H} + \dfrac{w}{H}\right) + \xi^2 \\ HY2_2 = -\cosh\left(\dfrac{\xi w}{H} - \dfrac{w}{H}\right) + \xi^2 \end{cases} \tag{14}$$

Another set of enrichment function is formed by the trigonometric functions proposed by Monteiro (2017), and given by

$$Fs_i = \sin\left(\frac{\pi}{2}(i - 2)(\xi + 1)\right), \qquad i = 3, 4, 5, 6 \tag{15}$$

These functions will be used as the γ_i enrichment functions in Eq. (11). Each analysis labeled by the prefix "GFEM" indicates that the solution is enriched, and the following letters indicate the enrichment function used, e.g. "GFEM L_2" stands for the solution provided by the GFEM considering the Lobatto function L_2, as presented in Eq. (12), while "GFEM L_4" is the enriched solution obtained using as enrichment functions L_2, L_3 and L_4, that is, it includes L_4 and all other Lobatto function with degree lower than L_4. The notation is similar regarding the sine functions in Eq. (15), so that analysis "GFEM Fs_5" considers functions Fs_3, Fs_4 and Fs_5, but not Fs_6.

3. NUMERICAL RESULTS

In the following two examples are presented. Example 1 consists of a cable subjected only to its own weight while in Example 2 a horizontally distributed load is applied.

3.1 Example 1

The first cable analyzed weights $w = 0{,}005$ kN/m, it spans a distance of $L = 20$ m and its sag is $f = 6$ m. The cross-section area is $A = 0{,}5$ cm² and the elastic modulus is $E = 165$ GPa. The cable is shown in Figure 3.
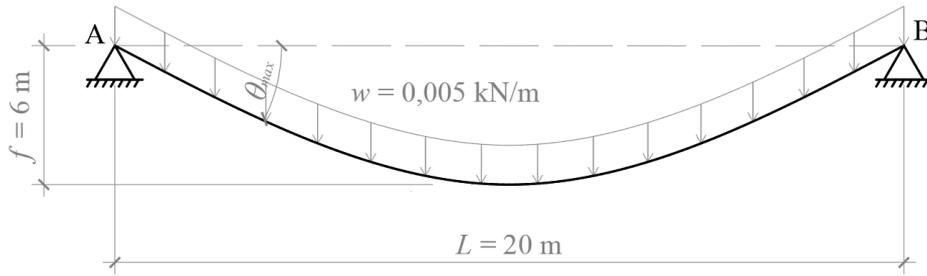
Figure 3. Cable under self-weight load.

The results are shown on Table 1, where the percentages in parentheses are the relative errors with respect to the linear analytic solution. The nonlinear solution presented by Pereira Junior (2002) is also shown. The variables analyzed are the cable horizontal thrust H, the cable's length L, the traction at support A T_A and the angle with the horizontal θ_{max}, as shown in Figure 3. The number of degrees of freedom is N_{dof} and N_{cond} is the condition number of the stiffness matrix obtained in the last iteration. All the results of the GFEM linear analyses were obtained using only one element.

Table 1. Results. Cable subjected to self-weight.

Solution	H (N)	S (m)	T_A (N)	θ_{max} (°)	N_{dof}	N_{cond}	Iterations
Linear analytic	45.944707	24.188203	75.944707	-52.772902			
Pereira Júnior (2002) (1000 elements)	45.94 (0.0102%)	24.1882 (0.0000%)	75.88 (0.0852%)	-52.74 (0.0623%)	-	-	-
FEM (2 elements)	45.944706 (0.0000%)	23.323903 (3.5732%)	53.580494 (29.4480%)	-30.964149 (41.3257%)	1	1.00E+00	5
FEM (10 elements)	45.944706 (0.0000%)	24.154860 (0.1379%)	69.879912 (7.9858%)	-48.891968 (7.3540%)	9	3.99E+01	5
GFEM L_2	46.653161 (1.5420%)	24.086943 (0.4186%)	72.870269 (4.0483%)	-50.191613 (4.8913%)	2	1.67E+00	6
GFEM L_3	45.937960 (0.0147%)	24.189581 (0.0057%)	75.875202 (0.0915%)	-52.739416 (0.0635%)	4	2.01E+15	5
GFEM L_4	45.937960 (0.0147%)	24.189581 (0.0057%)	75.875202 (0.0915%)	-52.739416 (0.0635%)	6	2.34E+15	5
GFEM L_5	45.944741 (0.0001%)	24.188312 (0.0004%)	75.938481 (0.0082%)	-52.769301 (0.0068%)	8	2.78E+16	5
GFEM L_6	45,944741 (0.0001%)	24,188312 (0.0004%)	75.938481 (0.0082%)	-52,769301 (0.0068%)	10	4,10E+15	5
GFEM HY_1	45.961746 (0.0371%)	24.185725 (0.0102%)	75.912237 (0.0428%)	-52.738122 (0.0659%)	2	9.55E+00	6
GFEM HY_2	46.653051 (1.5417%)	24.086959 (0.4186%)	72.870804 (4.0476%)	-50.192076 (4.8904%)	2	1.71E+00	6
GFEM HY_1-HY_2	45.947780 (0.0067%)	24.187752 (0.0019%)	75.966914 (0.0292%)	-52.782716 (0.0186%)	4	1.36E+04	5
GFEM Fs_3	47.659228 (3.7317%)	23.889048 (1.2368%)	65.490524 (13.7655%)	-43.303815 (17.9431%)	2	1.87E+00	6
GFEM Fs_4	45.659436 (0.6209%)	24.243551 (0.2288%)	72.074434 (5.0962%)	-50.690889 (3.9452%)	4	3.03E+02	5
GFEM Fs_5	45.983914 (0.0853%)	24.180770 (0.0307%)	75.169033 (1.0214%)	-52.284568 (0.9254%)	6	7.42E+03	5
GFEM Fs_6	45.938462 (0.0136%)	24.189524 (0.0055%)	75.756189 (0.2482%)	-52.670435 (0.1942%)	8	2.71E+05	5

It can be seen that, for Example 1, the results presented in Table 1 by the nonlinear analysis differ from the analytic solution only by small amounts.

According to Table 1, the standard FEM solutions showed the best approximation for H, However, the results for S, T_A and θ_{max} were not as good as those provided by the GFEM.

Considering the enriched analyses, the best approximation for every studied variable was obtained by GFEM L_5 and GFEM L_6, presenting relative errors less than 0.0082%. It is worth noting that, even though GFEM L_6 has 10 degrees of freedom (DOF), its results are not better than those of GFEM L_5, and that the condition number of GFEM L_5 is the highest one considering all linear analyses. Following GFEM L_5 and GFEM L_6, the solution GFEM HY_1-HY_2 presented good results as well, with all errors less than 0.0292%, especially when taking into account that GFEM HY_1-HY_2 uses fewer DOFs and has lower stiffness matrix condition number than GFEM L_5 and GFEM L_6.

Every analysis containing 2 DOFs required six iterations of the iterative scheme, while all other required five iterations. However, the additional iteration in GFEM L_2, GFEM HY_1, GFEM HY_2 and GFEM Fs_3 did not imply necessarily in greater program execution time compared to other solutions, as shown in Figure 4.
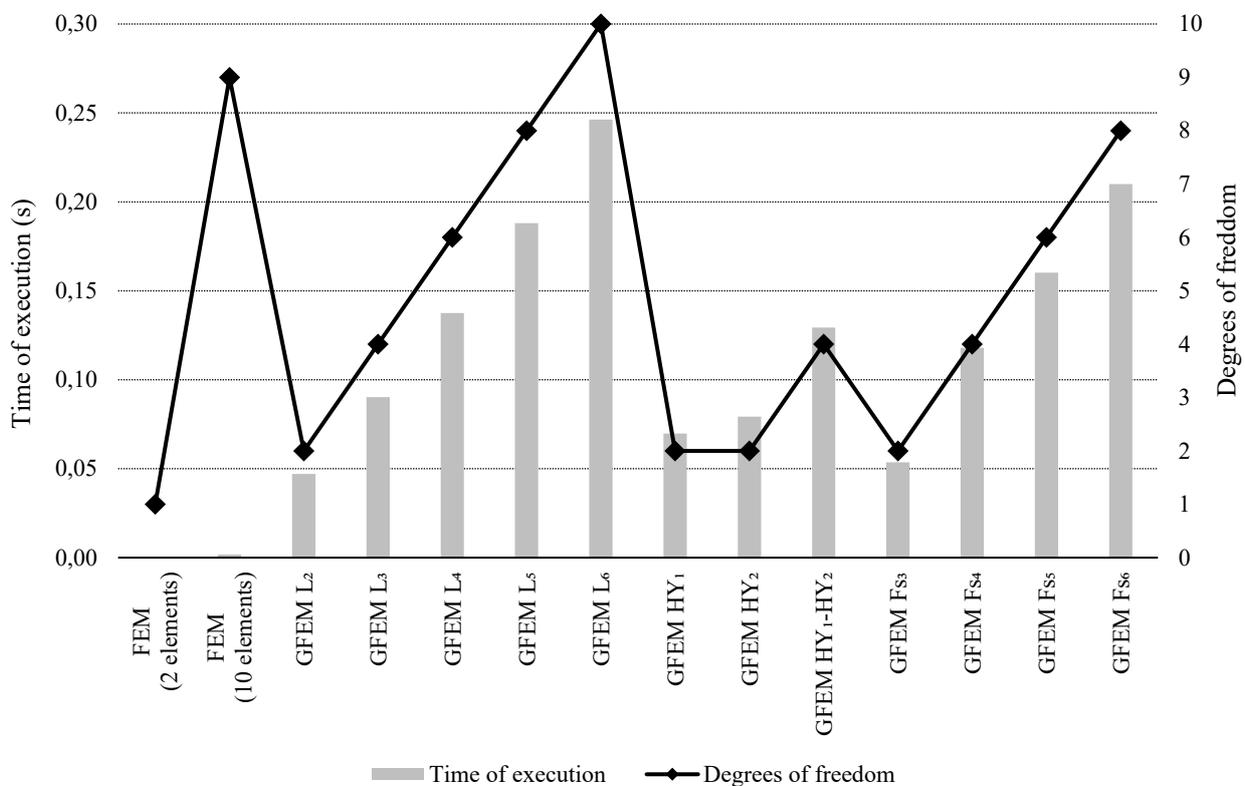

Figure 4. Example 1. Program execution time.

In fact, GFEM L_2, GFEM HY_1, GFEM HY_2 and GFEM Fs_3 were the fastest analyses, as shown in Figure 4, where, as expected, one can see that the higher the number of DOFs, the longer it takes the program to run. The standard FEM analyses were significantly faster than the GFEM ones, as FEM (2 elements) and FEM (10 elements) took 0.0005 and 0.0018 seconds, respectively

3.2 Example 2

The cable analyzed is shown in Figure 5. The span is $L = 36$ m and the sag is $f = 6$ m.
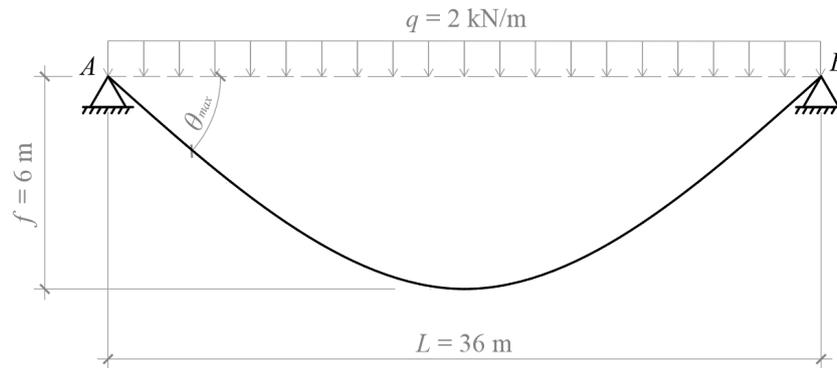
Figure 5. Example 2. Cable under distributed load along the span.

Since the self-weight is neglected in the present case, the solutions considering the enrichment function HY_2 were not computed. The results are shown in Table 2.

Table 2. Results. Cable under distributed load along the span.

Solution	H (kN)	S (m)	T_A (kN)	θ_{max} (°)	N_{dof}	N_{cond}	Iterations
Linear analytic	54.00000	38.51223	64.89992	-33.69007	-	-	-
FEM	54.00000	37.94733	57.49021	-18.43495	1	1.00E+00	1
(2 elements)	(0.000%)	(1.467%)	(11.417%)	(45.281%)			
FEM	54.00000	38.49002	63.60402	-30.96376	9	3.99E+01	1
(10 elements)	(0.000%)	(0.058%)	(1.997%)	(8.092%)			
GFEM L_2	54.00000	38.51223	65.54691	-33.68742	2	1.67E+00	1
	(0.000%)	(0.000%)	(0.997%)	(0.008%)			
GFEM L_3	54.00000	38.51223	65.54691	-33.68742	4	2.31E+15	1
	(0.000%)	(0.000%)	(0.997%)	(0.008%)			
GFEM L_4	54.00000	38.51223	65.54691	-33.68742	6	1.90E+15	1
	(0.000%)	(0.000%)	(0.997%)	(0.008%)			
GFEM L_5	54.00000	38.51223	65.54691	-33.68742	8	1.76E+16	1
	(0.000%)	(0.000%)	(0.997%)	(0.008%)			
GFEM L_6	54.00000	38.51223	65.54691	-33.68742	10	2.85E+15	1
	(0.000%)	(0.000%)	(0.997%)	(0.008%)			
GFEM HY_1	52.89539	38.59414	65.49864	-36.13985	2	9.55E+00	5
	(2.046%)	(0.213%)	(0.923%)	(7.272%)			
GFEM Fs_3	55.73065	38.35326	62.90812	-27.63683	2	1.87E+00	4
	(3.205%)	(0.413%)	(3.069%)	(17.967%)			
GFEM Fs_4	53.73539	38.53471	63.55345	-32.27285	4	3.03E+02	4
	(0.490%)	(0.058%)	(2.075%)	(4.207%)			
GFEM Fs_5	54.03538	38.50906	64.70084	-33.36778	6	7.42E+03	3
	(0.066%)	(0.008%)	(0.307%)	(0.957%)			
GFEM Fs_6	53.99445	38.51272	64.84122	-33.62104	8	2.71E+05	3
	(0.010%)	(0.001%)	(0.090%)	(0.205%)			

According to the linear theory of cables, a cable subjected only to a constant distributed load along its span features a parabolic shape. In Table 2, the horizontal thrust H obtained by GFEM L_2 is equal to the linear analytical solution, achieved in only one iteration. This result was expected, as the enrichment function L_2 is parabolic and the value for H_0 used in the iterative scheme is the thrust H of a parabolic shaped cable. Since all analyses using Lobatto polynomials as enrichment functions include L_2, they all present the same values for the variables considered.

The hyperbolic and trigonometric functions, on the other hand, were not as successful regarding the number of iterations, requiring a minimal of 3 iterations for the horizontal thrust H. Nevertheless, both GFEM Fs_5 and Fs_6 presented more accurate values for T_A than the solutions using Lobatto functions.

In Example 2, the standard FEM analyses also took 1 iteration, yet GFEM L_2 presented better results for every studied variable using only 2 DOFs, which is a prominent result when considering that it surpassed the standard FEM solution of 10 elements and 9 DOFs, while also presenting lower stiffness matrix condition number.

The program execution time for the analyses is shown in Figure 6.
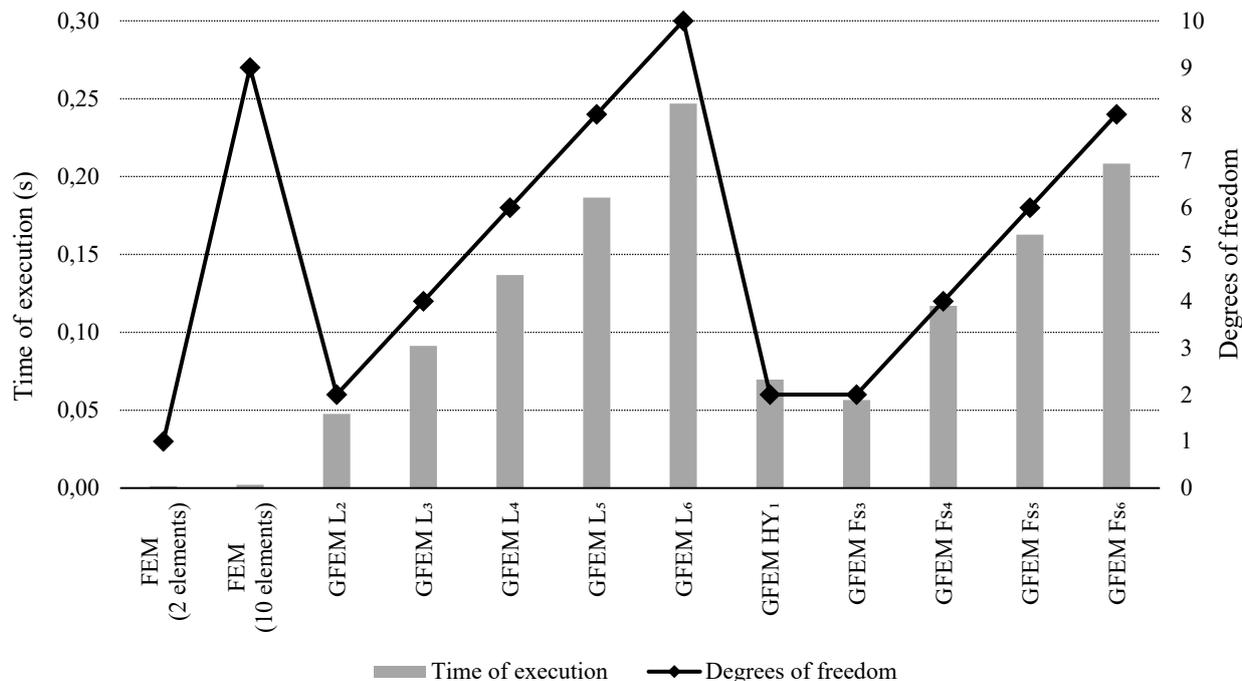

Figure 6. Example 2. Program execution time.

Once again, the standard FEM analyses were much faster than the GFEM ones, both presenting less than 0.0022 second of execution time.

It is possible to state that, regarding computational cost, the best cost-benefit enriched solution was GFEM L_2, since it achieved the best values for the studied variables H, S and θ_{max} requiring only 2 DOFs and it also presented the fastest execution time (considering enriched solutions only), according to Figure 6.

4. CONCLUSIONS

The enrichment functions used in the GFEM presented in this work were able to achieve satisfactory results while showing some advantages over the standard FEM. In general, most of the enriched solutions presented better results than those provided by the FEM with 9 DOFs, considering that the enriched analyses used only one element with, at most, 8 DOFs.

Even though GFEM HY_1-HY_2 included hyperbolic functions for the enrichment, which corresponds to the shape of a catenary that a cable under the self-weight assumes, the higher degree of enrichment provided better results for GFEM L_5 and L_6 in Example 1. Also, the consideration of nonlinear effects in this case did not imply in significantly different results compared to those of the linear theory of cables and of the solutions presented.

In almost all analyses the stiffness matrix condition number increased with the degree of enrichment, the exception being GFEM L_6, and the time of execution, in enriched analyses, was proportional to number of DOFs, noting that for the standard FEM, times of execution were much faster than the enriched ones.

5. REFERENCES

Arndt, M., 2009. *The Generalized Finite Element Method Applied to the analysis of free vibration of structural frames* (in Portuguese). Doctoral Dissertation, Federal University of Paraná, Curitiba, Brazil.

Arndt, M.; Machado, R. D.; Scremin, A., 2016. "Accurate assessment of natural frequencies for uniform and non-uniform Euler-Bernoulli beams and frames by adaptive generalized finite element method". *Engineering Computations*, p. 1586-1609.

Bathe, K.-J., 2014. Finite Element Procedures. Prentice Hall, New Jersey.

Costa, R. S., 2014. *Formulation for the advanced analysis of cables and spatial trusses for application in cable-stayed towers and transmission lines* (in Portuguese). Doctoral Dissertation, Graduate Program in Structural Engineering, Federal University of Minas Gerais, Belo Horizonte, Brazil.

Duarte, C. A.; Oden, J. T., 1996. "An *h-p* adaptative method using clouds". *Computer Methods in Applied Mechanics and Engineering*, v. 139, p. 237-262.

Irvine, H. M.,1981. Cable Structures. The MIT Press Series, Massachusetts.

Karoumi, R., 1999. "Some modelling aspects in the nonlinear finite element analysis of cable supported bridges". *Computers and Structures*, v.71, p. 397-412.

Kim, D. J.; Duarte, C. A.; Proença, S. P., 2009. "Generalized Finite Element Method with global-local enrichments for nonlinear fracture analysis". *Brazilian Society of Mechanical Sciences and Engineering*, p. 317-330.

Melenk, J.M.; Babuška, I., 1996. "The Partition of Unity Finite Element Method: Basic Theory and Applications". *Seminar für Angewandte Mathematik*, p. 1-31.

Monteiro, F. A. C., 2017. *Systematic construction of sets of hierarchical C^m functions according to the Fourier Series Theory* (in Portuguese). Doctoral Dissertation, Graduate Program in Aeronautical Infrastructure Engineering, Aeronautical Technology Institute, São José dos Campos, Brazil.

O'hara, P.; Duarte, C. A.; Eason, T., 2016. "A two-scale generalized finite element method for interaction and coalescence of multiple crack surfaces". *Engineering Fracture Mechanics*, v. 163, p.274-302.

Pereira Júnior, E. J. P., 2002. *A consistent formulation for the nonlinear analysis of suspended cable structures* (in Portuguese). Master Thesis, Graduate Program in Structural Engineering, Federal University of Minas Gerais, Belo Horizonte, Brazil.

Piedade Neto, D.; Proença, S. P. B., 2015. "Generalized Finite Element Method in linear and nonlinear structural dynamic analyses". *Engineering Computations,* v. 33, n. 3, p.806-830.

Proença, S. P. B; Torres, I. F. R., 2007. "Generalized finite element method for nonlinear three-dimensional analysis of solids". *International Journal of Computational Methods*, v. 5, n. 1, p.37-62.

Przybysz, J. C.; Machado, R. D.; Weinhardt, P. O.; Arndt, M., 2019. "Static analysis of inextensible suspended cables by the GFEM". In *XL CILAMCE Ibero-Latin american congress on computational methods in engineering*. Natal, Brazil.

Ren, W.-X.; Peng, X.-L., 2005. "Baseline finite element modeling of a large span cable-stayed bridge through field ambient vibration tests". *Computers & Structures*. v. 83, p. 536-550.

Schwebke, K.G.; Holzer, S.M., 2002. "Some Remarks on Generalized Finite Element Methods (GFEM) in Solid Mechanics". In *Fifth World Congress on Computational Mechanics*. Vienna, Austria.

Sukumar, N.; Moës, N.; Moran, B.; Belytschko, T., 1999. "Extended finite element method for three-dimensional crack modelling". *International Journal for Numerical Methods in Engineering*. p. 1549-1570.

Tibert, G., 1999. *Numerical analyses of cable roof structures*. Doctoral Dissertation, Royal Institute of Technology, Stockholm, Sweden.

Torii, A. J., 2012. *Structural dynamics analysis using the Generalized Finite Element Method* (in Portuguese). Doctoral Dissertation, Federal University of Paraná, Curitiba, Brazil.