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STABILIZED GALERKIN LEAST-SQUARE APPROXIMATIONS FOR SHEAR-THINNING VISCOSITY AND RELAXATION TIME WHITE-METZNER–LIKE VISCOELASTIC FLUID FLOW

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Abstract. *Despite the presence of viscoelastic fluid in Nature, the importance of this class of fluids in industrial applications and then that, the knowledge of the behavior of this materials under rheological and kinematics influence is the focus of interest in the past 300 years since NewtonâĂŹs first understanding of rheological characteristics, as viscosity. In the present work, the flow of a non-linear pseudoplastic viscoelastic fluid is given by the extended White-Metzner constitutive equation and the non-linear terms given by a Carreau-YasudaâĂŞlike model for viscosity and relaxation time. The solution to this problem is approximated by the coupled Petrov-Galerkin and stabilized Galerkin least-square variational formulation in terms of velocity, extra-stress and additive constant and the non-linear system solved by the Newton method via low- and equal-order finite Lagrangian finite elements. A simple flow is considered a problem domain to evaluate the influence of rheological parameters without kinematical interference. The preliminary results point to a fine agreement between analytical and approximate solutions for the constant viscosity and relaxation time case. Shows the influence of the shear-thinning viscosity and relaxation time in the longitudinal velocity. Also shows an important dependence of the shear-thinning relaxation time in both first normal and shear stress.*

Keywords: *GLS, non-linear, shear-thinning, viscoelastic, White-Metzner*

1. INTRODUCTION

Considering the industrial applications, e.g. in the oil and gas extraction, the production of paint and associated polymers, the production of food stuff and cosmetics, the polymer extrusion, production of textiles and cellulose, a considerable portion of the materials employed show discordant effects that those considered with linear behavior or newtonian behavior.

The observations of (Weissenberg, 1947), (Rivlin, 1948), (Oldroyd, 1950), (Mooney, 1951), (Braun and Reiner, 1952) and (Truesdell, 1952) and considered by (DeWitt, 1955) about the White-Metzner model, allow the formulation, from the considerations of Maxwell for the superposition between elastic and viscous effects. This considerations goes to a converging point between the classical viscoelastic theorys: the additional terms, that result a expressive normal stress in viscoelastic materials, results from second-order effects as a deviation of classical behavior.

The propose of (White and Metzner, 1963) of an constitutive equation capable to predict the non-linear behavior of polymers under simple and complex flows consider Maxwell theory of superposition of elastic and viscous effects considering the rheological variables dependent of the invariants of the rate of strain tensor. This model shows simplicity to the implementations in engineering projects, capable to predict second-order effects as first normal stress differences for lower rate of strain.

(Souvaliotis and Beris, 1992) extend the differential viscoelastic model and imposed the dependence of the rheological variables, the apparent viscosity and relaxation time, to the fluid microstructural parameter dependent of kinematical and rheological quantities. This permits to formulate a power-law–like model for the relaxation time and create a dependence of the apparent viscosity as suggest by (White and Metzner, 1963).

(Garduño *et al.*, 2017) consider the effect of shear-thinning fluids in the drag behavior an the influence of aspect ratio in the flow of viscoelastic fluid through imerse sphere in a planar channel. The viscoelastic model based in the White-Metzner considerations permits to observe a drag increase with the variation of the sphere-channel aspect ratio. Despite the relaxation time is considering constant, the non-linear effect of the shear-viscosity is considerable.

(Cortada-Garcia *et al.*, 2018), observing the shear-thinning viscosity fluids, compare the experimental results obtained via PIV with numerical simulation of a pseudoplastic fluid of Carreau-Yasuda model. The numerical simulations shows a fine agreement considering the velocity fields, demonstrating the accurance of the phenomenological Carreau-Yasuda model.

(Yamamoto, 2019) consider the White-Metzner viscoelastic model in order to predict the shear-thinning viscosity dependent of the volume fraction of cellulose nanofiber suspension and kinematical parameters also the relaxation time dependent of kinematical parameters.

More recent, (Yamamoto, 2020) consider the White-Metzner model to predict the flow of cellulose nanofiber suspension with shear-thinning viscosity and relaxation time dependent of the dimension of the fluid suspension agglomerate. The predictions goes in the same direction that (Yamamoto, 2019). Both shear-thinning viscosity and relaxation time respond to the kinematical influence and to temporal dissociation.

The present work observe the influence of rheological paramters on shear-thinning viscosity and relaxation time pseudoplastic viscoelastic fluid governing by the extended White-Metzner model. The approximated solution of the mechanical model is given by a coupled Petrov-Galerkin and stabilized Galerkin least-square variational formulation in terms of velocity, extra-stress and additive constant and the non-linear system is solved by the Newton method via low- and equal-order Lagrange finite elements. The simple flow is considered in order to observe the rheological influence without kinematical interference. The numerical method is verified in comparison with the analytical solution for UCM viscoelastic fluid.

2. MECHANICAL MODELING

Considering a regular fluid domain Ω with polygonal boundaries and subjected to boundary conditions of a steady state, isothermal and inertialess flow of a non-linear viscoelastic fluid described by an UCM–like extended White-Metzner constitutive equation (Souvaliotis and Beris, 1992).

$$\theta(\dot{\gamma})\left(\partial_{,t}\,\tau_{ij} + u_k\left(\partial_{,x_k}\tau_{ij}\right) - \tau_{ik}\left(\partial_{,x_j}u_k\right) - \left(\partial_{,x_i}u_k\right)\tau_{kj}\right) = \eta(\dot{\gamma})\,\dot{\gamma}_{ij} - \tau_{ij}, \quad \text{for} \quad i,j = 1,2,3 \quad \text{at} \quad \Omega \tag{1}$$

where τ_{ij} is the extra-stress tensor, $\partial_{,t}\,\tau_{ij}$ and $u_k\left(\partial_{,x_k}\tau_{ij}\right)$ are the time derivative and transport term of the extra-stress tensor, $\tau_{ik}\left(\partial_{,x_j}u_k\right)$ and $\left(\partial_{,x_i}u_k\right)\tau_{kj}$ are related to the application of material principle of indifference to the material derivative of the extra-stress tensor, $\dot{\gamma}_{ij}$ is the rate of strain tensor $\partial_{,x_j}u_i + \partial_{,x_i}u_j$ and $\dot{\gamma}$ is the rate of strain modulus $\sqrt{2\tau_{ij}\tau_{ji}}$.

The non-linear characteristics are given, in Eq. (1), by rheological variables dependent of invariants of the rate of strain tensor. The first is the non-newtonian apparent viscosity $\eta(\dot{\gamma})$ given by the Carreau-Yasuda viscosity function for pseudoplastic materials (Carreau, 1972) and (Yasuda *et al.*, 1981). This model predicts the newtonian behavior of the viscosity for the limits of low- and high-rate of strain; also, between those limits, the model predicts a non-newtonian behavior region that permits the shear-thinning of the viscosity give by a power-law–type model.

$$\frac{\eta\left(\dot{\gamma}\right) - \eta_\infty}{\eta_0 - \eta_\infty} = \left[1 + (\tilde{\lambda}\dot{\gamma})^a\right]^{\frac{(n-1)}{a}}, \quad \text{at} \quad \Omega \tag{2}$$

where $\eta\left(\dot{\gamma}\right)$, η_0 and η_∞ are the apparent viscosity and the viscosity for the newtonian plateau of low- and high-rate of strain modulus, $\tilde{\lambda}$ is the power-law period, a is the Yasuda parameter and n is the power-law index. This model have experimental ajust for a bunch of fluids; in numerical simulations, is useful using an analytical expression to model the non-linear viscosity behavior (Bird *et al.*, 1987).

The second one is the relaxation time $\theta(\dot{\gamma})$ given by the rate between the Carreau-Yasuda viscosity and a constant elastic modulus. This model permits the relaxation time to assume the same behavior observed in the apparent viscosity.

$$\frac{\theta\left(\dot{\gamma}\right) - \theta_\infty}{\theta_0 - \theta_\infty} = \left[1 + (\bar{\lambda}\,\dot{\gamma})^b\right]^{\frac{(k-1)}{b}}, \quad \text{at} \quad \Omega \tag{3}$$

where $\theta\left(\dot{\gamma}\right)$, θ_0 and θ_∞ are the apparent relaxation time and relaxation time for the plateau of low- and high-rate of strain modulus and $\bar{\lambda}$, b and k are, respectively, the position and behavior of the transition region and the power-law index, in the same sense that apparent viscosity.

Coupled with the constitutive equation, the momentum equation for inertialess and no body forces. In the momentum equation, the elastic-viscous split stress (Rajagopalan *et al.*, 1990) gives a elliptic characteristic adding a surface term related to the low-rate portion of the Carreau-Yasuda viscosity, avoiding the hyperbolic characteristic commonly related to viscoelastic problems and, consequently, giving some stability characteristics to the mechanical model.

$$\partial_{,x_i}\alpha = \eta_\infty \partial_{,x_j}\dot{\gamma}_{ij} + \partial_{,x_j}\tau_{ij}, \quad \text{for} \quad i = 1,2,3 \quad \text{at} \quad \Omega \tag{4}$$

where α is the additive constant, in terms of the extra-stress tensor, $p + 1/3\,\mathrm{tr}\,\tau_{ij}$ where p is the hidrodynamic pressure and $1/3\,\mathrm{tr}\,\tau_{ij}$ is the mean pressure.

Last, but not less important, the continuity equation for steady-state for incompressible materials.

$$\partial_{,x_i}u_i = 0 \quad \text{at} \quad \Omega \tag{5}$$

3. NUMERICAL MODELING

The approximate solution of the non-linear system, Eq. (1) to (5), is given by the variational formulation based in the three-field coupled Petrov-Galerkin and Galerkin least-square in terms of extra-stress, pressure and velocity (Behr *et al.*, 1993), (Franca *et al.*, 1992) and (Franca and Frey, 1992). The least-square terms are stabilized via mesh-dependent terms that permits capturing elasto- and advective-dominant regions and, due to the fine convergence, permits the use of low- and equal-order finite elements for all the three fields.

The mesh-dependent stability parameter for continuity and momentum GLS-equation take account the local element mesh size and the ratio between elastic and advective terms, called mesh-Reynolds number, in the same sense of the mesh-Peclet number (Franca and Frey, 1992). The stability parameter of the constitutive GLS-equation is given by (Coronado *et al.*, 2006).

The non-linear variational formulation is solved by the Newton iterative method that uses strategy of frozen Jacobian gradient refreshing the Jacobian matrix after a few iterations (Zinani and Frey, 2008).

4. COMPUTATIONAL FEATURES

4.1 Fluid domain and boundary conditions

The non-linear pseudoplastic viscoelastic fluid is observed in a simple flow. A planar channel with height and lenght h_c and $l_c = 50h_c$. This domain permits evaluated the origin of some non-linear behaviors in the fluid flow, particularly in the case of normal stress and viscosity and relaxation time, without any geometrical interference in the rate of strain field, also permits evaluate the influence of rheologial parameters in the flow pattern.

The boundary conditions of the domain are in the inlet: a fully-developed profile of a UCM viscoelastic fluid for longitudinal velocity, shear and first normal stresses; in the walls: no-slip and impermeability conditions; in the symmetry: zero-transverse velocity and shear stress reinforcing the imposition of essential and natural boundary conditions; and in the outlet: free-traction.

4.2 Non-dimensionalization of mechanical model

The non-dimensionalization of the mechanical model is given in the kinematic form using the dimensions of lenght, velocity and viscosity, also called characteristic dimensions. Considering the non-dimensional variables: The non-

$$x_i^* = \frac{x_i}{h_c} \qquad u_i^* = \frac{u_i}{U_c} \qquad t^* = t\dot{\gamma}_c$$
$$\dot{\gamma}_{ij}^* = \frac{\dot{\gamma}_{ij}}{\dot{\gamma}_c} \qquad \eta^* = \frac{\eta}{\eta_c} \qquad \tau_{ij}^* = \frac{\tau_{ij}}{\eta_c\dot{\gamma}_c}$$

dimensional mechanical model are state:

$$Wi(\dot{\gamma}^*)\left(\partial_{,t^*}\tau_{ij}^* + u_k^*(\partial_{,x_k^*}\tau_{ij}^*) - \tau_{ik}^*(\partial_{,x_j^*}u_k^*) - (\partial_{,x_i^*}u_k^*)\tau_{kj}^*\right) = \eta^*(\dot{\gamma}^*)\dot{\gamma}_{ij}^* - \tau_{ij}^*, \quad \text{for} \quad i^*, j^* = 1,2,3 \quad \text{at} \quad \Omega^* \quad (6)$$

$$\partial_{,x_i^*}\alpha^* = \eta_\infty^*\partial_{,x_j^*}\dot{\gamma}_{ij}^* + \partial_{,x_j^*}\tau_{ij}^*, \quad \text{for} \quad i^* = 1,2,3 \quad \text{at} \quad \Omega^* \quad (7)$$

$$\partial_{,x_i^*}u_i^* = 0, \quad \text{at} \quad \Omega^* \quad (8)$$

where, in Eq. (6), the non-dimensional Weissenberg number $Wi = \theta(\dot{\gamma})\,\dot{\gamma}_c$ have the same order that the relaxation time. Give by the ratio between the elastic, first normal stress difference, and viscous, shear, forces. In this case, the Weissenberg number indicates a global characteristic, but also can indicate a local value when is function of the rate of strain modulus $Wi(\dot{\gamma}^*) = \theta(\dot{\gamma})\,\dot{\gamma}^*$.

The non-dimensional form of the non-linear rheological parameters are state:

$$\frac{\eta^*(\dot{\gamma}^*) - \eta_\infty^*}{\eta_0^* - \eta_\infty^*} = \left[1 + (Cu_\eta\dot{\gamma}^*)^a\right]^{\frac{n-1}{a}}, \quad \text{at} \quad \Omega^* \quad (9)$$

$$\frac{Wi(\dot{\gamma}^*) - Wi_\infty}{Wi_0 - Wi_\infty} = \left[1 + (Cu_\theta\dot{\gamma}^*)^b\right]^{\frac{k-1}{b}}, \quad \text{at} \quad \Omega^* \quad (10)$$

where, in both Eq. (9) and (10), Carreau number Cu, $Cu_\eta = \tilde{\lambda}\,\dot{\gamma}_c$ and $Cu_\theta = \bar{\lambda}\,\dot{\gamma}_c$, have the same order of the period of power-law region for both apparent viscosity and relaxation time curves.

Considering the above non-dimensional form of the mechanical model, we can consider: Wi, as the governing parameter for elastic influence; Cu, both Cu_η and Cu_θ, the governing parameter for presence or no-presence of shear-thinning characteristic in apparent viscosity and relaxation time - discussed ahead; and n and k, the governing parameter for the intensity of shear-thinning behavior.

The last ones, Cu_η and Cu_θ and n and k, are considered rheological governing parameters related to the non-newtonian characteristics of the apparent viscosity and relaxation time. The non-dimensional period of power-law region take account the position of transition between the low-rate constant viscosity plateau and the power-law region, Fig. (1).

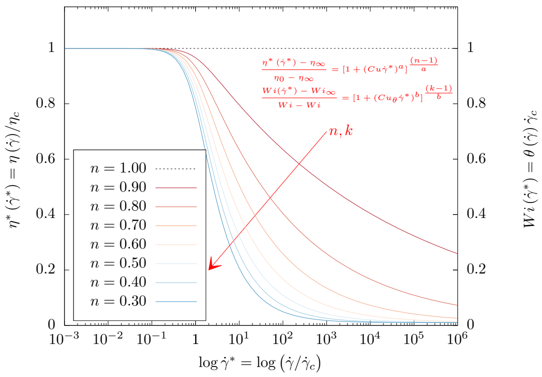


Figure 1: Representative behavior of non-dimensional apparent viscosity and relaxation time: influence of shear-thinning of viscosity and relaxation time parameter. *The arrow indicates the decrease* n *and* k.

4.3 Finite element mesh

The results show are based in a combination of bi-linear Lagrangian element function for the three primal variables of the variational formulation. This linear element-function permits a small computational time in comparison with high order element functions. But manipulations are necessary in the variational formulation in order to assure that a element function of the primal variables correspond to the functional spaces.

The results are mesh independent for a 8,181 finite elements totalizing 49,086 degrees of freedom for the components of the six variables - two velocity components, three stress components and one isotropic pressure.

5. RESULTS AND DISCUSSION

The results show in this section are relative to the preliminary study observing the influence of rheological parameters on the flow pattern of non-linear pseudoplastic viscoelastic fluids. In this work, the elastic influence and the shear-thinning behavior are investigated.

5.1 Numerical verification

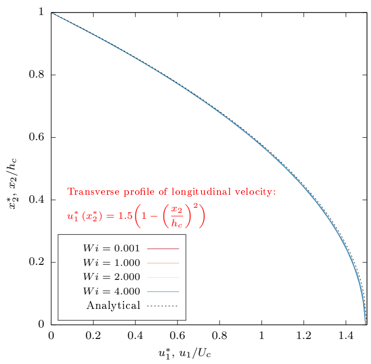
The constant viscosity and relaxation time viscoelastic fluid, given by the UCM constitutive equation, is compute considering the same domain with boundary conditions that permit the fluid development. In the above case, we take the governing parameters $Wi = 0.10$, $n = k = 1$ and $Cu_\eta = Cu_\theta = 0$ and $\eta_0^* = 1$ and $\eta_\infty^* = 0.001$. The approximated numerical solution of UCM fluids is compare to an analytic profile for UCM fluids (Behr *et al.*, 2005). In this comparison, the Wi will be the investigate parameter.

Is interesting notice that for low-rate of strain modulus, in the case of simple geometries without corners, obstacles and contractions the material derivative in terms of convected coordinates is capable to converge approximated solution for relatively high-values of elasticity. In cases like this, where the variation between the low- and high-rate of strain modulus don't surpass the order of 10, is easy the convergence for values of $Wi = 10$.

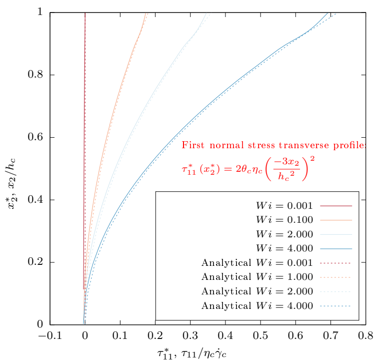
The numerical longitudinal velocity and shear stress for UCM fluid don't have influence of the fluid elasticity, Fig. (2a) and (2c), also as the analytic profiles. It's possible notice a fine agreement between the numerical approximations and analytic case. It's also possible observe, Fig. (2b) and (2c), a slight difference between the numerical and analytic profile near the channel wall ($x_2^* = 1$) that can be associate the high-rate in that region (300% higher than $\dot\gamma_c$) and high stress dissipation caused by the fluid viscosity.

5.2 Influence of shear-thinning behavior

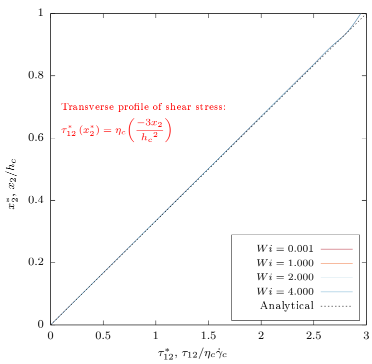
When we take $n < 1$ and $k < 1$ and $Cu_\eta \neq 0$ and $Cu_\theta \neq 0$, we have shear-thinning viscosity and relaxation time extended White-Metzner fluid. In the above cases, we take a base-fluid for the governing parameters $Wi = 0.10$, $n = k = 0.50$ and $Cu_\eta = Cu_\theta = 1$ and $\eta_0^* = 1$, $\eta_\infty^* = 0.001$. For the analysis of influence of those rheological parameters, we will froze two of those and vary another in order to verify the origin of such behavior in the flow of



(a)



(b)



(c)

Figure 2: Numerical transverse profiles of UCM fluids and comparison with analytical solution for UCM fluids: influence of Wi number. (a) non-dimensional longitudinal velocity; (b) non-dimensional first normal stress; and (c) non-dimensional shear stress.

non-linear pseudoplastic viscoelastic fluids.

Is easy to observe the behavior of shear-thinning fluids in the non-dimensional longitudinal velocity, Fig. (3a). The decrease of the velocity modulus in the channel symmetry line ($x_2^* = 0$), in comparison with the UCM fluid, is a consequence of the velocity increase in the channel wall ($x_2^* = 1$), that is possible thankful to the viscosity decrease in a high-rate region. In comparison with UCM fluid, this region have a increase of 20% in the high-rate value and a decrease of 52% of the viscosity. For the UCM base-fluid case: constant and equal to $\eta^* = \eta_0^* = 1$ and for the extended WM base-fluid case: shear-thinning and equal to $\eta^*(x_2^* = 1) = 0.483$.

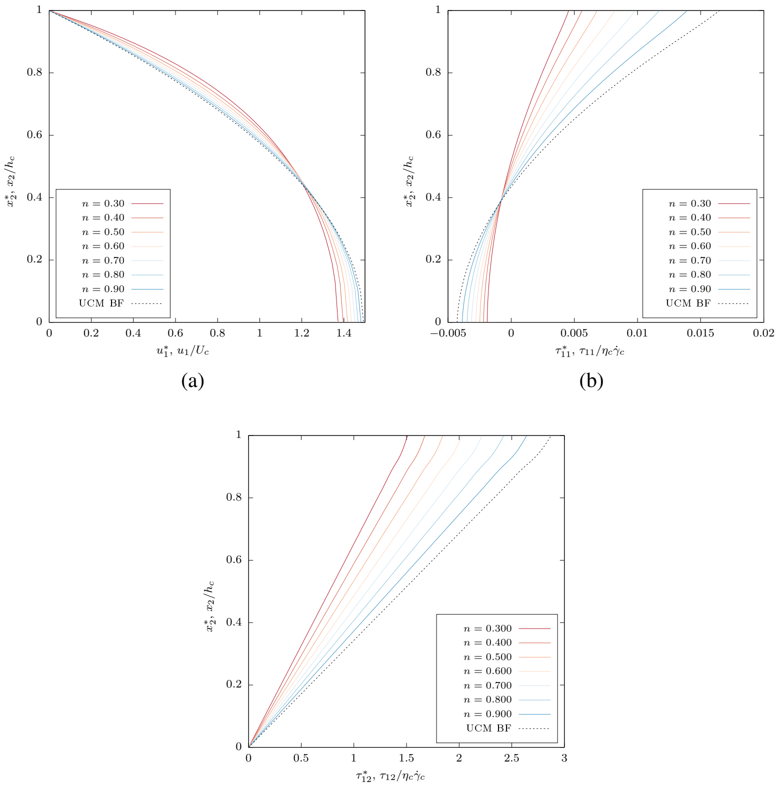
Observing the Fig. (3b) and (3c) we can attribute the difference, in comparison with UCM fluid, to different variables, but to the same behavior. Fig. (3b) is related directly to shear-thinning Wi. With the decrease n and k, τ_{11}^* decrease considering that the local $\eta^*(x_2^*)$ and $Wi(x_2^*)$ decrease. All the transverse profiles have different behavior. Considering the variation of n, the angle of the profiles has changed and the value in the wall that goes from $0.9\,\eta_0^*$ and $0.9\,Wi$ to $0.4\,\eta_0^*$ and $0.4\,Wi$ for both shear-thinning parameters for n going from 0.90 to 0.30.

In the case of viscoelastic fluids, the rate of strain tensor isn't symmetric, in comparison with purely viscous fluids. Adding elasticity to the fluid, the normal components of the rate of strain tensor are not null. This behavior permits the contribution for the decrease of the normal stress that came from the shear-thinning $\eta^*(\dot{\gamma}^*)$ and $Wi(\dot{\gamma}^*)$.

$$\tau_{11}^* = \eta^*(\dot{\gamma}^*)\dot{\gamma}_{ij}^* - Wi(\dot{\gamma}^*)\left(u_1^*(\partial_{,x_1^*}\tau_{11}^*) + u_1^*(\partial_{,x_2^*}\tau_{11}^*)\right) + Wi(\dot{\gamma}^*)\left(2\tau_{11}^*\dot{\gamma}_{11}^* + \tau_{12}^*(\dot{\gamma}_{21}^* + \dot{\gamma}_{22}^*)\right) \quad (11)$$

This variation is noticeable when we compare with the classical White-Metzner fluid, shear-thinning η^* and constant Wi. The $\tau_{11}^*(x_2^* = 1)$ vary, in the BF case, from 0.0165 for the UCM fluid, to 0.0130 for the classical WM and to 0.0068 for the extended WM. We can associate the elastic influence to 58% of the τ_{11}^*–decrease. In comparison with the classical WM, the elastic shear-thinning correspond to 64% of the influence. The viscosity shear-thinning to 36% of the influence. In Eq. (11) is possible notice the contribution of shear-thinning η^* and Wi to the τ_{11}^*–decrease.

The shear-thinning behavior of the fluid is also represented in η^* and Wi curve as function of $\dot{\gamma}^*$, Fig. (4). Is interesting to observe that the $\dot{\gamma}^*$ increase with the n decrease, contributing to the considerations of the Fig. (3a). We don't have significatively changes in $\dot{\gamma}^*$ in the highest values of n, this effect is more present in the lowest values.



(a) (b)

(c)

Figure 3: Numerical transverse profiles of extended White-Metzner fluids for $Wi = 0.10$ and $Cu_\eta = Cu_\theta = 1$ and comparison with numerical profiles of UCM fluids: influence of n. (a) non-dimensional longitudinal velocity; (b) non-dimensional first normal stress; and (c) non-dimensional shear stress.

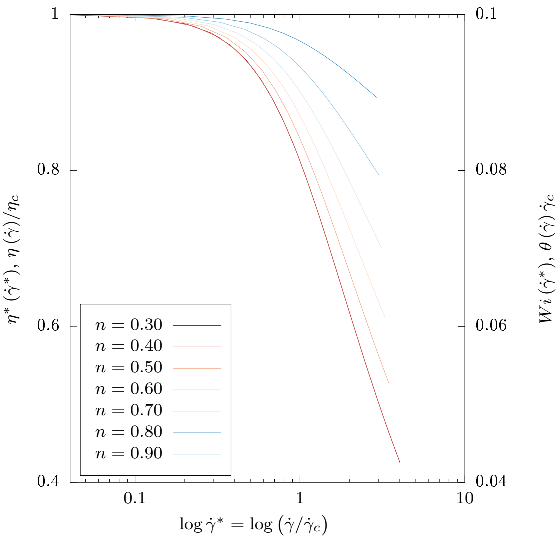


Figure 4: Numerical profiles of non-dimensional apparent viscosity and Weissenberg number versus logarithm non-dimensional rate of strain modulus: influence of n.

6. CONCLUSION

This work employs the coupled Petrov-Galerkin and the stabilized GLS variational formulation in terms of velocity, pressure and extra-stress for the approximated solution via equal-order bilinear finite elements of a shear-thinning viscosity and relaxation time of pseudoplastic viscoelastic fluid in a simple flow. In these preliminary results, is possible to observe the simplicity to converge high-elastic dominant problems. Despite the problems associated with the time-derivative of the extra-stress tensor, the simple flow allows a considerable low-rate of strain modulus and, consequently, doesn't permit a hyperbolic dominance of the problem. Also, the shear-thinning formulation for both non-linear rheological variables allow a slight increase of the high-rate of strain modulus, in the channel wall; but, perhaps more important than that, a significant decrease in the relaxation time in the same region. This behavior allows a small elastic dominance of the viscoelastic fluid in the high-rate of strain modulus region and, in the same sense that the viscosity decrease in this region, the relaxation time also decreases. The result of this is a physically accurate fluid behavior and, especially in high-rate of strain regions, a convergence characteristics to the numerical solution of viscoelastic problems, since high-elastic dominant problems in complex flows don't have this characteristic.

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