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CONVECTIVE-DIFFUSIVE EIGENVALUE PROBLEM BASES FOR INTEGRAL TRANSFORM SOLUTIONS IN TRANSPORT PHENOMENA

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Abstract. A transformation of coefficients strategy is revisited and generalized that allows for rewriting convection-diffusion problems so that advective and diffusive terms are partially or fully merged into a generalized diffusion problem. The aim is to enhance the application of the Generalized Integral Transform Technique (GITT) in the hybrid numerical-analytical solution of transport of a passive scalar quantity, by offering an eigenfunction expansion base that partially incorporates the convective effects, for improved convergence rates of the proposed expansions. The transformation strategy allows for different choices of reference coefficients, which lead to different effects on overall convergence enhancement. Transient laminar forced convection in a parallel plates channel is considered more closely to illustrate the different convective-diffusive eigenvalue problem choices and examine their convergence behavior in comparison to the more traditional GITT approach based on a purely diffusive eigenvalue problem.

Keywords: Convection-diffusion, hybrid methods, integral transforms, eigenvalue problems, transient convection

1. INTRODUCTION

The Generalized Integral Transform Technique (GITT) is an extension of the classical integral transform method (Mikhailov and Ozisik, 1984), advanced along the last few decades to provide hybrid numerical-analytical solutions for linear or nonlinear diffusion and convection-diffusion problems (Cotta, 1990; Cotta, 1993), as either a tool for benchmarking purposes or an alternative computational approach in transport phenomena on itself. The traditional solution path is based on the proposition of a diffusive eigenvalue problem with characteristic linear coefficients, extracted from the original formulation, which provides the base for the eigenfunction expansion that leads to the integral transformation of the proposed partial differential system. In recent years (Cotta et al., 2017; Knupp et al., 2020), a fairly straightforward coefficients transformation has been introduced which combines convective and diffusive terms, either fully or partially, providing a reformulated problem that naturally leads to the proposition of a generalized diffusive eigenvalue problem encompassing at least part of the influence of convection in the eigenfunction expansion base. The expected final outcome is the improvement on convergence rates when employing this more informative eigenfunction expansion proposal, both in linear and nonlinear convection-diffusion problems. However, since the choice of reference coefficient in the transformation is, to some extent, also a choice in the proposed reformulation strategy, it may lead to different equation operator coefficients and source terms, thus offering different combinations of eigenvalue problem and transformed potential ODE systems. Naturally, these choices shall imply different convergence behaviors of the proposed eigenfunction expansions within the domain and understanding these possibilities and their effects on computational performance are an essential step in the implementation of a more systematic approach. The present work aims at shedding light on the adoption of convective-diffusive eigenvalue problems in eigenfunction expansions for convergence enhancement. First, it provides a unified framework for this reformulation strategy which merges advective and diffusive terms for a general problem that governs the transport of a passive scalar quantity. It is then pointed out the importance of choosing the most adequate transformation coefficients for the proposed reformulation strategy. Finally, an example related to transient convection within parallel plate channels (Gondim et al., 2007) is employed to illustrate the methodology and investigate the main aspects behind the eigenvalue problem choices.

2. ANALYSIS

A rather general model for the convective-diffusive transport of a passive scalar quantity can be written as:

$$w(\mathbf{x})\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla \cdot [k(\mathbf{x})\nabla T] - d(\mathbf{x})T(\mathbf{x},t) + P(\mathbf{x},t,T), \mathbf{x} \in V \quad (1.a)$$

with initial and boundary conditions given by,

$$T(\mathbf{x},0) = T_0(\mathbf{x}), \mathbf{x} \in V \quad (1.b)$$

$$\alpha(\mathbf{x})T(\mathbf{x},t) + \beta(\mathbf{x})k(\mathbf{x})\frac{\partial T}{\partial \mathbf{n}} = \phi(\mathbf{x},t,T), \mathbf{x} \in S \quad (1.c)$$

Considering that the convective term coefficient vector \mathbf{u} can be represented in the three-dimensional situation by the three components $\{u_x, u_y, u_z\}$, here illustrating the transformation in the Cartesian coordinates system, $\mathbf{x}=\{x, y, z\}$, then eq.(1.a) is rewritten in the generalized diffusive form as (Cotta et al., 2017; Ozisik et al., 2017, Knupp et al., 2020):

$$\begin{aligned} \hat{w}(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t} = & \hat{k}_y(\mathbf{x})\hat{k}_z(\mathbf{x})\frac{\partial}{\partial x}[\hat{k}_x(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial x}] + \hat{k}_x(\mathbf{x})\hat{k}_z(\mathbf{x})\frac{\partial}{\partial y}[\hat{k}_y(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial y}] + \hat{k}_x(\mathbf{x})\hat{k}_y(\mathbf{x})\frac{\partial}{\partial z}[\hat{k}_z(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial z}] \\ & - \hat{d}(\mathbf{x})T(\mathbf{x},t) + \hat{P}(\mathbf{x},t,T), \mathbf{x} \in V, t > 0 \end{aligned} \quad (2.a)$$

where,

$$\begin{aligned} \hat{k}(\mathbf{x}) = & \hat{k}_x(\mathbf{x})\hat{k}_y(\mathbf{x})\hat{k}_z(\mathbf{x}); \hat{w}(\mathbf{x}) = w(\mathbf{x})\hat{k}(\mathbf{x})/k(\mathbf{x}); \hat{d}(\mathbf{x}) = d(\mathbf{x})\hat{k}(\mathbf{x})/k(\mathbf{x}); \hat{P}(\mathbf{x},t,T) = P(\mathbf{x},t,T)\hat{k}(\mathbf{x})/k(\mathbf{x}); \\ \mathbf{u}^*(\mathbf{x}) = & \frac{1}{k(\mathbf{x})}[\mathbf{u}(\mathbf{x}) - \nabla k(\mathbf{x})]; \hat{k}_x(\mathbf{x}) = e^{-\int u_x^*(\mathbf{x})dx}; \hat{k}_y(\mathbf{x}) = e^{-\int u_y^*(\mathbf{x})dy}; \hat{k}_z(\mathbf{x}) = e^{-\int u_z^*(\mathbf{x})dz}; \end{aligned} \quad (2.b-i)$$

For the general situation of eq. (2.a), the separation of variables as applied to the homogeneous version of the problem, leads to a non-classic eigenvalue problem, hence an appropriate orthogonality property for the eigenfunctions needs to be developed, and then the Generalized Integral Transform Technique (GITT) can be directly employed. Also, as pointed out in (Cotta et al., 2017), when the transformed diffusion coefficients are functions of only the corresponding space coordinate, or $\hat{k}_x(\mathbf{x}) = \hat{k}_x(x)$, $\hat{k}_y(\mathbf{x}) = \hat{k}_y(y)$, $\hat{k}_z(\mathbf{x}) = \hat{k}_z(z)$, with the consequent restrictions on the choices of the characteristic linear coefficients $k(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$, a generalized diffusion formulation is constructed, which leads to a classical self-adjoint eigenvalue problem, given by:

$$\hat{w}(\mathbf{x})\frac{\partial T(\mathbf{x},t)}{\partial t} = \nabla \cdot [\hat{k}(\mathbf{x})\nabla T(\mathbf{x},t)] - \hat{d}(\mathbf{x})T(\mathbf{x},t) + \hat{P}(\mathbf{x},t,T), \mathbf{x} \in V, t > 0 \quad (3.a)$$

where

$$\hat{k}(\mathbf{x}) = \hat{k}_x(x)\hat{k}_y(y)\hat{k}_z(z) \quad (3.b)$$

The coefficients transformation strategy leading to a classical self-adjoint eigenvalue problem, such as in eqs.(3) above, may alternatively be constructed through the formalism that follows. Let $F(\mathbf{x})$ be a class C1 function. Multiplying eqs. (1.a,c) by $F(\mathbf{x})$, we then have,

$$\hat{w}(\mathbf{x})\frac{\partial T}{\partial t} = F(\mathbf{x})\nabla \cdot [k(\mathbf{x})\nabla T] - F(\mathbf{x})\mathbf{u} \cdot \nabla T - \hat{d}(\mathbf{x})T(\mathbf{x},t) + \hat{P}(\mathbf{x},t,T), \mathbf{x} \in V \quad (4.a)$$

with initial and boundary conditions given by,

$$T(\mathbf{x},0) = T_0(\mathbf{x}), \mathbf{x} \in V \quad (4.b)$$

$$\hat{\alpha}(\mathbf{x})T(\mathbf{x},t) + \beta(\mathbf{x})\hat{k}(\mathbf{x})\frac{\partial T}{\partial \mathbf{n}} = \hat{\phi}(\mathbf{x},t,T), \mathbf{x} \in S \quad (4.c)$$

where,

$$\hat{w}(\mathbf{x}) = w(\mathbf{x})F(\mathbf{x}); \hat{k}(\mathbf{x}) = k(\mathbf{x})F(\mathbf{x}); \hat{d}(\mathbf{x}) = d(\mathbf{x})F(\mathbf{x}) \quad (4.d-f)$$

$$\hat{P}(\mathbf{x}, t, T) = P(\mathbf{x}, t, T)F(\mathbf{x}); \hat{\alpha}(\mathbf{x}) = \alpha(\mathbf{x})F(\mathbf{x}); \hat{\phi}(\mathbf{x}, t, T) = \phi(\mathbf{x}, t, T)F(\mathbf{x}) \quad (4.g-i)$$

Consider the following vector calculus identity,

$$\nabla \cdot [F(\mathbf{x})k(\mathbf{x})\nabla T] = F(\mathbf{x})\nabla \cdot [k(\mathbf{x})\nabla T] + k(\mathbf{x})\nabla F \cdot \nabla T \quad (5)$$

In order to incorporate the advective term into the diffusive term after substituting eq. (5) into eq. (4.a), it suffices that the function $F(\mathbf{x})$ satisfies the relation below:

$$k(\mathbf{x})\nabla F = -F(\mathbf{x})\mathbf{u}(\mathbf{x}) \Rightarrow \frac{\nabla F}{F} = \nabla(\ln F) = -\frac{\mathbf{u}(\mathbf{x})}{k(\mathbf{x})} \quad (6)$$

For a non-trivial function $F(\mathbf{x})$, eq. (6) implies $\mathbf{u}(\mathbf{x})/k(\mathbf{x})$ must be a conservative vector field. This assumption can be made by noting that the uncompliant portions of the advective and diffusive terms can be added to the source term $P(\mathbf{x}, t, T)$ without loss of generality. For a class C1 function, $\zeta(\mathbf{x})$, one may write:

$$\frac{\mathbf{u}(\mathbf{x})}{k(\mathbf{x})} = -\nabla\zeta \quad \text{thus, } F(\mathbf{x}) = \exp(\zeta(\mathbf{x})) \quad (7,8)$$

Substituting eq. (5) into eq. (4.a), for a function $F(\mathbf{x})$ in the form described in eq. (8), the convective-diffusive transport model then becomes,

$$\hat{w}(\mathbf{x})\frac{\partial T}{\partial t} = \nabla \cdot [\hat{k}(\mathbf{x})\nabla T] - \hat{d}(\mathbf{x})T(\mathbf{x}, t) + \hat{P}(\mathbf{x}, t, T), \mathbf{x} \in V \quad (9.a)$$

with initial and boundary conditions given by,

$$T(\mathbf{x}, 0) = T_0(\mathbf{x}), \mathbf{x} \in V \quad (9.b)$$

$$\hat{\alpha}(\mathbf{x})T(\mathbf{x}, t) + \beta(\mathbf{x})\hat{k}(\mathbf{x})\frac{\partial T}{\partial \mathbf{n}} = \hat{\phi}(\mathbf{x}, t, T), \mathbf{x} \in S \quad (9.c)$$

Proceeding with the GITT formalism, a Sturm-Liouville eigenvalue problem with convective information embedded in the diffusive term, either in the solution of problems (3) or (9), is obtained as follows:

$$\nabla \cdot [\hat{k}(\mathbf{x})\nabla \psi_i] + [\hat{w}(\mathbf{x})\mu_i^2 - \hat{d}(\mathbf{x})]\psi_i(\mathbf{x}) = 0, \mathbf{x} \in V \quad (10.a)$$

$$\hat{\alpha}(\mathbf{x})\psi_i(\mathbf{x}) + \beta(\mathbf{x})\hat{k}(\mathbf{x})\frac{\partial \psi_i}{\partial \mathbf{n}} = 0, \mathbf{x} \in S \quad (10.b)$$

with normalization and orthogonality property given by,

$$\int_V \hat{w}(\mathbf{x})\psi_i(\mathbf{x})\psi_j(\mathbf{x})dv = \delta_{ij}N_i, \quad N_i = \int_V \hat{w}(\mathbf{x})\psi_i(\mathbf{x})^2 dv \quad (10.c,d)$$

By virtue of the orthogonality property of eq. (10.e), a transform-inverse pair can be obtained in the form,

$$\bar{T}_i(t) = \int_V \hat{w}(\mathbf{x})\psi_i(\mathbf{x})T(\mathbf{x}, t)dv; \quad (11.a)$$

$$T(\mathbf{x}, t) = \sum_{i=1}^{\infty} \frac{1}{N_i}\psi_i(\mathbf{x})\bar{T}_i(t) \quad (11.b)$$

Applying $\int_V \tilde{\psi}_i(\mathbf{x})(\cdot)dv$ to eq. (9.a), the transformed problem below results:

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$$\frac{d\bar{T}_i}{dt} + \mu_i^2 \bar{T}_i(t) = \bar{g}_i(t, \bar{T}_j) \quad (12.a)$$

$$\bar{T}_i(0) = \bar{f}_i = \int_V \hat{w}(\mathbf{x}) \psi_i(\mathbf{x}) T_0(\mathbf{x}) dv \quad (12.b)$$

with integral coefficients given by,

$$\bar{g}_i(t, \bar{T}_j) = \int_S \left[\frac{\psi_i(\mathbf{x}) - \hat{k}(\mathbf{x}) \partial \psi_i / \partial \mathbf{n}}{\hat{\alpha}(\mathbf{x}) + \beta(\mathbf{x})} \right] \hat{\phi}(\mathbf{x}, t, T) ds + \int_V \psi_i(\mathbf{x}) \hat{P}(\mathbf{x}, t, T) dv \quad (12.c)$$

The diffusion problems (3) or (9) incorporated a generalized diffusive operator, thereby conveying relevant information on the convective effects, as specified in the chosen linear convective term coefficients that undergo the exponential transformation, which can provide a desirable convergence enhancement effect in the integral transform solution of the original multidimensional convection-diffusion problem. Nevertheless, the operators coefficients and the source terms are modified by the choice of transformation coefficients, and the effects of such choices on the eigenfunction expansions convergence behavior need investigation, as discussed in the application that follows.

3. APPLICATION

Transient laminar forced convection of a Newtonian fluid in a parallel plates channel (Gondim et al., 2007) is here analyzed, through the energy equation for temperature field, $T(x, y, t)$, with initial and boundary conditions written in dimensionless form as:

$$\frac{\partial T}{\partial t} + U(y) \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2}, \quad x > 0, \quad 0 < y < 1 \quad (13.a)$$

$$T(x, y, 0) = 1, \quad (13.b)$$

$$T(0, y, t) = 0, \quad \left. \frac{\partial T}{\partial x} \right|_{x=L} = 0, \quad \left. \frac{\partial T}{\partial y} \right|_{y=0} = 0, \quad T(x, 1, t) = 0 \quad (13.c-f)$$

$$U(y) = \frac{3Pe}{8} (1 - y^2); \quad Pe = \frac{u_w D_h}{\alpha}; \quad y = \frac{y^*}{h}; \quad x = \frac{x^*}{h}; \quad t = \frac{\alpha t^*}{h^2}; \quad T = \frac{T^* - T_w}{T_0 - T_w}; \quad (13.g-l)$$

where h is half the distance between the plates and the hydraulic diameter is given by $D_h = 4h$. From direct correspondence with the general formulation in eqs.(1), we have:

$$\mathbf{x} = \{x, y\}, \quad w(\mathbf{x}) = 1, \quad \mathbf{u}(\mathbf{x}) = \{U(y), 0\}, \quad k(\mathbf{x}) = 1, \quad P(\mathbf{x}, t, T) = 0, \quad f(\mathbf{x}) = 1, \quad \phi(\mathbf{x}, t, T) = 0 \quad (14)$$

with the boundary conditions coefficients $\alpha(\mathbf{x}), \beta(\mathbf{x})$ being either 0 or 1 accordingly for each boundary surface.

3.1 SOLUTION WITH NON-CLASSIC EIGENVALUE PROBLEM

First, consider the general situation given in eq. (2.a), which leads to a non-classic eigenvalue problem (Knupp et al., 2020). For the application considered in this work, the following coefficients are employed:

$$u_x^* = U(y), \quad u_y^* = 0 \quad (15.a,b)$$

resulting:

$$\hat{k}_x(x, y) = e^{-\int u_x^* dx} = e^{-U(y)x}, \quad \hat{k}_y(x, y) = 1, \quad \hat{w}(x, y) = \hat{k}_x(x, y) \quad (16.a-c)$$

leading to the following generalized diffusive form of the problem considered:

$$\hat{w} \frac{\partial T}{\partial t} = \hat{k}_y \frac{\partial}{\partial x} \left(\hat{k}_x \frac{\partial T}{\partial x} \right) + \hat{k}_x \frac{\partial}{\partial y} \left(\hat{k}_y \frac{\partial T}{\partial y} \right), \quad x > 0, 0 < y < 1 \quad (17.a)$$

Separation of variables is then applied to problem (17.a) and boundary conditions given by eqs. (13.c-f), with $T(x, y) = \Gamma(t)\varphi(x, y)$, yielding the following problem for $\Gamma(t)$:

$$\frac{d\Gamma(t)}{dt} + \beta^2 \Gamma(t) = 0, \quad \text{providing } \Gamma(t) = A e^{-\beta^2 t} \quad (18.a,b)$$

and the non-classic eigenvalue problem for $\varphi(x, y)$:

$$\hat{k}_y \frac{\partial}{\partial x} \left(\hat{k}_x \frac{\partial \varphi}{\partial x} \right) + \hat{k}_x \frac{\partial}{\partial y} \left(\hat{k}_y \frac{\partial \varphi}{\partial y} \right) + \beta^2 \hat{w} \varphi = 0 \quad (19.a)$$

$$\varphi(0, y) = 0, \quad \left. \frac{\partial \varphi}{\partial x} \right|_{x=L} = 0, \quad \left. \frac{\partial \varphi}{\partial y} \right|_{y=0} = 0, \quad \varphi(x, 1) = 0 \quad (19.b-e)$$

The eigenvalue problem given by eqs. (19.a-e) do not follow the same orthogonality property as for the classical Sturm-Liouville problem. Also, the corresponding eigenvalues spectrum is not known *a priori* and eventually complex quantities may be present. At this point, assuming this eigenvalue problem originates a complete set of eigenfunctions, the sought solution can be written as:

$$T(x, y, t) = \sum_{i=1}^{\infty} A_i \varphi_i(x, y) e^{-\beta_i^2 t} \quad (20)$$

where the expansion coefficients A_i must be determined from the initial condition. Hence, operating on eq. (20) with

$\int_0^1 \int_0^L \hat{w}(x, y) \varphi_j(x, y) (\cdot) dx dy$ at $t = 0$ yields the following linear algebraic system:

$$\int_0^1 \int_0^L \hat{w}(x, y) \varphi_j(x, y) dx dy = \sum_{i=1}^{\infty} A_i \int_0^1 \int_0^L \hat{w}(x, y) \varphi_i(x, y) \varphi_j(x, y) dx dy, \quad i = 1, 2, 3, \dots, \quad j = 1, 2, 3, \dots \quad (21)$$

which, after truncated to a finite order N , can be solved for the coefficients A_i , for $i = 1, 2, 3, \dots, N$, and the expansion given by Eq. (20) can be readily used to calculate the dimensionless temperature T at any position (x, y) and time t .

The derived non-classic eigenvalue problem does not allow for a closed form solution, but the GITT can be employed to achieve an analytical-based solution. Consider a simpler basis, given by the following classical Sturm-Liouville eigenvalue problem

$$\nabla^2 \Omega_n + \lambda_n^2 \Omega_n(x, y) = 0, \quad (x, y) \in V \quad (22.a)$$

$$\Omega_n(0, y) = 0, \quad \left. \frac{\partial \Omega_n(x, y)}{\partial x} \right|_{x=L} = 0, \quad \left. \frac{\partial \Omega_n(x, y)}{\partial y} \right|_{y=0} = 0, \quad \Omega_n(x, 1) = 0 \quad (22.b-e)$$

Also, consider the following integral transformation pair,

$$\varphi_i(x, y) = \sum_{m=0}^{\infty} \tilde{\Omega}_m(x, y) \bar{\varphi}_{i,m}, \quad \text{inverse} \quad (23.a)$$

$$\bar{\varphi}_{i,m} = \int_0^1 \int_0^L \varphi_i(x, y) \tilde{\Omega}_m(x, y) dx dy, \quad \text{transform} \quad (23.b)$$

Operating on eq. (19.a) with $\int_0^1 \int_0^L \Omega_j(x, y) (\cdot) dx dy$ and making use of the inversion formula lead to the following algebraic problem:

$$(\mathbf{C} - \beta^2 \mathbf{D}) \bar{\boldsymbol{\varphi}} = 0 \quad (24.a)$$

with:

$$C_{ij} = \int_0^L \int_0^L \hat{k}_y \frac{\partial}{\partial x} \left(\hat{k}_x \frac{\partial \tilde{\Omega}_i}{\partial x} \right) \tilde{\Omega}_j dx dy + \int_0^L \int_0^L \hat{k}_x \frac{\partial}{\partial y} \left(\hat{k}_y \frac{\partial \tilde{\Omega}_i}{\partial y} \right) \tilde{\Omega}_j dx dy, \quad D_{ij} = - \int_0^L \int_0^L \hat{w} \tilde{\Omega}_i \tilde{\Omega}_j dx dy \quad (24.b,c)$$

The algebraic problem (24.a) can be numerically solved to provide results for the corresponding eigenvalues β^2 and eigenvectors $\bar{\boldsymbol{\varphi}}$, upon truncation to a sufficiently large finite order N_I . The eigenvectors $\bar{\boldsymbol{\varphi}}$ form the expansion coefficients for the eigenfunctions $\varphi(x, y)$.

3.2 SOLUTION WITH CLASSIC EIGENVALUE PROBLEM

According to eq. (7), the function $\zeta(\mathbf{x})$ should satisfy:

$$-\frac{\partial \zeta}{\partial x} = U(y), \quad -\frac{\partial \zeta}{\partial y} = 0 \quad (25.a,b)$$

However, these two conditions cannot be satisfied simultaneously, but eq.(13.a) can be rewritten so as to split the convective term into a compliant portion and merging the remaining portion in the source term, in the form:

$$\frac{\partial T}{\partial t} + U^* \frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} + P(x, y, t), \quad x > 0, \quad 0 < y < 1 \quad (26.a)$$

$$P(x, y, T) = (U^* - U(y)) \frac{\partial T}{\partial y} \quad (26.b)$$

where U^* is a chosen constant reference velocity. Then,

$$\zeta(x, y) = -U^* x \quad \text{and} \quad F(x, y) = e^{-U^* x} \quad (27.a,b)$$

and the condition (14.b) is readily satisfied. The corresponding self-adjoint eigenvalue problem, eqs. (10.a,b), is then defined by the coefficients:

$$\hat{k}(\mathbf{x}) = \hat{w}(\mathbf{x}) = e^{-U^* x}, \quad \hat{d}(\mathbf{x}) = 0 \quad (28.a-c)$$

while the transformed system, eqs. (12.a-c), is defined by the coefficients

$$\bar{f}_i = \int_V \hat{w}(\mathbf{x}) \psi_i(\mathbf{x}) dv, \quad \text{since} \quad T_0(\mathbf{x}) = 1 \quad (28.d,e)$$

$$\bar{g}_i(t, \bar{T}_j) = \int_V \psi_i(\mathbf{x}) \hat{P}(\mathbf{x}, t, T) dv, \quad \text{where} \quad \hat{P}(\mathbf{x}, t, T) = e^{-U^* x} P(x, y, T) \quad (28.f,g)$$

Although the resulting self-adjoint eigenvalue problem, eqs. (10), with the coefficients defined in eqs. (28.a-c), has an exact analytical solution in this particular case, a more general approach for solving eigenvalue problems with arbitrarily variable coefficients, based on the GITT itself [Cotta, 1993; Cotta et al., 2016], is here implemented aiming at demonstrating the flexibility of the present approach. The idea is to express the original eigenfunctions as an eigenfunction expansion based on a simpler auxiliary eigenvalue problem of known analytical solution, as in the previous section. Then, through integral transformation over problem (10), an algebraic eigenvalue problem is obtained and numerically solved to provide the eigenvalues and corresponding eigenvectors that composed the expansion for the desired eigenfunctions. The auxiliary eigenvalue problem here considered is the same given by eqs. (22.a-e). The proposed expansion for the original eigenfunctions is given through the integral transform pair below:

$$\psi_i(x, y) = \sum_{n=1}^{\infty} \tilde{\Omega}_n(x, y) \bar{\psi}_{i,n}, \quad \text{inverse} \quad (29.a)$$

$$\bar{\psi}_{i,n} = \int_V \tilde{\Omega}_n(x, y) \psi_i(x, y) dv, \quad \text{transform} \quad (29.b)$$

Equation (10.a) is then integral transformed through the operator $\int_V \tilde{\Omega}_n(x, y)(-)\,dv$ to yield the algebraic eigenvalue problem as (Cotta et al., 2016):

$$(\mathbf{A} + \mathbf{C})\{\bar{\boldsymbol{\psi}}\} = \mu^2 \mathbf{B}\{\bar{\boldsymbol{\psi}}\} \quad (30.a)$$

with the elements of the matrices given by:

$$A_{nm} = \int_V \left(\hat{k}(\mathbf{x}) - 1 \right) \nabla \tilde{\Omega}_n(\mathbf{x}) \cdot \nabla \tilde{\Omega}_m(\mathbf{x}) \, dv, \quad C_{nm} = \lambda_n^2 \delta_{nm}, \quad B_{nm} = \int_V \hat{w}(\mathbf{x}) \tilde{\Omega}_n(\mathbf{x}) \tilde{\Omega}_m(\mathbf{x}) \, dv \quad (30.b-d)$$

Numerical solution by standard routines for algebraic eigenvalue problems such as eq. (30.a) provides the eigenvalues and eigenvectors that will compose the inverse formula, eq.(20.a), to compute the required eigenfunctions. Also, the transformed ODE system of eqs. (12) becomes:

$$\frac{d\bar{T}_i}{dt} + \sum_{j=1}^{\infty} E_{i,j} \bar{T}_j(t) = 0 \quad (31.a)$$

$$\bar{T}_i(0) = \bar{f}_i \quad (31.b)$$

with the transformed initial conditions, norms, and the matrix \mathbf{E} coefficients are computed from:

$$\bar{f}_i = \sum_{n=1}^{\infty} \int_V \hat{w}(x) \tilde{\Omega}_n(x, y) \, dv \bar{\psi}_{i,n} \quad (31.c)$$

$$N_i = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_V \hat{w}(x) \tilde{\Omega}_n(x, y) \tilde{\Omega}_m(x, y) \, dv \bar{\psi}_{i,n} \bar{\psi}_{i,m} \quad (31.d)$$

$$E_{i,j} = \frac{1}{N_j} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int_V \hat{w}(x) (U^* - U(y)) \tilde{\Omega}_n(x, y) \frac{\partial \tilde{\Omega}_m(x, y)}{\partial y} \, dv \bar{\psi}_{i,n} \bar{\psi}_{j,m} \quad (31.e)$$

where all the above integrals are obtained analytically. System (31) can be readily solved by the appropriate matrix eigenvalue analysis or more compactly through the matrix exponential function, in the form:

$$\bar{\mathbf{T}} = \exp[-t \mathbf{E}] \bar{\mathbf{f}} \quad (32)$$

and the inverse formula, eq. (11.b), provides the final temperature field.

4. RESULTS AND DISCUSSION

Problem (13) has been solved through the GITT for four different choices of convective-diffusive eigenvalue problem. First, the generalized diffusion formulation of eq.(2.a) was implemented, with $\{u_x, u_y, u_z\} = \{U(y), 0, 0\}$, as described in Section 3.1. The alternative solution path, partially taking into account the advective term into the eigenfunction basis, as described in Section 3.2, was also considered and the formulation in eq.(26.a) was solved with the different choices of $U^* = 0, U_{av}$, and U_{max} . The choice of $U^* = 0$ ($U(y)$ at $y=1$) corresponds to a purely diffusive eigenvalue problem, while the choices of $U^* = U_{av}$, and U_{max} correspond, respectively, to the average and maximum (at $y=0$) values of $U(y)$. The influence of the Peclet number was investigated through three different values, $Pe=12.5, 25$, and 50 , covering a range where both the convective and diffusive effects in the axial direction are simultaneously important.

Table 1 first illustrates the convergence behaviour for the eigenvalue problem (10) when solved by GITT itself, and critically compared to the exact solution obtained by separation of variables. The first ten eigenvalues μ_i 's are shown for truncation orders $N_T=50, 100, 150, 200$, and 250 . Through direct comparison to the exact solution in the last column, for $Pe=12.5$, one can see that the truncated solution obtained through the matrix eigenvalue problem (30) is fully converged to at least four significant digits at truncation orders as low as 150 and to the fifth digit at truncation order of 200. In the remaining calculations, the truncation order was thus fixed at $N_T=150$, which provides sufficiently accurate results for converging the temperature field to within four significant digits.

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Table 1. Convergence behaviour of the first ten eigenvalues, μ_i of problem (10) for different truncation orders N_I ($Pe=12.5, U^*=U_{av}$)

i	N_I	50	100	150	200	250	Exact
1		3.11567	3.11561	3.11559	3.11558	3.11557	3.11556
2		5.42647	5.42644	5.42642	5.42642	5.42642	5.42641
3		5.48261	5.48227	5.48220	5.48217	5.48215	5.48211
4		7.05678	7.05652	7.05647	7.05644	7.05643	7.05640
5		8.30211	8.30209	8.30208	8.30208	8.30207	8.30207
6		8.34605	8.34564	8.34543	8.34538	8.34535	8.34528
7		9.44863	9.44844	9.44839	9.44838	9.44837	9.44834
8		9.45493	9.45457	9.45438	9.45434	9.45431	9.45425
9		11.3200	11.3200	11.3200	11.3200	11.3200	11.3200
10		11.3523	11.3520	11.3518	11.3518	11.3518	11.3517

Tables 2.a-c below provide an illustration of the convergence behavior for the cases of $Pe=12.5, 25,$ and $50,$ respectively, at $t=0.02$ and $x=0.2,$ comparing the temperature $T(x, y, t)$ at selected y positions as obtained from the expansions based on a purely diffusive eigenvalue problem ($U^*=0$), a convective eigenvalue problem with $U^*=U_{max},$ a convective eigenvalue problem with $U^*=U_{av},$ and the convective eigenvalue problem with $U^*=U(y)$ from the formulation in eq.(2.a). Also shown are the fully converged GITT results for the diffusive eigenvalue problem choice with a large enough truncation order ($N=350$), which is already fully converged to the four digits shown. In addition, for comparison purposes, the purely numerical solution obtained with the Method of Lines implemented in the NDSolve routine of the *Mathematica* v.12 system (Wolfram, 2019), with maximum spatial step of 0.025, is presented in the last line of Tables 1. It is evident from the analysis of Tables 2 that the convective base considering the full velocity profile, $U(y),$ presents the best overall convergence, with four significant digits being fully converged much earlier than the other three choices. This is a clear verification of the importance of the transformed source term in delaying convergence, which remains equal to zero in the non-classic formulation, while new source terms are produced by the portion of the convective term not accounted for in the base. The smallest overall deviation of the three sets of eigenvalue problems with constant reference velocities ($U^*=0, U_{av},$ and U_{max}), with respect to the benchmark fully converged results, is achieved by the solution with the diffusive eigenvalue problem, followed by the solution employing the average dimensionless velocity as a reference value for the coefficients transformation, as observed from the relative deviations presented. It can also be observed that convergence is faster close to $y=0$ for the choice with $U^*=U_{max}$ and slower close to $y=1,$ while for $U^*=U_{av},$ the convergence is more uniform at the three y positions selected.

Table 2.a. Convergence behaviour of the dimensionless temperature for different choices of eigenvalue problem base ($Pe=12.5, t=0.02, x=0.2,$ and $N_I=150$)

N	Diffusive eig. problem $U^*=0$			Convective eig. problem $U^*=U_{max}$			Convective eig. problem $U^*=U_{av}$			Convective eig. problem $U^*=U(y)$		
	$y=0$	$y=0.6$	$y=0.8$	$y=0$	$y=0.6$	$y=0.8$	$y=0$	$y=0.6$	$y=0.8$	$y=0$	$y=0.6$	$y=0.8$
10	0.5383	0.5646	0.4135	0.5133	0.54	0.393	0.5216	0.5479	0.3993	0.5191	0.5539	0.4091
20	0.5175	0.5511	0.4188	0.5225	0.5599	0.4258	0.5201	0.5561	0.4228	0.5237	0.5560	0.4217
30	0.5194	0.552	0.4198	0.5234	0.5579	0.4253	0.5215	0.5553	0.423	0.5239	0.5558	0.4216
40	0.5218	0.5539	0.4205	0.5234	0.5563	0.423	0.5226	0.5552	0.4220	0.5239	0.5558	0.4216
50	0.5236	0.5553	0.4211	0.5239	0.5551	0.4217	0.5238	0.5552	0.4215	0.5239	0.5558	0.4216
60	0.5245	0.556	0.4214	0.5237	0.5546	0.4204	0.5241	0.5552	0.4209	0.5239	0.5558	0.4216
70	0.5245	0.556	0.4213	0.5235	0.5547	0.4201	0.5240	0.5553	0.4206			
80	0.5247	0.5562	0.4214	0.5234	0.5545	0.4198	0.5240	0.5552	0.4205			
90	0.5245	0.556	0.4213	0.5235	0.5546	0.4199	0.5239	0.5553	0.4206			
100	0.5245	0.556	0.4213	0.5235	0.5546	0.4200	0.5240	0.5553	0.4206			
110	0.5242	0.5558	0.4212	0.5235	0.5549	0.4204	0.5238	0.5553	0.4208			
120	0.5241	0.5558	0.4212	0.5235	0.5549	0.4204	0.5238	0.5553	0.4208			
350 Diff.	0.5240	0.5556	0.4211	0.5240	0.5556	0.4211	0.5240	0.5556	0.4211	0.5240	0.5556	0.4211
% deviat.	0.028	0.032	0.026	-0.101	-0.129	-0.162	-0.040	-0.054	-0.074	-0.022	0.043	0.109
NDSolve*	0.5249	0.5562	0.4221	0.5249	0.5562	0.4221	0.5249	0.5562	0.4221	0.5249	0.5562	0.4221

Table 2.b. Convergence behaviour of the dimensionless temperature for different choices of eigenvalue problem base
($Pe = 25$, $t=0.02$, $x = 0.2$, and $N_I=150$)

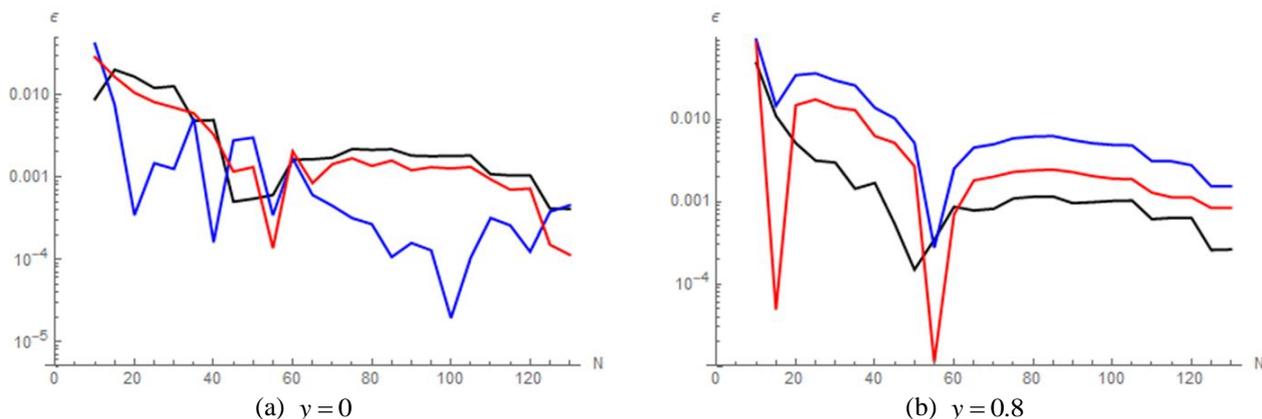
N	Diffusive eig. problem $U^*=0$			Convective eig. problem $U^*=U_{max}$			Convective eig. problem $U^*=U_{av}$			Convective eig. problem $U^*=U(y)$		
	y=0	y=0.6	y=0.8	y=0	y=0.6	y=0.8	y=0	y=0.6	y=0.8	y=0	y=0.6	y=0.8
20	0.3527	0.4471	0.3697	0.3559	0.462	0.3845	0.3528	0.4538	0.3766	0.3580	0.4527	0.3732
30	0.3546	0.4482	0.3710	0.3575	0.4576	0.3831	0.3552	0.4522	0.3769	0.3579	0.4525	0.3732
40	0.3567	0.4507	0.3721	0.3576	0.4542	0.3777	0.3567	0.4521	0.3748	0.3579	0.4525	0.3731
50	0.3582	0.4525	0.3729	0.3590	0.4520	0.3746	0.3585	0.4521	0.3737	0.3579	0.4525	0.3731
60	0.359	0.4534	0.3733	0.3586	0.4509	0.3716	0.3588	0.4521	0.3725	0.3579	0.4525	0.3731
70	0.359	0.4534	0.3733	0.3581	0.4512	0.3706	0.3585	0.4523	0.3719			
80	0.3592	0.4536	0.3734	0.3578	0.4507	0.3701	0.3585	0.4522	0.3717			
90	0.359	0.4534	0.3733	0.3578	0.4512	0.3703	0.3585	0.4523	0.3718			
100	0.359	0.4534	0.3733	0.3579	0.4511	0.3706	0.3585	0.4523	0.3719			
110	0.3588	0.4531	0.3731	0.3580	0.4518	0.3714	0.3583	0.4524	0.3722			
120	0.3587	0.4531	0.3732	0.3579	0.4516	0.3715	0.3583	0.4523	0.3723			
130	0.3585	0.4528	0.3730	0.3577	0.4523	0.3721	0.3580	0.4524	0.3724			
140	0.3585	0.4528	0.3730	0.3578	0.4522	0.3726	0.3580	0.4523	0.3726			
150	0.3583	0.4526	0.3729	0.3579	0.4523	0.3728	0.3580	0.4524	0.3728			
350 Diff.	0.3585	0.4529	0.3731	0.3585	0.4529	0.3731	0.3585	0.4529	0.3731	0.3585	0.4529	0.3731
% deviat.	-0.057	-0.066	-0.063	-0.164	-0.122	-0.076	-0.151	-0.119	-0.079	-0.163	-0.093	0.022
NDSolve*	0.3605	0.4538	0.3741	0.3605	0.4538	0.3741	0.3605	0.4538	0.3741	0.3605	0.4538	0.3741

Table 2.c. Convergence behaviour of the dimensionless temperature for different choices of eigenvalue problem base
($Pe = 50$, $t=0.02$, $x = 0.2$, and $N_I=200$)

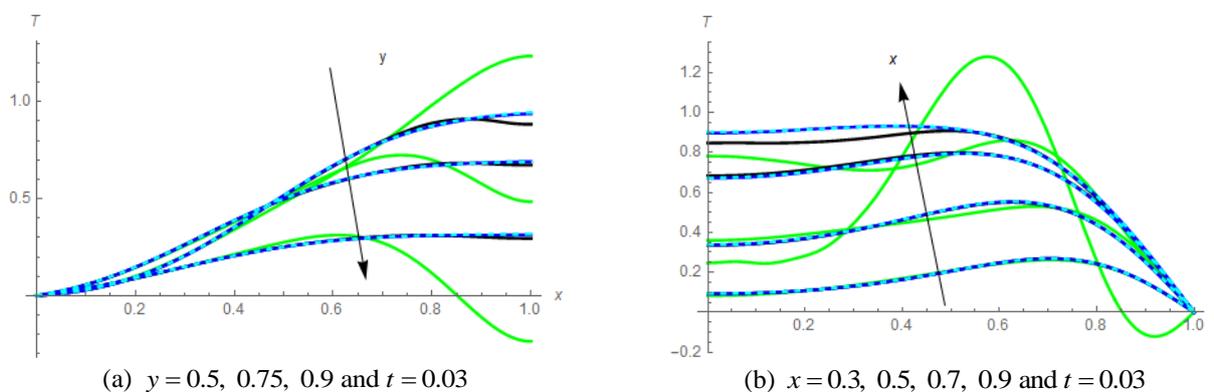
N	Diffusive eig. problem $U^*=0$			Convective eig. problem $U^*=U_{max}$			Convective eig. problem $U^*=U_{av}$			Convective eig. problem $U^*=U(y)$		
	y=0	y=0.6	y=0.8	y=0	y=0.6	y=0.8	y=0	y=0.6	y=0.8	y=0	y=0.6	y=0.8
20	0.1094	0.2562	0.2725	0.1063	0.2770	0.3049	0.1056	0.2623	0.2836	0.1101	0.2602	0.2754
30	0.1099	0.2565	0.2732	0.1092	0.2680	0.3010	0.1085	0.2591	0.2840	0.1106	0.2599	0.2751
40	0.1105	0.2585	0.2741	0.1094	0.2619	0.2881	0.1096	0.2590	0.2798	0.1106	0.2598	0.2751
50	0.1106	0.2597	0.2745	0.1135	0.2583	0.2808	0.1116	0.2592	0.2775	0.1106	0.2598	0.2751
60	0.1108	0.2602	0.2750	0.1124	0.2569	0.2729	0.1115	0.2594	0.2748	0.1106	0.2598	0.2751
70	0.1110	0.2601	0.2751	0.1109	0.2578	0.2699	0.1110	0.2597	0.2737			
80	0.1109	0.2603	0.2752	0.1101	0.2571	0.2689	0.1106	0.2596	0.2734			
90	0.1109	0.2601	0.2752	0.1102	0.2582	0.2695	0.1107	0.2598	0.2735			
100	0.1109	0.2602	0.2752	0.1104	0.2579	0.2705	0.1107	0.2597	0.2739			
110	0.1108	0.2599	0.2750	0.1109	0.2593	0.2728	0.1108	0.2599	0.2745			
120	0.1108	0.2600	0.2750	0.1103	0.2590	0.2732	0.1106	0.2598	0.2746			
130	0.1108	0.2597	0.2749	0.1099	0.2602	0.2743	0.1104	0.2600	0.2749			
140	0.1108	0.2597	0.2749	0.1106	0.2596	0.2756	0.1106	0.2598	0.2753			
150	0.1107	0.2596	0.2748	0.1101	0.2606	0.2762	0.1104	0.2599	0.2754			
350 Diff.	0.1107	0.2597	0.2749	0.1107	0.2597	0.2749	0.1107	0.2597	0.2749	0.1107	0.2597	0.2749
% deviat.	0.0413	-0.040	-0.052	-0.568	0.329	0.481	-0.271	0.077	0.182	-0.049	0.045	0.063
NDSolve*	0.1130	0.2607	0.2759	0.1130	0.2607	0.2759	0.1130	0.2607	0.2759	0.1130	0.2607	0.2759

Figures 1.a,b illustrate the evolution of the absolute errors of three alternative solutions ($U^*=0$, U_{av} , and U_{max}) for $Pe = 25$, $t=0.01$ and $x = 0.2$, taking the solution with $N=150$ as the reference result, respectively for $y=0$, at the center of the channel, and for $y=0.8$, closer to the channel wall. It can be concluded that the convective eigenvalue problem choice with $U^*=U_{max}$ (blue curve) presents an improved convergence behavior at the channel center, where the flow velocity assumes the maximum velocity that is adopted as reference velocity in the problem transformation. On the other hand, close to the channel wall, where the velocities approach zero, the diffusive eigenvalue problem choice (black curve) still provides the overall smallest absolute errors out of the three expansion base choices. Thus, best convergence among these choices is achieved with $U^*=U(y)$ at any position y , a feasible parametric implementation when analytical solutions for the eigenvalue problems are available.

Figures 2.a,b present a graphical illustration of the excellent convergence behavior achieved when employing the non-classic convective eigenvalue problem with $U^*=U(y)$ for $Pe = 50$, by presenting some temperature profiles along x and y , respectively, for the truncation orders $N = 10$, 15 and 20 , in comparison with the benchmark results. One may first observe a remarkable increase in the solution quality from $N = 10$ to $N = 15$ terms. It is worth noting that with $N = 15$ terms the solution is adherent to the benchmark with the exception of lower values of y and higher values of x . With only $N = 20$ terms the solution is fully adherent to the benchmark results throughout.

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Figures 1.a,b – Absolute error evolution for $Pe=25$, $x=0.2$ and $t=0.01$: Diffusive eigenvalue problem (black), convective eigenvalue problem with $U^*=U_{max}$ (blue), convective eigenvalue problem with $U^*=U_{av}$ (red)



Figures 2.a,b – Temperature profiles along x and y for $Pe = 50$ employing the convective eigenvalue problem with $U^*=U(y)$ for $N = 10$ (green), $N = 15$ (black) and $N = 20$ (cyan), and the benchmark results (dashed blue).

5. ACKNOWLEDGEMENTS

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6. REFERENCES

- Cotta, R.M., 1990. "Hybrid numerical-analytical approach to nonlinear diffusion problems", *Num. Heat Transfer, Part B*, Vol. 127, pp. 217-226.
- Cotta, R.M., 1993. *Integral Transforms in Computational Heat and Fluid Flow*, CRC Press, Boca Raton, USA.
- Cotta, R.M., Knupp, D.C., Naveira-Cotta, C.P., 2016. *Analytical Heat and Fluid Flow in Microchannels and Microsystems*, Mechanical Eng. Series, Springer-Verlag.
- Cotta, R.M., Naveira-Cotta, C.P., Knupp, D.C., 2017. "Convective eigenvalue problems for convergence enhancement of eigenfunction expansions in convection-diffusion problems", *ASME J. Thermal Science and Eng. Appl.*, Vol.10, n.2, pp.021009.
- Knupp, D.C., Cotta, R.M., and Naveira-Cotta, C.P., 2020. "Conjugated heat transfer analysis via integral transforms and eigenvalue problems", *J. Eng. Physics & Thermophysics*, Vol.93, no.1, pp.65-77.
- Gondim, R.R., Macedo, E.N., Cotta, R.M., 2007. "Hybrid solution for transient internal convection with axial diffusion: integral transforms with local instantaneous filtering", *Int. J. Num. Meth. Heat & Fluid Flow*, Vol.17, pp.405-417.
- Ozisk, M.N., Orlande, H.R.B., Colaço, M.J., and Cotta, R.M., 2017, *Finite Difference Methods in Heat Transfer*, 2nd Edition, CRC Press, Boca Raton, FL.
- Wolfram, S., 2019. *Mathematica v.12 – A system for doing mathematics by computer*, Wolfram Research Inc., IL.

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