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# ON THE USE OF EXPLICIT FORMULATIONS IN THE HIGH ORDER DISCRETE ORDINATES SOLUTION OF THE RADIATIVE TRANSFER EQUATION IN ANISOTROPIC SCATTERING MEDIA

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***Abstract.*** *Studies on the solution of the radiative transfer equation in multidimensional geometry are subject to permanent research and development due to requirements on the modeling of many problems of interest in several fields. According to the literature, one aspect that still deserves special attention is the case of anisotropic scattering media. In this work, the Analytical Discrete Ordinate Method (ADO Method), along with nodal techniques, is applied to obtain the solution of the two-dimensional discrete-ordinates radiative transfer equation in anisotropic scattering media. The complexity of the model is reduced through a transverse-integration process over each spatial direction and explicit solutions are then derived, for the discrete ordinates approximation of the original model. Particular attention is given to the use of alternative quadrature schemes that allow higher order schemes than the classical ones. Numerical results are provided for radiation density and radiative heat flux in a medium with anisotropic scattering. A detailed analysis is performed on the effects on the radiative heat flux of the optical thickness of the medium as well as reflectivity on the boundary of the domain. Comparisons with results available in the literature reinforce the good performance of the method in coarser meshes and indicate the importance of using higher order quadrature schemes to better describe models with anisotropy.*

***Keywords:*** *Radiative transfer, anisotropic scattering, high order quadrature scheme, ADO method*

## 1. INTRODUCTION

In high temperature engineering systems, such as furnaces and combustion engines, and in tomographic modeling, the dominant heat transfer process is the transport of thermal radiation (Howell *et al.*, 2016). In these applications, for the study of the propagation of electromagnetic radiation in a medium, the solution of the radiative transfer equation (RTE) is necessary. The RTE mathematically describes the interaction between emission, absorption and scattering by which the medium affects the transfer of radiation. In the literature we can find several works that solve the RTE, such as Fiveland (1988); Klose *et al.* (2002); Liu *et al.* (2002); Guo and Kumar (2002); Altaç (2003); Boulet *et al.* (2007); Coelho (2014); Aghanajafi and Abjadpour (2016); Hunter and Guo (2016); Addoum *et al.* (2018); Ndjanda Heugang *et al.* (2020), however when addressing the RTE solution in multidimensional geometry most of the works are restricted to nonscattering or isotropic scattering models.

With respect to the effects of anisotropy, we can mention the work of Kim and Lee (1988) in which the effects of anisotropic scattering on radiative heat transfer in two-dimensional domains were analyzed and numerical results were presented for problems with anisotropy degree up to twelve. Later, Trivic *et al.* (2004) presented comparisons with those found on Kim and Lee (1988) when analyzing the same effects. According to Trivic *et al.* (2004), the results given by Kim and Lee (1988) were the only available on the subject at that time. In 2016, more recently, Moghadam *et al.* (2016) still used the results given by Kim and Lee (1988) as a reference result, when working on transient heat transfer. Comparisons between results provided by these three approaches have, in general, only two digits in agreement.

One of the classic forms of representation of the angular variables in the radiative transfer equation (RTE) is the approximation in discrete ordinates through a deterministic method: the Discrete Ordinates method ($S_N$). It was introduced by Wick (1943) and Chandrasekhar (1950) and it is based on the discretization of the particle directions (the angular variables in the RTE), in the unitary sphere. In this context, Barichello and Siewert (1999) proposed the Analytical Discrete Ordinates (ADO) method, whose solution is constructed from eigenvalues and eigenfunctions and the quadrature scheme used to represent the discrete directions is arbitrary. This method, which in unidimensional geometry determines explicit solutions of the RTE in the spatial variable and no spatial discretization is needed, was expanded, along with nodal techniques, for the treatment of a class of multidimensional problems (Barichello *et al.*, 2017; Cromianski *et al.*, 2019; Barichello *et al.*, 2020). In the case of nuclear applications, the extension of the ADO approach to two-dimensional problems was restricted to media with linearly anisotropic scattering (Picoloto *et al.*, 2017). Bearing in mind that the effects of anisotropy are still a relevant aspect to be investigated in the context of the multidimensional RTE equation, first we extended the ADO formulation for problems with second degree of anisotropy (Rui and Barichello, 2018). And recently, the derivation of the eigenvalue problem, which is fundamental in the case of a spectral methodology as the ADO method, was obtained for a more general expansion of the scattering law (Rui *et al.*, 2020). In the work (Rui *et al.*, 2020) comparisons were established with the Refs. Kim and Lee (1988); Trivic *et al.* (2004); Moghadam *et al.* (2016) not only as a verification of our formulation, but also to contribute to the analysis of models of thermal radiation with anisotropy, for which, there are few reference results available in the literature, as far as we know. The ADO-Nodal formulation was tested in problems with anisotropy degree up to twelve for different orders of the classical Level Symmetric Quadrature Scheme LQN (Lewis and Miller, 1984).

The RTE solution with general anisotropy through the ADO-Nodal method can be a useful tool in modeling radiative heat transfer in high temperature engineering systems, such as combustion of pulverized coal, and in applications with biological tissues, in optical tomography, where the presence of particles is an important factor and their scattering can be considered to correctly determine heat transfer, since the scattering medium is highly anisotropic and has a great effect on radiative transfer (Gronarz *et al.*, 2018; Hunter and Guo, 2012). However, to deal with higher order expansions of phase functions, the use of higher order quadrature schemes is desirable, in order to provide a more conclusive analysis of the computational performance of our formulation, as well as to obtain benchmark results for test problems. We note that all the three references we referred to deal with anisotropic scattering (Kim and Lee, 1988; Trivic *et al.*, 2004; Moghadam *et al.*, 2016) worked only up to $S_{14}$ approximation. In this work, we investigate the use of such a quadrature scheme known as Quadruple Range ($QR$) (Abu-Schumays, 1977) in the solution of problems in anisotropic scattering medium that allow us the use of higher order discrete ordinates approximations. We also discuss the effects of the radiative heat flux for different optical thickness of the medium as well as reflectivity on the boundary of the domain.

Therefore, in the next section, Section 2, we introduce the formulation of the problem along the ADO-Nodal solution, where we describe the eigenvalue problem and the general solution of the problem. Next, in Section 3 we presented the numerical results and we conclude with our remarks in Section 4.

## 2. FORMULATION OF THE PROBLEM

We consider the problem of solving the discrete ordinates approximation of the two-dimensional RTE, in a rectangular medium, $x \in [0, a]$ and $y \in [0, b]$ (Kim and Lee, 1988),

$$\mu_i \frac{\partial}{\partial x} I(x, y, \mathbf{\Omega}_i) + \eta_i \frac{\partial}{\partial y} I(x, y, \mathbf{\Omega}_i) + \beta I(x, y, \mathbf{\Omega}_i) = \kappa I_b(x, y) + \frac{\sigma_s}{4\pi} \sum_{n=1}^{M} w_n I(x, y, \mathbf{\Omega}_n) \Phi(\mathbf{\Omega}_n \cdot \mathbf{\Omega}_i), \tag{1}$$

for $i = 1, \ldots, M$, in which $M$ is the number of discrete directions; $w_n$ are the weights (normalized to $4\pi$) associated to the angular directions $\mathbf{\Omega}_i$; $I(x, y, \mathbf{\Omega}_i)$ is the radiation intensity; $\beta = \kappa + \sigma_s$ is the extinction coefficient, $\kappa$ and $\sigma_s$ are the absorption and scattering coefficients of the medium, respectively; $I_b(x, y)$ is the intensity of radiation from a blackbody. Still, the angular directions are such that $\Omega_x = \mu = \sqrt{1 - \xi^2} \cos(\varphi)$; $\Omega_y = \eta = \sqrt{1 - \xi^2} \sin(\varphi)$ and $\Omega_z = \xi = \cos(\theta)$, where $\theta$ is the polar angle measured from the $z$-axis and $\varphi$ is the azimuthal angle measured from the $x$-axis; and $\Phi(\mathbf{\Omega}_n \cdot \mathbf{\Omega}_i)$ is the scattering phase function.

We follow Chandrasekhar (1950); Cacuci (2010) and we express $\Phi(\mathbf{\Omega_n} \cdot \mathbf{\Omega_i})$ by a finite Legendre polynomial expansion in terms of the cosine of the scattering angle. Then, we write the equation for the radiation intensity $I(x, y, \mathbf{\Omega}_i)$ as

$$\mu_i \frac{\partial}{\partial x} I(x, y, \mathbf{\Omega}_i) + \eta_i \frac{\partial}{\partial y} I(x, y, \mathbf{\Omega}_i) + \beta I(x, y, \mathbf{\Omega}_i) = \kappa I_b(x, y) + \\ \frac{\sigma_s}{4\pi} \sum_{l=0}^{L} \sum_{\substack{p=0 \\ (l+p \text{ even})}}^{l} (2 - \delta_{0,p}) C_l^p P_l^p(\xi_i) \sum_{n=1}^{M} w_n P_l^p(\xi_n) \cos[p(\varphi_n - \varphi_i)] I(x, y, \mathbf{\Omega}_n). \tag{2}$$

where $C_l^p = C_l (l-p)!/(l+p)!$, $\delta_{0,p}$ is the Kronecker delta (equal to 1 when $p = 0$, and equal to zero otherwise) and the $P_l^p$'s are the associated Legendre polynomials of order $l$ (Kim and Lee, 1988).

We assume boundary conditions, written as

$$I(x,y,\mathbf{\Omega}_i) = \epsilon_w I_{bw}(x,y) + \frac{\rho}{\pi} \sum_{\boldsymbol{n}\cdot\mathbf{\Omega}_j>0} w_j I(x,y,\mathbf{\Omega}_j)|\,\boldsymbol{n}\cdot\mathbf{\Omega}_j|, \tag{3}$$

for $(x, y)$ on the contour, where $\epsilon_w$ is the surface emissivity; $\rho$ is the surface reflectivity; $\mathbf{\Omega}_i$ (with $\boldsymbol{n}\cdot\mathbf{\Omega}_i < 0$, $\boldsymbol{n}$ being the unit outer normal vector at the boundary) denote the incoming flux directions and $\mathbf{\Omega}_j$ are the outgoing flux directions ($\boldsymbol{n}\cdot\mathbf{\Omega}_j > 0$).

We consider the domain subdivided in $r = 1, \ldots, R$ rectangular regions (nodes) defined by $x \in [a^r_{h-1}, a^r_h]$ and $y \in [b^r_{k-1}, b^r_k]$ with $0 \le a^r_{h-1} < a^r_h \le a$ and $0 \le b^r_{k-1} < b^r_k \le b$, where $h = 1, \ldots, H$ and $k = 1, \ldots, K$ indicate a number of subdivisions considered in the axes $x$ and $y$, respectively (Barichello *et al.*, 2017). Then as in Barichello *et al.* (2011) we associate the directions $\mathbf{\Omega}_i$, with $i = 1, \ldots, M/2$, to the coordinates $\eta_i > 0$; $i = M/2 + 1, \ldots, M$ to the coordinates $\eta_i < 0$ and we associate indices $i = 1, \ldots, M/2$ to directions where $\mu_i > 0$ and indices $i = M/2+1, \ldots, M$ to directions where $\mu_i < 0$ to express the average intensity radiation along the $x$ and $y$ direction in region $r$: $I_{xr}(y, \mathbf{\Omega}_i)$ and $I_{yr}(x, \mathbf{\Omega}_i)$, respectively.

### 2.1 The ADO-Nodal solution in a region $r$: eigenvalue problem

To obtain the one dimensional transverse-integrated equations, in $y$ and $x$ directions, we integrate the Eq. (2) for all $x \in [a^r_{h-1}, a^r_h]$ and $y \in [b^r_{k-1}, b^r_k]$, respectively. In the integration process, additional unknowns in the contours of each region are introduced in the system. From the boundary conditions of the problem, some of these variables may be known, for example, in the incoming directions, whereas for other directions and on the edges of the internal regions those terms have to be approximated (Rui and Barichello, 2018). In this work, we express the unknown intensities in each region $r$ by constant functions, although other approximations have been tested (Prolo Filho and Barichello, 2014; Cromianski *et al.*, 2019).

From the application of nodal schemes in the two-dimensional equation, we obtain a system of unidimensional equations transversally integrated, in the variables $x$ and $y$, which are then solved by the ADO method. Due to the limit on the number of pages we will not repeat the steps of the ADO-Nodal solution process here; for more details please see Rui *et al.* (2020). Thus, we seek solutions of the homogeneous problem for the $x$ direction, in terms of eigenvalues and eigenfunctions (Barichello *et al.*, 2017), for $i = 1, \ldots, M$, as

$$I^H_{yr}(x, \mathbf{\Omega}_i) = \Phi_{yr}(\nu_r, \mathbf{\Omega}_i) e^{-x/\nu_r}, \tag{4}$$

where $\nu_r$ is the separation constant associated with eigenfunctions $\Phi_{yr}(\nu_r, \mathbf{\Omega}_i)$. Substituting the Eq. (4) in the system of unidimensional equations transversally integrated, after some manipulations, we derive the eigenvalue problem

$$[\mathbf{A}_{yr}\mathbf{B}_{yr}]\,\mathbf{U}_{yr} = \lambda_r \mathbf{U}_{yr}, \tag{5}$$

with

$$\lambda_r = \frac{1}{\nu_r^2}. \tag{6}$$

Here $\mathbf{A}_{yr}$ and $\mathbf{B}_{yr}$ are $M/2 \times M/2$ matrices defined as

$$\mathbf{A}_{yr}(i,j) = \begin{cases} \dfrac{\sigma_{sr}}{2\pi\mu_i} \displaystyle\sum_{l=0}^{L} \sum_{\substack{p=0 \\ (l+p \text{ even})}}^{l} (2-\delta_{0,p}) C^p_l P^p_l(\xi_i) w_j P^p_l(\xi_j) \zeta^p_{yl}(\mathbf{\Omega}_j) - \dfrac{\beta_r}{\mu_i}, & \text{if} \quad i = j, \\ \dfrac{\sigma_{sr}}{2\pi\mu_i} \displaystyle\sum_{l=0}^{L} \sum_{\substack{p=0 \\ (l+p \text{ even})}}^{l} (2-\delta_{0,p}) C^p_l P^p_l(\xi_i) w_j P^p_l(\xi_j) \zeta^p_{yl}(\mathbf{\Omega}_j), & \text{otherwise}, \end{cases} \tag{7}$$

where

$$\zeta^p_{yl}(\mathbf{\Omega}_j) = \begin{cases} \sin(p\varphi_j)\sin(p\varphi_i), & \text{if} \quad p \quad \text{is even}, \\ \cos(p\varphi_j)\cos(p\varphi_i), & \text{if} \quad p \quad \text{is odd}, \end{cases} \tag{8}$$

and

$$\mathbf{B}_{yr}(i,j) = \begin{cases} \dfrac{\sigma_{sr}}{2\pi\mu_i} \displaystyle\sum_{l=0}^{L} \sum_{\substack{p=0 \\ (l+p \text{ even})}}^{l} (2-\delta_{0,p}) C^p_l P^p_l(\xi_i) w_j P^p_l(\xi_j) \Gamma^p_{yl}(\mathbf{\Omega}_j) - \dfrac{\beta_r}{\mu_i}, & \text{if} \quad i = j, \\ \dfrac{\sigma_{sr}}{2\pi\mu_i} \displaystyle\sum_{l=0}^{L} \sum_{\substack{p=0 \\ (l+p \text{ even})}}^{l} (2-\delta_{0,p}) C^p_l P^p_l(\xi_i) w_j P^p_l(\xi_j) \Gamma^p_{yl}(\mathbf{\Omega}_j), & \text{otherwise}, \end{cases} \tag{9}$$

where

$$\Gamma^p_{yl}(\mathbf{\Omega}_j) = \begin{cases} \cos(p\varphi_j)\cos(p\varphi_i), & \text{if} \quad p \quad \text{is even}, \\ \sin(p\varphi_j)\sin(p\varphi_i), & \text{if} \quad p \quad \text{is odd}, \end{cases} \tag{10}$$

for $i = 1, ..., M/2$, $j = 1, ..., M/2$.

Conversely, to derive the eigenvalue problem and the homogeneous solution in $y$ direction we proceed in a similar way. For a region $r$, we propose solutions of the homogeneous problem, for $i = 1, \dots M$, as (Barichello *et al.*, 2017)

$$I^H_{xr}(y, \mathbf{\Omega}_i) = \Phi_{xr}(\gamma_r, \mathbf{\Omega}_i)e^{-y/\gamma_r}, \tag{11}$$

where $\gamma_r$ is a separation constant associated with eigenfunctions $\Phi_{xr}(\gamma_r, \mathbf{\Omega}_i)$. Substituting the Eq. (11) in the system of unidimensional equations transversally integrated, after some manipulations, we derive the eigenvalue problem

$$\left[\mathbf{A}_{xr}\mathbf{B}_{xr}\right]\mathbf{U}_{xr} = \lambda^*_r\mathbf{U}_{xr}, \tag{12}$$

with

$$\lambda^*_r = \frac{1}{\gamma^2_r}. \tag{13}$$

$\mathbf{A}_{xr}$ and $\mathbf{B}_{xr}$ are $M/2 \times M/2$ matrices defined as

$$\mathbf{A}_{xr}(i,j) = \begin{cases} \dfrac{\sigma_{sr}}{2\pi\eta_i}\displaystyle\sum_{l=0}^{L}\sum_{\substack{p=0 \\ (l+p \text{ even})}}^{l}(2-\delta_{0,p})C^p_l P^p_l(\xi_i)w_j P^p_l(\xi_j)\zeta^p_{xl}(\mathbf{\Omega}_j) - \dfrac{\beta_r}{\mu_i}, & \text{if} \quad i = j, \\ \dfrac{\sigma_{sr}}{2\pi\eta_i}\displaystyle\sum_{l=0}^{L}\sum_{\substack{p=0 \\ (l+p \text{ even})}}^{l}(2-\delta_{0,p})C^p_l P^p_l(\xi_i)w_j P^p_l(\xi_j)\zeta^p_{xl}(\mathbf{\Omega}_j), & \text{otherwise}, \end{cases} \tag{14}$$

where

$$\zeta^p_{xl}(\mathbf{\Omega}_j) = \sin(p\varphi_j)\sin(p\varphi_i), \tag{15}$$

and

$$\mathbf{B}_{xr}(i,j) = \begin{cases} \dfrac{\sigma_{sr}}{2\pi\eta_i}\displaystyle\sum_{l=0}^{L}\sum_{\substack{p=0 \\ (l+p \text{ even})}}^{l}(2-\delta_{0,p})C^p_l P^p_l(\xi_i)w_j P^p_l(\xi_j)\Gamma^p_{xl}(\mathbf{\Omega}_j) - \dfrac{\beta_r}{\mu_i}, & \text{if} \quad i = j, \\ \dfrac{\sigma_{sr}}{2\pi\eta_i}\displaystyle\sum_{l=0}^{L}\sum_{\substack{p=0 \\ (l+p \text{ even})}}^{l}(2-\delta_{0,p})C^p_l P^p_l(\xi_i)w_j P^p_l(\xi_j)\Gamma^p_{xl}(\mathbf{\Omega}_j), & \text{otherwise}, \end{cases} \tag{16}$$

where

$$\Gamma^p_{xl}(\mathbf{\Omega}_j) = \cos(p\varphi_j)\cos(p\varphi_i), \tag{17}$$

for $i = 1, ..., M/2$ and $j = 1, ..., M/2$.

We derive homogeneous solutions in terms of eigenvalues and eigenfunctions and in this work, particular solutions for a constant source. Therefore, we write the explicit solution in terms of each of the spatial variables for $I_{xr}$ and $I_{yr}$. We note that a sweep technique is not required, as is usual in most of the methodologies. Therefore, the general solution for the average angular intensity in the $y$ direction and $x$ direction, respectively, in the region $r$, may be written in the form

$$I_{yr}(x, \mathbf{\Omega}_i) = \sum_{j=1}^{M/2}\left[A_{j,r}\Phi_{yr}(\nu_{jr}, \mathbf{\Omega}_i)e^{-(x-a^r_{h-1})/\nu_{jr}} + A_{j+M/2,r}\Phi_{yr}(\nu_{jr}, \mathbf{\Omega}_{i+M/2})e^{-(a^r_h - x)/\nu_{jr}}\right] + K_{i,r}, \tag{18}$$

$$I_{yr}(x, \mathbf{\Omega}_{i+M/2}) = \sum_{j=1}^{M/2}\left[A_{j,r}\Phi_{yr}(\nu_{jr}, \mathbf{\Omega}_{i+M/2})e^{-(x-a^r_{h-1})/\nu_{jr}} + A_{j+M/2,r}\Phi_{yr}(\nu_{jr}, \mathbf{\Omega}_i)e^{-(a^r_h - x)/\nu_{jr}}\right] + K_{i+M/2,r}, \tag{19}$$

and,

$$I_{xr}(y,\mathbf{\Omega}_i) = \sum_{j=1}^{M/2} \left[B_{j,r}\Phi_{xr}(\gamma_{jr},\mathbf{\Omega}_i)e^{-(y-b^r_{k-1})/\gamma_{jr}} + B_{j+M/2,r}\Phi_{xr}(\gamma_{jr},\mathbf{\Omega}_{i+M/2})e^{-(b^r_k-y)/\gamma_{jr}}\right] + W_{i,r}, \quad (20)$$

$$I_{xr}(y,\mathbf{\Omega}_{i+M/2}) = \sum_{j=1}^{M/2} \left[B_{j,r}\Phi_{xr}(\gamma_{jr},\mathbf{\Omega}_{i+M/2})e^{-(y-b^r_{k-1})/\gamma_{jr}} + B_{j+M/2,r}\Phi_{xr}(\gamma_{jr},\mathbf{\Omega}_i)e^{-(b^r_k-y)/\gamma_{jr}}\right] + W_{i+M/2,r}, \quad (21)$$

for $i = 1, ..., M/2$, $x \in [a^r_{h-1}, a^r_h]$ and $y \in [b^r_{k-1}, b^r_k]$. The unknown coefficients $A_{j,r}$, $A_{j+M/2,r}$ and $B_{j,r}$, $B_{j+M/2,r}$ are arbitrary coefficients of the homogeneous solution and $K_{i,r}$, $K_{i+M/2,r}$ and $W_{i,r}$, $W_{i+M/2,r}$ are constants related to the particular solutions. These unknown coefficients are to be determined.

To establish the general solution, for $y$ and $x$ direction, we need to determine all the unknown coefficients of the homogeneous problem and the particular solution in Eqs. (18), (19), (20) and (21). To do that, we generate a linear system that is composed of sets of equations that must be associated with the integrated one-dimensional intensities, in $x$ and $y$. These sets of equations are derived from boundary conditions; continuity conditions at the interfaces of the regions; auxiliary equations that approximate unknown intensities in the contours and the equations for defining the particular solution (since the source is unknown, as it depends on the intensities in the contours). The order of linear system is $4M(HK)$ (Rui *et al.*, 2020).

## 3. NUMERICAL RESULTS

In order to establish numerical comparisons with the literature (Kim and Lee, 1988) we assume a square domain and a pure scattering medium (albedo $\omega = 1$). The walls and the medium are kept cold and have prescribed emissive power $E_w = E_b = 0$, except by the bottom boundary that is kept hot and their emissive power is $E_{wb} = 1.0$. Also, we consider $a = b = 1.0$ (nondimensional variables), a black medium, $\epsilon_w = 1$ and $\rho = 0$ and the coefficients of expansion of the scattering law in terms of Legendre polynomials are taken from the same reference.

We evaluate the average radiation density and the radiative heat flux, respectively as

$$\phi_{xr}(y) = \frac{1}{4}\sum_{n=1}^{M/2} w_n[I_{xr}(y,\mathbf{\Omega}_n) + I_{xr}(y,\mathbf{\Omega}_{n+M/2})], \quad (22)$$

$$q_{yr} = \sum_{n=1}^{M} w_n\eta_n I_{xr}(y,\mathbf{\Omega}_n). \quad (23)$$

Our results were generated using the classic quadrature scheme $LQ_N$ (Lewis and Miller, 1984) which allows only the use of orders up to $N \leq 16$ (corresponding to 36 directions per octant) but seemed to be more similar to the schemes used in the other references. In addition, we used the Quadruple Range $QR$ (Abu-Schumays, 1977) scheme which allows to implement higher order approximations. We used $\sigma_s = 0.999$ to avoid degenerate eigenvalues (when $\sigma_s = 1$). We present comparisons with results provided by Kim and Lee (1988). In that work, the authors obtained the solution for the RTE considering pure scatterring medium, using the finite volume method where the domain was subdivided in $26 \times 26$ and the $S_{14}$ approximation, not making it clear which quadrature scheme was used. The numerical results were generated in a program implemented in Fortran 90 and to solve the eigenvalue problems and the linear system we used the DGEEV and DGETRF/DGETRS subroutines from the LAPACK library (Anderson *et al.*, 1999). We used an Intel Core $i7$, $3.60GHz$, $8.00\ GiB\ RAM$ computer to generate the results. Due to the restriction of computational memory to store the linear system, we were able to obtain results only for meshes up to size $6 \times 6$. For larger systems we have to use other storage and solution resources (Moura *et al.*, 2018).

We show results for the average radiation density along the centerline ($x = 0.5$) of the domain in Table 1 for second order of anisotropy ($L = 2$) and in Table 2 for forward scattering $L = 8$. The values obtained in this study, using coarser meshes, agree in two digits with the results found in the literature Kim and Lee (1988); Trivic *et al.* (2004) using a more refined mesh. Trivic *et al.* (2004) compares its results with Kim and Lee (1988) and in general they also agree up to two significant digits. Comparing the results obtained by the ADO method via $QR$ scheme, with the increase in the number of directions per octant ($DO$) we observed up to four digits of agreement. When comparing the density values obtained with $LQ_N(DO = 36)$ and with $QR(DO = 32)$ we observed an agreement of up to three digits. Further research in this direction is necessary to fully establish reference solutions to this problem. From the tabulated results, we can see

Table 1. Average radiation density along the centerline ($x = 0.5$), $L = 2$.

| y | $LQ_N$ | $QR$ | | | | | Kim e Lee (1988) |
|---|---|---|---|---|---|---|---|
| | $DO = 36$ | $DO = 8$ | $DO = 18$ | $DO = 32$ | $DO = 50$ | $DO = 72$ | $DO = 28$ |
| 0.00 | 0.6647 | 0.6643 | 0.6647 | 0.6646 | 0.6646 | 0.6646 | 0.660444 |
| 0.38 | 0.3172 | 0.3221 | 0.3185 | 0.3189 | 0.3190 | 0.3190 | 0.317122 |
| 0.58 | 0.2066 | 0.2049 | 0.2073 | 0.2068 | 0.2068 | 0.2068 | 0.212815 |
| 0.74 | 0.1474 | 0.1505 | 0.1479 | 0.1482 | 0.1482 | 0.1482 | 0.152204 |
| 0.94 | 0.0873 | 0.0823 | 0.0879 | 0.0871 | 0.0871 | 0.0872 | 0.091917 |
| 1.00 | 0.0693 | 0.0606 | 0.0703 | 0.0687 | 0.0689 | 0.0689 | 0.074333 |

Table 2. Average radiation density along the centerline ($x = 0.5$), $L = 8$.

| y | $LQ_N$ | $QR$ | | | | | Kim e Lee (1988) |
|---|---|---|---|---|---|---|---|
| | $DO = 36$ | $DO = 8$ | $DO = 18$ | $DO = 32$ | $DO = 50$ | $DO = 72$ | $DO = 28$ |
| 0.00 | 0.5588 | 0.5593 | 0.5595 | 0.5594 | 0.5594 | 0.5594 | 0.557988 |
| 0.38 | 0.3001 | 0.3060 | 0.3021 | 0.3025 | 0.3026 | 0.3026 | 0.299127 |
| 0.58 | 0.2183 | 0.2160 | 0.2188 | 0.2183 | 0.2183 | 0.2183 | 0.222990 |
| 0.74 | 0.1758 | 0.1790 | 0.1765 | 0.1767 | 0.1767 | 0.1767 | 0.179558 |
| 0.94 | 0.1331 | 0.1263 | 0.1335 | 0.1326 | 0.1326 | 0.1327 | 0.137162 |
| 1.00 | 0.1194 | 0.1072 | 0.1200 | 0.1182 | 0.1184 | 0.1184 | 0.125029 |

the importance of considering the highest quadrature order, since the values agree in a larger number of digits with the increase in the quadrature order.

In Figure 1 we show the radiative heat flux along the centerline for different values of reflectivity $\rho$, generated considering a gray medium for $L = 2$ and $DO = 50$ of the quadrature $QR$. We observe that as $\rho$ increases, the surface emissivity becomes smaller, the angular distribution of the heat flux becomes more uniform and the flux decreases.

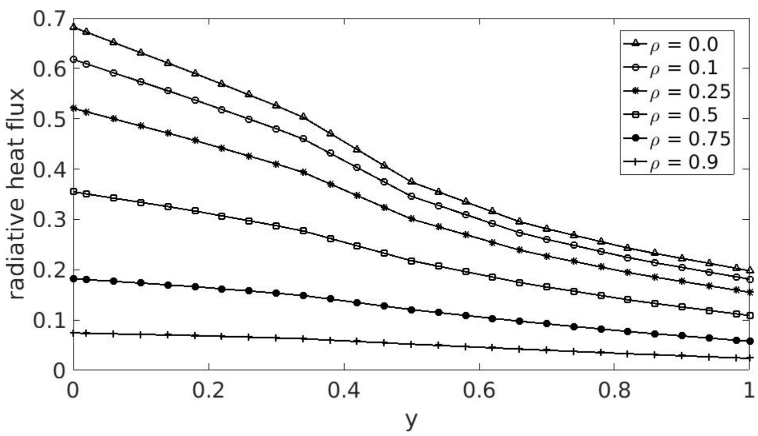


Figure 1. Radiative heat flux along the centerline ($x = 0.5$) for different values of reflectivity, $L = 2$, $DO = 50$.

In Figure 2 we show the radiative heat flux along the centerline for different optical thickness $\tau$ ($y = \tau_y/\tau$), generated considering $\rho = 0$, for $L = 2$ and $DO = 50$ of the quadrature $QR$. We observe that the radiative heat flux is greater for shorter optical thickness and becomes more flat as $\tau$ is greater. We note that with the use of the quadrature $QR$ and a coarser mesh, we obtain the same graph profile generated by Kim and Lee (1988) using a more refined mesh.

## 4. CONCLUDING REMARKS

Having established the ADO-Nodal formulation to solve two-dimensional radiative transfer problems for arbitrary degree of anisotropic scattering, in this work we extended the analysis of the average radiation density and the radiative heat flux via higher order quadrature schemes. To our knowledge this is not available in the literature for this problem. From the analysis performed, we observed that comparing to the results obtained by the ADO formulation between different order of the quadrature scheme and keeping the mesh fixed, we verified a good agreement in digits in the ADO-Nodal solution. Studies to obtain benchmark solutions to this problem continue and we are focusing our attention on making use of more refined meshes together with the use of high order quadratures. These studies, in addition to

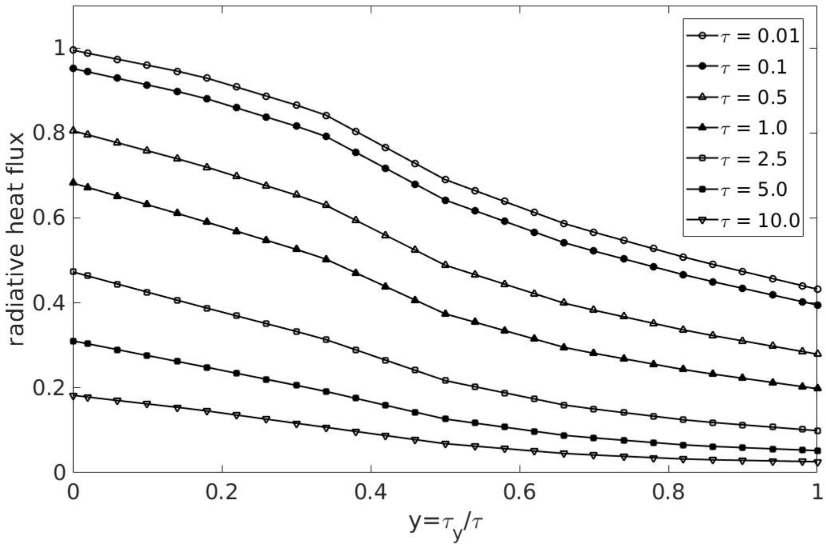


Figure 2. Radiative heat flux along the centerline ($\tau_x = 0.5\tau$) for different optical thickness, $L = 2$, $DO = 50$.

expanding the class of problems that can be addressed by the ADO methodology, move towards allowing us, for example, to treat problems of interest in tomographic reconstructions.

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