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Asymptotic analysis for a Carreau-Yasuda fluid driven by gravity over an inclined plane

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Abstract. This work is based, and contain fragments, of the study of a film flow driven by gravity over an inclined plane where the fluid rheology is given by the Carreau-Yasuda model, presented in Chimetta and Franklin (2020). The authors present an review with the most relevant works in rheology and hydrodynamic stability that were necessary for this work. A brief review of the Carreau-Yasuda model is presented, together with the continuity and momentum equations. For the development of the temporal stability analysis, a small perturbation technique was used. An asymptotic solution is presented for the reference flow and the stability problem. This solution is obtained with a symbolic code written in the Wolfram Mathematica environment. Reference flow gives the velocity profile and the film thickness, while the instability problem produces the celerity and growth rate for the surface waves.

Keywords: Gravitational flow, non-Newtonian fluid, Carreau-Yasuda model, asymptotic solution

1. Introduction

Gravitational flows are one of the most interesting subjects in fluid mechanics. This type of flow describes a wide variety of natural and industrial phenomena as mud and lava flows, glaciers, coating, reactor colling, friction-reducing effects, and so on. While the Newtonian aspects of this type of flow are well known and were vastly studied, the non-Newtonian approach is still in development and represents, in several cases, a challenge because of the complexity of the constitutive equations. A good example of a complex system is described by Weinstein (1990) on wave propagation in gravitational flows for fluids with shear-thinning behavior. The system deals with a multilayer flow of Carreau fluids in the presence of an inclined plane, with the last layer above in contact having a free surface. In his study, he demonstrates that waves generated on the free surface have a behavior similar to that of Newtonian systems, but interfacial waves between two adjacent layers of fluids depend strongly on the local interfacial viscosity. One of the most interesting features of his work was the application of an asymptotic expansion to solve the reference flow problem. However, the solution is only valid for fluids with weak shear-thinning behavior. Five years later Pinarbasi and Liakopoulos (1995) presented a paper on the linear stability in the interface between two non-Newtonian fluids flowing into a closed channel. These liquids followed the rheological behavior of the Carreau-Yasuda and Bingham models. They used a pseudospectral method based on Chebyshev polynomials for the problem discretization and a QZ algorithm for the generalized eigenvalue problem. Their results showed that for a two Newtonian fluid system, replacing the fluid below for one with viscoplastic behavior generates a stabilizing effect at the interface for long and intermediate wavelengths. Another interesting result occurred in the configuration of two fluids with shear-thinning characteristics, where variations in the Yasuda parameter a can stabilize or destabilize the interface depending on the wavelength and thickness of the fluid layer. The temporal stability of a Carreau fluid flowing down an inclined plane was considered by Rousset *et al.* (2007). With the Carreau model, it is possible to predict a power-law region and still predict a viscosity that remains finite while the shear rate approaches zero, which makes it suitable for free surface flows. A standard linear stability approach was used by the authors to study the flow. Two methods were implemented to solve the Orr-Sommerfeld equation and the boundary conditions: an asymptotic and spectral collocation method, the latter based on Chebyshev polynomials for the discretization, using Gauss-Lobatto collocation points. For the analytical solution, they carried out the solutions until the first-order. Results showed that for a shear-thinning case ($n < 1$) the critical Reynolds number is less than that for the Newtonian one with a larger phase speed, but remains proportional to $\cot\beta$.

2. Formulation of the physical problem

The present work deals with an incompressible fluid flow driven by gravity over an inclined plane. A two-dimensional flow of a generalized Newtonian fluid model was considered. The shear stress τ is a function of the shear rate $\dot{\gamma}$. This class of fluids satisfy the following constitutive equation $\tau = -\eta(\dot{\gamma})\dot{\gamma}$ (Morrison (2001), Macosko (1994)), where $\eta(\dot{\gamma})$ is a scalar function and $\dot{\gamma} = |\dot{\gamma}|$. The Carreau model is suitable for problems where the shear rate tends to zero since this is what happens at the free surface, the Carreau equation being thus the best choice for modeling the physical problem. In this work, it was considered the Carreau-Yasuda model, a five parameter constitutive equation given by,

$$\eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty)[1 + (\dot{\gamma}\lambda)^a]^{\frac{n-1}{a}} \quad (1)$$

where a predicts the transition behavior between the regions of zero-shear-rate plateau and power-law-like of the viscosity versus shear-rate curvature, and λ determines the shear rate value at which the transition to the power-law portion occurs. It also describes the gradual change between the power-law region to $\eta = \eta_\infty$. The index n governs the inclination of the rapidly decreasing portion of the η curve. In this model, the viscosity tends to η_∞ as the shear rate $\dot{\gamma}$ becomes larger. When the shear rate gets smaller the viscosity tends to η_0 . The dimensionless quantities are based on the film thickness scale $h_s = [\frac{\eta_0 U_s h_s}{\rho g \sin(\theta)}]^{\frac{1}{3}}$ and the characteristic velocity scale $U_s = \frac{\rho g h_s^3 \sin(\theta)}{\eta_0}$. As the base state, we considered the momentum and continuity equations in dimensionless form. The reference flow is parallel and steady and the velocity profile $\bar{u} = \bar{U}$ is a function of \bar{y} only. The component v of the velocity and the pressure gradient in the x -direction are both equal to zero. Under these considerations, the continuity equation is identically satisfied. The momentum and viscosity equations lead to,

$$\bar{U}(\bar{y}) = 0 \text{ at } \bar{y} = \bar{h} \quad (2)$$

where $\bar{h} = h/h_s$.

$$\left\{ I + (1 - I) \left[1 + \left(L \frac{d\bar{U}}{d\bar{y}} \right)^a \right]^{\frac{n-1}{a}} \right\} \frac{d\bar{U}}{d\bar{y}} = -\bar{y} \text{ for } \bar{y} \in [0; \bar{h}] \quad (3)$$

where, $I = \frac{\eta_\infty}{\eta_0}$ and $L = \frac{\lambda U_s}{h_s}$. A final relation can be considered by the definition of the dimensionless flow rate $\bar{Q} = Q/U_s h_s$ in the integral form. Applying a normalization with $\bar{Q} = 1$ it is possible to write the following integral,

$$\int_0^{\bar{h}} \bar{U} d\bar{y} = 1 \quad (4)$$

Equations 2, 3, and 4 establish a nonlinear problem for the reference flow and the film thickness. There is no exact solution in the general case for this system.

3. Temporal stability analysis

In order to study the linear stability of the flow, we make use of small perturbation technique for the velocity (\hat{u}, \hat{v}) and pressure (\hat{P}), together with the interface disturbance ($\hat{\xi}$), and neglect all terms $\mathcal{O}(\varepsilon^2)$, where $\mathcal{O}(\varepsilon) = \mathcal{O}(\hat{u}, \hat{v}, \hat{P})$ and the symbol \mathcal{O} refers to the asymptotic order. The kinematic condition at the free surface represents the impermeability, and the two dynamic conditions that are related to the continuity of the tangential and normal stresses at the interface. The first represents viscous effect and the second one rises from the Laplace-Young relation. Once that we are considering a two-dimensional problem we can use a stream function $\hat{\Psi}$ defined as $(\hat{u}, \hat{v}) = (\frac{\partial \hat{\Psi}}{\partial \bar{y}}, -\frac{\partial \hat{\Psi}}{\partial \bar{x}})$ and apply dimensionless normal modes as $\hat{\Psi}(\bar{x}, \bar{y}, \bar{t}) = \tilde{\Psi}(\bar{y})e^{i\alpha(\bar{x}-c\bar{t})}$ and $\hat{\xi}(\bar{x}, \bar{t}) = \tilde{\xi}e^{i\alpha(\bar{x}-c\bar{t})}$, where $\alpha = kh_s \in R$, in which k corresponds to the wave number, and $c = \frac{\omega}{k} \in C$, where ω is the complex frequency. The Orr-Sommerfeld equation for a Carreau-Yasuda model can be written as,

$$(D^2 + \alpha^2)[D^2\bar{\epsilon} + 2D\bar{\epsilon}D + \bar{\epsilon}(D^2 + \alpha^2)]\tilde{\Psi} - 4\alpha^2 D(\bar{\eta}D\tilde{\Psi}) = i\alpha Re[(\bar{U} - c)(D^2 - \alpha^2) - D^2\bar{U}]\tilde{\Psi} \quad (5)$$

where $D^m = \frac{\partial^m}{\partial \bar{y}^m}$. The boundary conditions at the wall are given by,

$$D\tilde{\Psi} = 0 \text{ at } \bar{y} = \bar{h} \quad (6)$$

$$\tilde{\Psi} = 0 \text{ at } \bar{y} = \bar{h} \quad (7)$$

combining the kinematic and the two dynamic conditions we find,

$$[1 + \bar{\epsilon}(\bar{U} - c)(D^2 + \alpha^2)]\tilde{\Psi} = 0 \text{ at } \bar{y} = 0 \quad (8)$$

$$i\alpha Re[(c - \bar{U})D + D\bar{U}]\tilde{\Psi} - 4\alpha^2\bar{\eta}D\tilde{\Psi} + (D^2 + \alpha^2)\left[D\bar{\epsilon} + \bar{\epsilon}D + i\alpha\bar{\epsilon}\left(\cot(\theta) + \frac{\alpha^2}{We_m}\right)\right]\tilde{\Psi} = 0 \text{ at } \bar{y} = 0 \quad (9)$$

where the Reynolds number is given by $Re = \frac{\rho U_s h_s}{\eta_0}$ and $We_m = \eta_0 Q(h_s \gamma)^{-1}$ is a modified Weber number. The function $\bar{\epsilon}$ is written as $\bar{\epsilon} = I + (1 - I)[1 + n(LD\bar{U})^a][1 + (LD\bar{U})^a]^{\frac{n-a-1}{a}}$

4. Asymptotic solutions

4.1 Reference flow

Next, we present an asymptotic solution for the reference flow by considering the limit case when L tends to zero. We write the solutions for the velocity profile and film thickness as exponential expansions. For the zeroth order, $\mathcal{O}(L^0)$, we obtain Eq. (10),

$$\bar{U}_0(y) = \frac{(\bar{h}_0^2 - \bar{y}^2)}{2} ; \bar{h}_0 = \sqrt[3]{3} \quad (10)$$

Proceeding next with the terms of $\mathcal{O}(L^a)$, we find Eq. (11).

$$\bar{U}_1(\bar{y}) = \frac{(-1)^a(1-I)(1-n)(\bar{h}_0^{a+2} - \bar{y}^{a+2})}{a(a+2)} + \bar{h}_0\bar{h}_1 ; \bar{h}_1 = \frac{(-1)^{a+1}(1-I)(1-n)\bar{h}_0^{a+1}}{a(a+3)} \quad (11)$$

Eqs. (10) and (11) are a generalization of the solution for the base flow proposed by Weinstein (1990) for the special case when $a = 2$.

4.2 Stability problem

We present the zeroth and first-order solutions for long-wavelength instabilities, obtained by expanding Ψ and c as power series of the wavenumber α , with $\alpha \rightarrow 0$. By considering that $\tilde{\Psi} = 1$ at $\bar{y} = 0$ at the zeroth order, $\tilde{\Psi} = 0$ at $\bar{y} = 0$ at the first order, as done by Rousset *et al.* (2007), and by using Eqs. (10) and (11), we obtain the zeroth and first order solutions for perturbations. A Mathematica script was written in order to solve both systems and it is available on Mendeley Data (Chimetta and Franklin, 2019). The zeroth order solution is given by,

$$c_0 = 3^{\frac{2}{3}} + \frac{(-1)^a 3^{\frac{a+2}{3}}(a+1)(I-1)(n-1)L^a}{a(a+3)} \quad (12)$$

and the first-order solution is written as,

$$c_1 = \frac{i[6Re - 5\cot(\theta)]}{5} + \frac{i(-1)^a 3^{\frac{a+3}{3}}(I-1)(n-1)L^a\{2[15 + a(6a+37)]Re - 5a(a+5)\cot(\theta)\}}{5a(a+3)(a+5)} \quad (13)$$

The initial stability of a falling film of a general liquid described by the Carreau-Yasuda model is determined by Eqs. (12) and (13), which are generalizations of the solutions proposed by Rousset *et al.* (2007) for the especial case when $a = 2$ (for $a = 2$, Eqs. (12) and (13) become exactly the same as Eqs. (42) and (49) of Rousset *et al.* (2007)). The critical Reynolds number at the onset of instability is given by Eq. (14), obtained by considering $c_i = \alpha c_1 = 0$.

$$Re_{cr} = \frac{5\cot(\theta)}{6} \left\{ 1 + \frac{(-1)^{a+1} 3^{\frac{a}{3}}[15 + a(3a+22)](I-1)(n-1)L^a}{a(a+3)(a+5)} \right\} \quad (14)$$

For $a = 2$, Eq.(14) becomes exactly the same as Eq. (50) of Rousset *et al.* (2007).

5. Results and discussion

Next, we present the critical conditions for the onset of film instabilities considering $1 < a < 10$ and $0 < n < 1$, which represents a gradual variation from shear-thinning to Newtonian flows. The set of parameters were $a = 2$, $n = 0.8$ and $L = 0.5$. Figure 1 shows the film thickness solution \bar{h} for the reference flow as a function of n and a , for fixed values of I and L . Our results show that the film thickness increases as the fluid rheology obtains stronger shear-thinning characteristics ($a \rightarrow 1$ and $n \rightarrow 0$). This increment is monotonic with n and non-monotonic with a . Figure 2 shows the critical Reynolds number Re_{cr} as a function of the slope angle θ and a for shear-thinning fluids. Also, it is observed that the critical Reynolds number decreases with the slope angle, similarly as in Newtonian cases and that it varies non-monotonically as the fluid rheology tends to stronger shear-thinning characteristics ($a \rightarrow 1$). When a reaches values above 7, the shear-thinning effects steadily lose their relevance and the fluid rheology tends to a Newtonian behavior. Remarkably, fluids with rheology given by $a \rightarrow 1$ are the most stable, while the Carreau fluids, which have $a = 2$, are the most unstable of shear-thinning fluids.

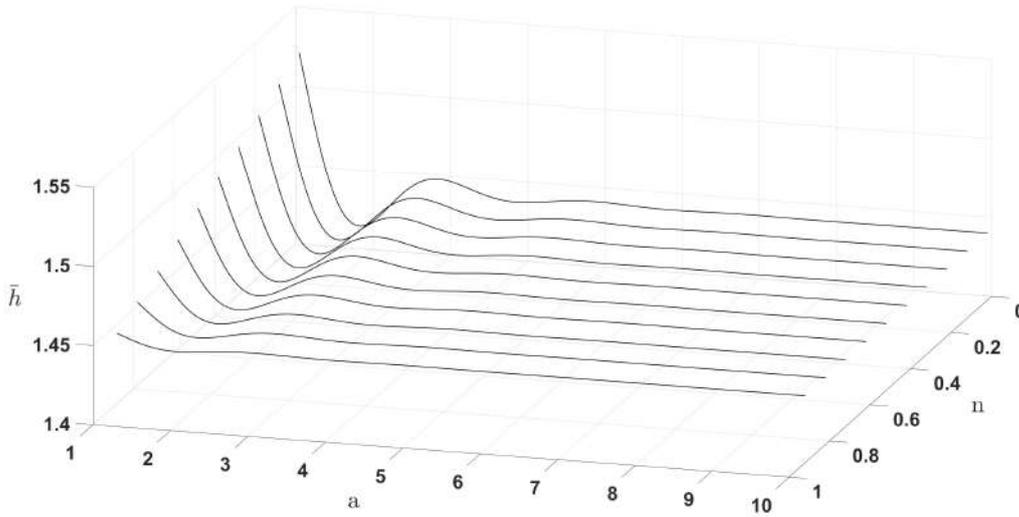


Figure 1. Film thickness \bar{h} as a function of the parameters n and a for a shear-thinning fluid with $L = 0.4$ and $I = 0.5$. Figure extracted from Chimetta and Franklin (2020).

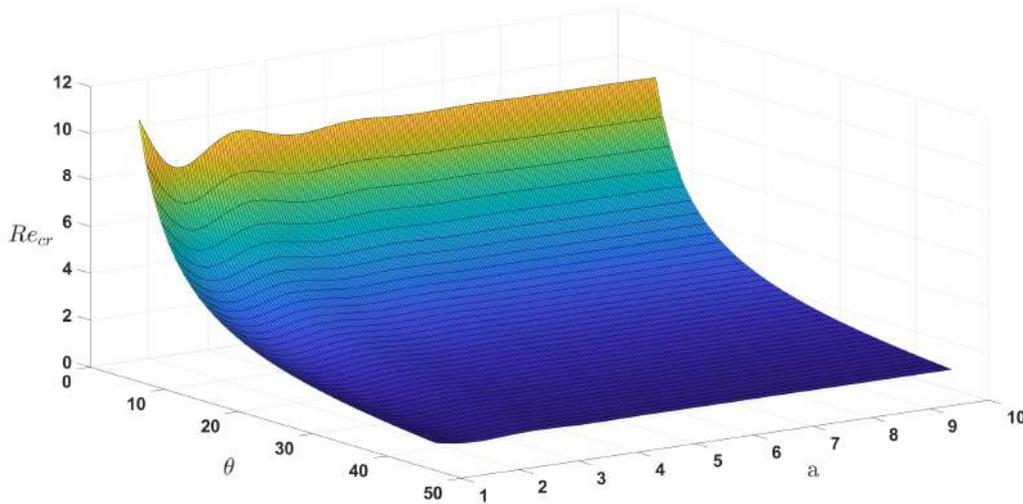


Figure 2. Critical Reynolds number Re_{cr} as a function of the slope angle θ and a for a shear-thinning fluid with $I = n = 0.5$ and $L = 0.4$. Figure extracted from Chimetta and Franklin (2020).

6. Conclusions

This paper presented a summary of the analytical approaches showed in Chimetta and Franklin (2020). The reference flow and stability problem solutions were obtained through asymptotic expansions. Our investigations for the onset of film instabilities were carried out by varying the parameters $0 < n < 1$ and $1 < a < 10$, which represents a gradual variation from shear-thinning to Newtonian flows. Results have shown that as the fluid rheology obtains stronger shear-thinning characteristics the film thickness increases. This increment is non-monotonic with a and monotonic with n . Further investigations showed that the critical Reynolds number decreases with the slope angle, the same behavior being present in Newtonian cases, and it varies non-monotonically as the fluid rheology tends to stronger shear-thinning characteristics. Values around $a = 2$, which represent the Carreau model, are the most unstable of shear-thinning fluids.

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