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THE NYSTRÖM METHOD FOR THE ONE-DIMENSIONAL RADIATIVE TRANSPORT PROBLEM

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Abstract. *In this work a methodology for solving the thermal radiative transfer problem in one-dimensional domain and one-band frequency was developed. This problem is characterized by a coupled system between radiative transfer and convection-diffusion equation. The radiative transfer equation is rewritten in terms of integral operators and it is discretized by Nyström method. The heat equation is discretized by finite difference method. The results obtained are compared with those found in the literature.*

Keywords: *Thermal Radiative, Transfer Equation, Convection-diffusion equation, Nyström Method.*

1. INTRODUCTION

The radiation is a phenomenon which can not be neglected in models of heat transfer involving high temperatures. Due to this, the understanding of this mechanism is important for the planning and development of sectors whose activities are related to this type of heat transfer process. Furnaces, burning, nuclear reactions, gas turbines and glass cooling are some examples of situations where radiation is a significant process and needs to be taken into account in the mathematical model that describe the flow of thermal energy together with the conduction and the convection.

The model for thermal radiative transfer problem is composed of coupled system between diffusion-convection equation and radiative transfer equation where the coupling is nonlinear and occurs due the source therm, that in this case is given by a Planck function. In last decades, this model has been largely studied both numerical and analytical point of views. Imposing some restrictive conditions in the coefficients of the problem, Thompson *et al.* (2008) established the theory of existence and uniqueness for the general radiative transfer coupled problem. Posteriorly, the theory of existence for the one-dimensional case on the C^α space was developed by de Azevedo *et al.* (2011).

In the numerical point of view, several methodologies has used in Seaïd *et al.* (2004) to obtain solutions for the coupled system in different dimensions. More recently, it has been developed techniques that approach through integral operators. The first step in these metodologies is to find two integral operators and then rewrite the radiative transfer equation in the integral formulation. In this context de Azevedo *et al.* (2011) develop the Green's Function Decomposition (GFD) method. Sauter *et al.* (2013) established simulations and errors estimates for the GFD methodology and de Almeida Konzen *et al.* (2016) featured numerical solutions for the transient one-dimensional model using the GFD to the radiative transport equation and Finite Element Method (FEM) to the diffusion-convection equation.

The integral formulation to the transport problem is a Fredholm integral equation of the second kind and, because this, the Nyström method can be used for get solutions of the problem. This methodology was introduced by Nyström (1930) and has been used to solve the transport equation in works Loyalka (1975) and de Azevedo *et al.* (2018). More details about the Nyström method can be find in works Atkinson (1997) and Press *et al.* (2007).

In this present work was proposed a methodology for solve the thermal radiative transfer problem in one-dimensional domain in a participative medium and considering a one-band in the spectrum frequency. For this purpose the Nyström method and classical difference schemes were used respectively in the the radiative transfer and diffusion-advection equation.

The one-dimensional formulation was obtained from the general model proposed by Frank *et al.* (2004) and can be write as:

$$\frac{\partial T}{\partial t} - k_0 \frac{\partial^2 T}{\partial x^2} = 4\kappa\pi(\hat{I}(x) - F(T)) \tag{1}$$

$$\mu\frac{\partial}{\partial x}I(x,\mu)+\lambda I(x,\mu)=\sigma\hat{I}(x)+\kappa F(T), \tag{2}$$

where $I(x,\mu)$ is the radiative intensity, k_0 is the thermal conductivity, σ is the scattering coefficient, κ is the absorption coefficient, λ is the total absorption given by relationship $\lambda=\sigma+\kappa$ and μ is the cossine of the incidence angle. The source term $F(T)$ is given by:

$$F(T)=\frac{\sigma_s T^4}{\pi} \tag{3}$$

where $\sigma_s=5.670373\times10^{-8}\ W/(m^2.K^4)$ is the Stefan-Boltzmann constant. This relationship represent the Planck function integrated over all frequency spectrum (see Howell *et al.* (2016)). And $\hat{I}(x)$ is the mean intensity given by:

$$\hat{I}(x)=\int_{-1}^{1}I(x,\mu)\ d\mu \tag{4}$$

The boundary conditions for the transfer equation given in Eq. (2) include semi-reflective conditions:

$$I(0,\mu)=\rho_0 I(0,-\mu,t)+(1-\rho_0)F(Tb_0),\quad \mu>0 \tag{5}$$

$$I(L,-\mu)=\rho_L I(L,\mu,t)+(1-\rho_L)F(Tb_L),\quad \mu>0 \tag{6}$$

where L is the length of one-dimensional domain, ρ_0 and ρ_L are the reflection coefficients; Tb_0 and Tb_L are the boudaries temperature in the left and right side, respectively. The model is complemented with the boundaries conditions for the thermal flux

$$T(0,t)=T_0(t),\qquad T(L,t)=T_L(t) \tag{7}$$

and the initial condition

$$T(x,0)=T^0(x). \tag{8}$$

The original model was inicially proposed for gas turbines and glass cooling problems (see Frank *et al.* (2004) and Klar and Siedow (1998)). In works cited were considered the isotropic case, that is, the absortion and scattering coefficients were considered constants. For comparison and numerical validation purposes the present work was proposed following this same model, but the method can be implemented in more general problems with variable coefficients (see de Azevedo *et al.* (2011)). The representation of the integral operators for this kind of problems and some results to radiative transfer equation in one-dimensional domain considering a anisotropic medium already were presented by de Azevedo *et al.* (2013) and Sauter *et al.* (2013) using the GFD method. Since the Nyström method is also based on the integral represetation, the metodology presented in this work could be adapted for the anisotropic case and expected that in this case the results present a good accuracy.

In the Section 2 are presented the methodology used to finding the integral formulation to the radiative transport problem and the Nyström method implementation. The results are presented in Section 3 and the conclusions in Section 4.

2. THE NYSTRÖM METHOD

2.1 Integral Formulation

Consider the general one-dimensional radiative transfer equation with arbitrary source term $S(x)$ given by

$$\mu\frac{\partial}{\partial x}I(x,\mu)+\lambda I(x,\mu)=\sigma\hat{I}(x)+S(x) \tag{9}$$

and the semi-reflective conditions:

$$I(0,\mu)=\rho_0(\mu)I(0,-\mu)+(1-\rho_0(\mu))B_0(\mu) \tag{10}$$

$$I(L,-\mu)=\rho_L(\mu)I(L,\mu)+(1-\rho_L(\mu))B_L(\mu), \tag{11}$$

where B_0 and B_L are integrable functions representing the border contributions.

Defining $Q(x)=\sigma\hat{I}(x)+S(x)$:

$$\mu\frac{\partial}{\partial x}I(x,\mu)+\lambda I(x,\mu)=Q(x). \tag{12}$$

The equation above can be solved using the integrating factor which result in the equations

$$I(x,\mu) = \frac{1}{\Delta}\left((1-\rho_0)B_0 + \rho_0(1-\rho_L)B_L e^{-\frac{\lambda L}{\mu}} + \frac{\rho_0}{\mu}\int_0^L Q(s)e^{-\frac{\lambda s}{\mu}}\, ds + \right.$$
$$\left. + \frac{\rho_0\rho_L}{\mu}\int_0^L Q(s)e^{(s-2L)\frac{\lambda}{\mu}}\, ds\right) e^{-\frac{\lambda}{\mu}x} + \frac{1}{\mu}\int_0^x Q(s)e^{-\frac{\lambda}{\mu}(x-s)}\, ds, \tag{13}$$

$$I(x,-\mu) = \frac{1}{\Delta}\left((1-\rho_L)B_L + \rho_L(1-\rho_0)B_0 e^{-\frac{\lambda L}{\mu}} + \frac{\rho_0\rho_L}{\mu}\int_0^L Q(s)e^{-(L+s)\frac{\lambda}{\mu}}\, ds + \right.$$
$$\left. + \frac{\rho_L}{\mu}\int_0^L Q(s)e^{(s-L)\frac{\lambda}{\mu}}\, ds\right) e^{\frac{\lambda}{\mu}(x-L)} + \frac{1}{\mu}\int_x^L Q(s)e^{\frac{\lambda}{\mu}(x-s)}\, ds. \tag{14}$$

Now, from the Eq. (4), the mean intensity is rewritten as:

$$\hat{I}(x) = \int_{-1}^{1} I(x,\mu)\, d\mu = \frac{1}{2}\int_0^1 \left(I(x,\mu) + I(x,-\mu)\right)\, d\mu. \tag{15}$$

Equation (15) can be rewritten in terms of two integral operators: the L_g operator acting over source term and the L_b operator acting over border (see de Azevedo *et al.* (2011)). The expressions of this operators are determined directly by replacing the Eq. (13) and Eq. (14) in the mean intensity equation above. Therefore:

$$\hat{I}(x) = L_g Q(x) + L_b B, \tag{16}$$

where B is the vector of integrable functions $[B_0, B_L]$, L_g and L_b are two integral operators. Here, the L_g operator is written as:

$$L_g Q(x) = \int_0^L K(x,s)Q(s)\, ds, \tag{17}$$

where $K(x,s)$ is the kernel of the operator and it is given by:

$$K(x,s) = \int_0^1 \left[\frac{1}{2\mu}\left(\frac{K_1+K_2+K_3+K_4}{\Delta} + K_5\right)\right] d\mu. \tag{18}$$

Here,

$$\Delta = 1 - \rho_0\rho_L e^{-\frac{2\lambda L}{\mu}}, \tag{19}$$

$$K_1 = \rho_0 e^{-\frac{\lambda(s+x)}{\mu}}, \tag{20}$$

$$K_2 = \rho_0\rho_L e^{-\frac{\lambda}{\mu}(x+2L-s)}, \tag{21}$$

$$K_3 = \rho_0\rho_L e^{-\frac{\lambda}{\mu}(2L+s-x)}, \tag{22}$$

$$K_4 = \rho_L e^{-\frac{\lambda}{\mu}(2L-s-x)} \tag{23}$$

and

$$K_5 = e^{-\frac{\lambda}{\mu}|s-x|}. \tag{24}$$

Also, the L_b operator acting over boundary is given by:

$$L_b B(x) = \int_0^1 \frac{1}{2\Delta}\left(\tilde{B_0} + \tilde{B_L}\right)\, d\mu \tag{25}$$

with

$$\tilde{B_0} = (1-\rho_0)(e^{-\frac{\lambda x}{\mu}} + \rho_L e^{-\frac{\lambda}{\mu}(2L-x)})B_0, \tag{26}$$

and

$$\tilde{B_L} = (1-\rho_L)(\rho e^{-\frac{\lambda}{\mu}(x+L)} + e^{-\frac{\lambda}{\mu}(L-x)})B_L. \tag{27}$$

Remembering that were considered $Q(x) = \sigma\hat{I}(x) + S(x)$, so from Eq. (16):

$$\begin{aligned}\hat{I}(x) &= L_g(\sigma\hat{I}(x) + S(x)) + L_b B \\ &= \sigma L_g\hat{I}(x) + L_g S(x) + L_b B(x),\end{aligned} \tag{28}$$

i.e.,

$$\hat{I}(x) = \sigma L_g\hat{I}(x) + g(x) \tag{29}$$

where $g(x) = L_g S(x) + L_b B(x)$.

The Eq. (29) is called integral formulation for the radiative transfer problem and can be rewritten as:

$$(\mathbf{1} - \sigma L_g)\hat{I}(x) = g(x) \tag{30}$$

with $\mathbf{1}$ been the indentity operator. This equation has a solution whenever that $\sigma||L_g|| < 1$, where $||\cdot||$ it is a norm of the functional space considered. Indeed, Eq. (30) has a solution if the operator $(1 - \sigma L_g)$ can be inverted, i.e., if the Neumann series of its inverse operator $(1 - \sigma L_g)^{-1}$ converges. Estimates for L_g were established in C^0 and C^α spaces in works de Azevedo *et al.* (2011).

Now, note that the Eq. (29) can be rewritten in terms of kernel of the operators as

$$\hat{I}(x) = \sigma\int_0^L K(x,s)\hat{I}(s)\ ds + g(x). \tag{31}$$

From this point the Nyström method can be described. Basically it is compose of two steps: in the first moment the integral that appear in the Eq. (32) is approximate by a quadrature and then, the expression resulting, is evaluated in all mesh points. More details about this implementation are given in the next section.

2.2 Nyström Method

Observe that the operator has a singularity whenever $x = s$ and therefore its necessary to use the singularity removal technique described in Press *et al.* (2007). Like this, the Eq. (31) is rewritten as:

$$\hat{I}(x) = \sigma\int_0^L K(x,s)(\hat{I}(s) - \hat{I}(x))\ ds + \sigma\hat{I}(x)R(x) + g(x), \tag{32}$$

where $R(x) = \int_0^L K(x,s)\ ds$. The same technique could be used in the operator $L_g S(x)$ that appear in the expression $g(x) = L_g S(x) + L_b B(x)$.

As mentioned before, the first step of the Nyström is approximate the integral term in Eq. (32) by a quadrature scheme. So considering that $\{s_j, \omega_j\}_{j=1}^N$ are the nodes and weights of the quadrature and despise the error of numerical truncation, the Eq. (32) can be rewriten as:

$$\hat{I}(x) = \sigma\sum_{j=1}^N \omega_j K(x,s_j)[\hat{I}(s_j) - \hat{I}(x)] + \sigma\hat{I}(x)R(x) + g(x). \tag{33}$$

Now, the second step is to evaluate the expression (33) in all mesh points of quadrature scheme, so:

$$\hat{I}(x_i) = \sigma\sum_{j=1}^N \omega_j K(x_i,s_j)[\hat{I}(s_j) - \hat{I}(x_i)] + \sigma\hat{I}(x_i)R(x_i) + g(x_i), \qquad 1 \le i \le N, \tag{34}$$

for simplicity do $\hat{I}(x_i) = \hat{I}_i$, $K(x_i,s_j) = K_{ij}$, $R(x_i) = R_i$ and $g(x_i) = g_i$, so:

$$\hat{I}_i = \sigma\sum_{j=1}^N \omega_j K_{ij}\left[\hat{I}_j - \hat{I}_i\right] + \sigma\hat{I}_i R_i + g_i, \qquad 1 \le i \le N. \tag{35}$$

The expression above can be rewritten in terms of matrices as:

$$[D]\,[\hat{I}] = [g], \tag{36}$$

where D is the matrix given by:

$$\begin{cases} d_{ij} = -\sigma\omega_j K_{ij}, & i \neq j \\ d_{ij} = (1 - \sigma R_i) + \sigma\sum_{l\neq i}^N \omega_l K_{i,l}, & i = j \end{cases} \tag{37}$$

$[\hat{I}] = [\hat{I}_1 \quad \hat{I}_2 \quad \cdots \quad \hat{I}_N]^T$ and $[g] = [g_1 \quad g_2 \quad \cdots \quad g_N]^T$.

Note that if $\sigma = 0$ the system (36) has a unique solution because, in this case, the matrix is a identity. So in a neighborhood of the value $\sigma = 0$ the system has a unique solution too.

Besides that, the solution obtained in Eq. (36) is valid only at points that belong the mesh given by quadrature scheme, to calculate the mean intensity at points outside mesh the interpolation formula could be used. This formula is obtained from Eq. (34) and given by

$$\hat{I}(x) \approx \frac{\sigma \sum_{j=1}^{N} \omega_j K(x, s_j) \hat{I}(s_j) + g(x)}{1 + \sigma \sum_{j=1}^{N} \omega_j K(x, s_j) - \sigma R(x)}. \tag{38}$$

The Nyström method has already been properly validated for the Gauss-Legendre and Boole quadratures in work Dalmolin *et al.* (2017a). As our goal is solve the coupled system (1)-(2), the radiative transfer is discretized by Nyström method described above and the diffusion-advection equation is discretized by progressive difference scheme. So in the next section will be showed this discretizetion.

2.3 Coupled Discretization

Progressive difference centered in point x_i in time t_j is used in order to discretizate the convection-diffusion equation:

$$\frac{T_i^{j+1} - T_i^j}{h_t} - k_0 \left(\frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{h_s^2} \right) + 4\kappa\pi h_t \left(\hat{I}_i^j - B(T_i^j) \right), \tag{39}$$

where $T_i^j = T(x_i, t_j)$, h_t and h_s are the steps in the time and space mesh, respectivelly. This scheme can be writen explicitly as

$$T_i^{j+1} = (1 - 2\delta) T_i^j + \delta \left(T_{i+1}^j + T_{i-1}^j \right) + 4\kappa\pi h_t \left(\hat{I}_i^j - B(T_i^j) \right), \tag{40}$$

where $\delta = (k_0 h_t)/h_s^2$ and $1 \leq i \leq N - 1$. This method converges whenever $\delta < 0.5$ (see Burden and Faires (2013)).

Besides that, the stationary temperature is reached when the Euclidian norm of the difference between two successive iterations reach a given tolerance 10^{-8}, this is:

$$||T_i^{j+1} - T_i^j||_2 \leq 10^{-8}, \quad 1 \leq i \leq N. \tag{41}$$

The term $\hat{I}_i^j$ in Eq. (40) can be obtained directly of the application of the Nyström method in the Eq. (2):

$$\hat{I}(x_i) = \sigma \sum_{j=1}^{N} \omega_j K(x_i, s_j)[\hat{I}(s_j) - \hat{I}(x_i)] + \sigma \hat{I}(x_i) R(x_i) + \kappa \sum_{j=1}^{N} \omega_j K(x_i, s_j)(F(T_j) - F(T_i)) + \\ + \kappa F(T_i) R(x_i) + L_b \tilde{B}(x_i), \tag{42}$$

where $\tilde{B}$ is the vector of boundary contributions.

Note that the mean intensity discretization described here is a little different from the one shown in the previous section. Originally, the L_g operator acting over the source term in the Nyström method is solved numerically, with *GSL integration* package, and here it also was approximate by the same quadrature rule used in the mean intensity. This occurs because, in this case, the source term depend of the temperature to be calculated in the de same points of the intensity.

3. RESULTS

The code was developed in C++ language. The numerical integrations and matrices operations were realized with the *GSL Integration* package of the *GNU Scientific Library*.

Two different cases were simulated: the fisrt with fixed temperature profile and, the second, calculating the temperature profile in the coupled system. For the first case a linear profile in which the temperature varies from 1000 K to 1800 K was chose and the contour conditions were considered no reflectives ($\rho_0 = \rho_L = 0$). So the problem is given by:

$$\mu \frac{\partial I}{\partial x} + \lambda I = \frac{\sigma}{2} \int_{-1}^{1} I \, d\mu + \kappa F(T) \tag{43}$$

$$I(0, \mu) = F(Tb_0); \qquad I(L, -\mu) = F(Tb_L), \tag{44}$$

with $F(T)$ is a source term given by Eq. (3).

The solution of this problem is given directly by the Nyström method in Eq. (42). The Boole quadrature was used with 100 points and the domain length $L = 1$ ($x \in [0, 1]$). The scattering and absorption coefficient is given by: $\sigma = \kappa = 1$

and also $\sigma = 0.1$ and $\kappa = 0.01$, remembering that $\lambda = \sigma + \kappa$. Also, the fixed profile temperature chosen is given by $T(x) = 1000 + 800x$ for $0 \leq x \leq 1$. The results were compared with the showed in the reference Seaïd *et al.* (2004).

The Fig. (1) - (2) show the results for the two set of parameters. The continuous line represents the results obtained by the Nyström method and the points (•) is the results obtained by direct numerical schemes Seaïd *et al.* (2004). In this figures it is possible perceive, as expected, that the intensity is growing when the thermal profile is growing and, moreover, both results feature a good agreement with the data obtained in the literature.

In the second case evaluated the thermal profile is variable in time. This case is described in the system (1)-(2) with Dirichlet conditions in boundary, initial linear thermal profile, no-reflective conditions for intensity and one-band model in frequency, like this:

$$\frac{\partial T}{\partial t} - k_0 \frac{\partial^2 T}{\partial x^2} = 2\kappa\pi \int_{-1}^{1} I \, d\mu - 4\kappa\pi F(T) \tag{45}$$

$$\mu \frac{\partial I}{\partial x} + \lambda I = \frac{\sigma}{2} \int_{-1}^{1} I \, d\mu + \kappa F(T), \tag{46}$$

with conditions:

$$T(x,0) = T_0, \qquad T(0,t) = 1000, \qquad T(L,t) = 1800 \tag{47}$$

$$I(0,\mu) = F(Tb_0), \qquad I(L,-\mu) = F(Tb_L). \tag{48}$$

The initial profile is given by $T(x,0) = 1000 + 800x$ for $0 \leq x \leq 1$. Here were used the parameters $k_0 = 1$, $\sigma = 0$ and $\kappa = 1$, and also use a Boole quadrature with 128 points. The results were compared with the Finite Element Method (FEM) described in de Almeida Konzen *et al.* (2016) and direct numerical schemes described in Seaïd *et al.* (2004). The heat equation is treated with scheme Eq. (40).

In the Fig. (3) are the results for the steady state temperature in comparison with de FEM method, and in the Fig. (4) show the results for the total intensity of this case. Here can be seen that both results obtained with the method developed feature good agreement with cases described in the literature.

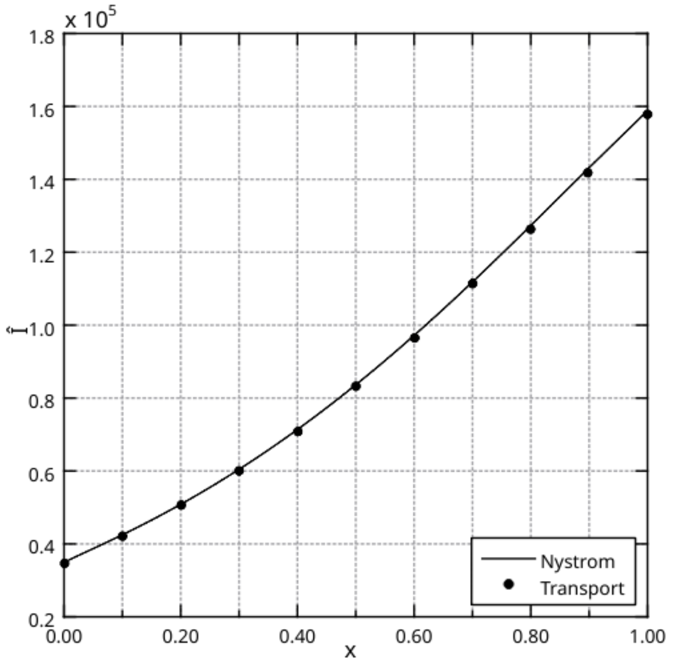

Figure 1. Mean intensity with fixed temperature profile, $\sigma = \kappa = 1$.

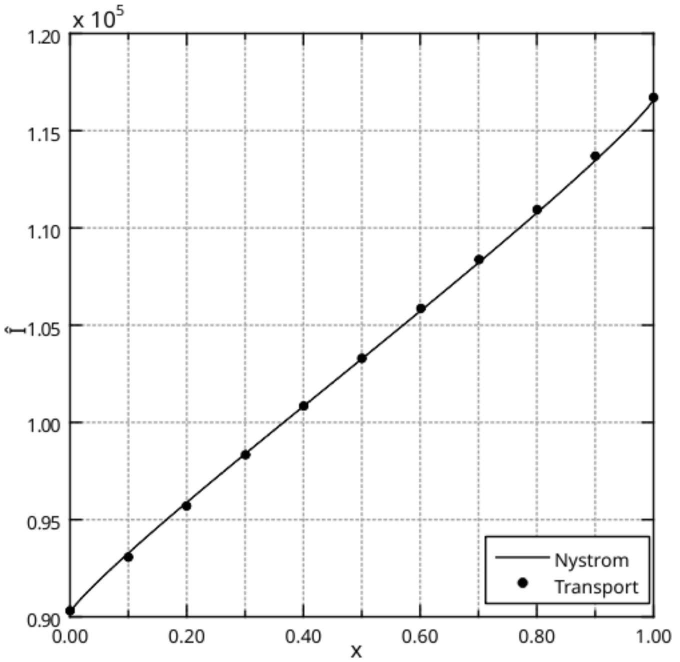

Figure 2. Mean intensity with fixed temperature profile, $\sigma = 0.1$ and $\kappa = 0.01$.

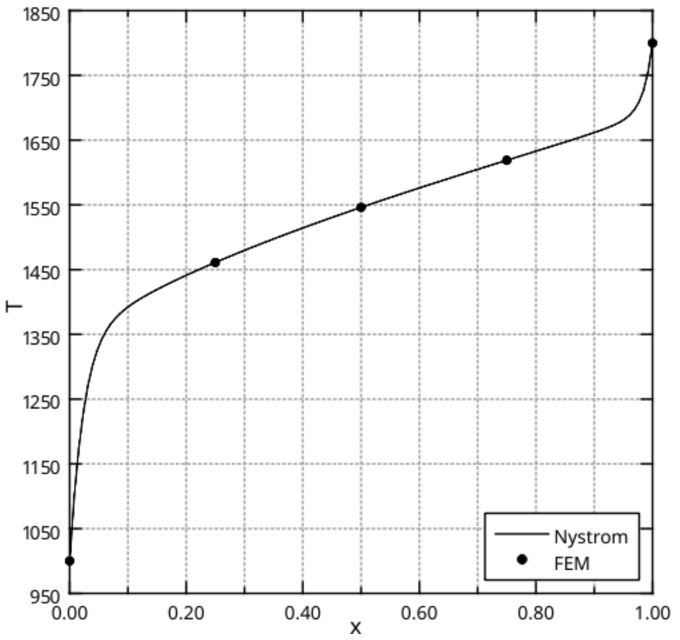

Figure 3. Steady state temperature for the second case, $\sigma = 0$ and $\kappa = 1$ in comparison with de Almeida Konzen *et al.* (2016).

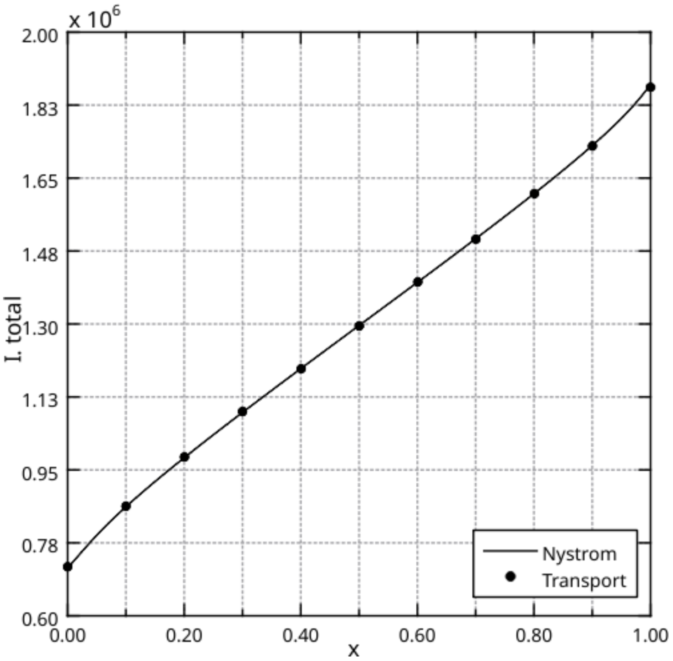

Figure 4. Total intensity for the second case, $\sigma = 0$ and $\kappa = 1$ in comparison with Seaïd *et al.* (2004).

4. Conclusions

The methodology developed in this work which consist of using the Nyström method combined with the finite difference method showed up efficient to solve the thermal radiative transfer problem in one-dimensional domain. Indeed, the methodology is computationally efficient and its implementation is simple. Furthermore, the results obtained is very accurate and have a good agreement with the others methods described in the literature according the Fig. (1) - (4).

One of the advantages of the Nyström method in one-dimensional models is the computational effort. Recently published articles show that this method for the radiative transfer equation present same precision in comparison with others methods but with shorter computational time (see Dalmolin *et al.* (2017b)). The same occurs with the coupled problem presented in present work which results were generated in a few seconds while maintaining a good accuracy compared to results obtained in the literature. One of the disadvantages is that for the problems in larger dimensions the computational effort reduces due to high number of operations necessary to generate results. There are already articles that approach the Nyström method for the neutron transport equation in bidimensional geometry (see de Azevedo *et al.* (2018)), but for the coupled system the authors has no knowledge of works with this approach kind.

About this present work, have been chosen few points in the mesh because the discretization by progressive differences scheme used in the convection-diffusion equation demands a time step much smaller than the step mesh, wich can result in significant increase at computational time if a mesh very tiny is considered. Note that the reduction of effort computational in this case is due to discretization scheme considered, if a unconditionaly scheme is used the Nyström can be a powerful tool in terms of computational efficiency. Furthermore, the Boole quadrature was used because the approach of the thermal equation demands that the quadrature chosen has nodes equally spaced.

For the future works will be done the Nyström method for the eight band model in the frequency spectrum. Furthermore the convection-diffusion equation will be treated with discretizations which are unconditionally stable. In this context, the numerical implementation of the Cranck-Nicholson method is already being developed, this method has the intended feature.

5. ACKNOWLEDGEMENTS

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