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SOME EXACT SOLUTIONS FOR A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS ARISING FROM THE MATHEMATICAL DESCRIPTION OF POROUS FINS

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Abstract. *This article presents analytical solutions for a rectangular profile porous fin. These solutions are proved to be unique and are associated with a convex functional. The original fin is subjected to natural convection and thermal radiation. Closed analytical solutions for two limiting cases, namely: 1) no thermal radiation and 2) the fin confined in an atmosphere-free space (no convection), are compared with results obtained numerically via minimization of a functional that accounts for both natural convection and thermal radiation simultaneously, showing excellent agreement. These easily obtained solutions provide an exceptional approximation for porous fins subjected to natural convection and thermal radiation.*

Keywords: *Nonlinear Ordinary Differential Equations, porous fins, exact solutions.*

1. INTRODUCTION

Fins and, particularly, porous fins interest many researchers due to performance and heat transfer optimization issues, as in Poulikakos and Bejan (1982). They sought to optimize geometric parameters to minimize the entropy generation in forced convection, while Popiel and Blanch (2007) analyzed the efficiency of horizontal fins in a natural convection environment with radiation heat exchange. Mueller and Abu-Mulaweh (2006), similarly, proposed a methodology for predicting the temperature distribution in a fin and a natural convection environment with radiation. Kiwan and Al-Nmir (2001) showed that the use of porous fins could considerably enhance heat transfer. The thermal performance of porous fins with distinct profiles was considered by Ma et al. (2017), who considered conductive, convective and radiative heat transfer, and accounted for the temperature-dependency in trapezoidal, convex parabolic, and concave parabolic profiles porous fins.

Gorla and Bakier (2011), Kiwan (2007), as well as Maheria (2010), used the concept of a volumetric fraction to study the temperature distribution along a rectangular porous fin, simplifying the momentum equation with Darcy's model. Natural convection could also be modeled using a Darcy-Forchheimer approach, instead of a Darcy law employed by Gorla and Bakier (2011) and in this work. Asl et al (2019), studied an inclined rectangular enclosure with porous fins attached to the hot wall, subjected to natural convection. The energy equation, in turn, was simplified through the hypothesis of local thermal equilibrium of the phases in addition to the hypothesis that the temperature variation occurred only along the length of the fin. Kiwan (2007) developed the basis of the scaling methodology used by Gorla and Bakier (2011) and Maheria (2010). The scaling methodology created by Kiwan (2007) allowed the analysis of different geometric and operational parameters. A porous dimensionless parameter S_H, accounted for the effect of Darcy and Rayleigh numbers, thermal conductivity, and fin thickness, allowing the analysis of the impact of these parameters variation in the temperature distribution and, consequently, in the heat transfer. Gorla and Bakier

(2011) expanded the formula proposed by Kiwan (2007) to include the effects of radiation heat transfer by inserting the geometric and radiation properties in the dimensionless parameter *G*, thus finding the impact of thermal radiation. Maheria (2010) used the same methodology as Gorla and Bakier (2011), obtaining similar data.

This work considers the physical idea used by Gorla and Bakier (2011), namely a rectangular profile porous fin, though which the fluid flows. A very long fin subjected to natural convection and radiation is considered, and the same scaled parameters are employed. Exact solutions for both absence of radiation heat transfer and the fin confined in an atmosphere-free space are proposed. A generalized functional to both these exact solutions is also proposed. Results are compared with those obtained by minimizing a convex functional presented via a Conjugate Gradient method (Martins-Costa et al., 2020), which, in turn, are compared with those by Gorla and Bakier (2011), showing good agreement.

2. MECHANICAL MODEL

Consider a rectangular profile porous fin (constant cross-section area, length *L*, width *W*, thickness *t*), through which the fluid flows (Gorla and Bakier, 2011) according to Darcy's law, in which the porous fin is supposed homogeneous, isotropic saturated and in thermal equilibrium with the flowing fluid. The radiant surface exchange is neglected, and the temperature is assumed to vary along the fin length only, so the steady-state the scaled energy balance equation (Gorla and Bakier, 2011) is

$$\frac{d^2\theta}{dX^2} - S_H\,\theta^2 - G\left[\left(\theta + C_T\right)^4 - C_T^4\right] = 0$$
$$\theta = \frac{T - T_\infty}{T_b - T_\infty} \quad X = \frac{x}{L} \quad S_H = \frac{Da \times Ra}{k_r}\left(\frac{L}{t}\right)^2 \tag{1}$$
$$G = \frac{2\sigma\varepsilon}{k_{eff}\,t}L^2\left(T_b - T_\infty\right)^3 \quad C_T = \frac{T_\infty}{T_b - T_\infty}$$

where T_b and T_∞ are the temperature at the fin base and the environment temperature, respectively and θ is the scaled temperature. X is the dimensionless axial coordinate. The porous medium effective thermal conductivity is given by $k_{eff} = \left(1-\varphi\right)k_s + \varphi k_f$, where φ is the fin porosity, k_f is the fluid thermal conductivity, and k_s is the porous matrix thermal conductivity, besides $k_r = k_{eff} / k_f$ represents the conductivity ratio. This allows defining the porous parameter S_H which accounts for both permeability and buoyancy effects, with the help of Darcy and Rayleigh numbers, given by

$$Da = \frac{K}{t^2} \quad Ra = \frac{g\beta}{k_f\nu}\left(T_b - T_\infty\right)t^3 \tag{2}$$

where K is the porous matrix specific permeability, β is the thermal expansion coefficient, g is the gravitational acceleration, and ν is the kinematic viscosity.

Using the Stefan-Boltzmann constant σ, the porous fin emissivity ε and other previously defined variables, a radiation parameter G, that accounts for the effect of the fin surface emissivity, is defined. Finally, C_T is the temperature ratio.

It can be noted that Eq. (1) is a differential equation that accounts for both natural convection and radiation heat transfer. In the absence of radiation heat transfer, $G = 0$, and Eq. (1) reduces to

$$\frac{d^2\theta}{dX^2} - S_H\,\theta^2 = 0, \quad \theta\left(0\right) = 1, \quad \theta\left(\infty\right) = 0 \tag{3}$$

On the other hand, if the fin is confined in an atmosphere-free space, there no convection, so C_T and S_H are zero, giving rise to

$$\frac{d^2\theta}{d\,X^2} - G\theta^4 = 0, \quad \theta(0) = 1, \quad \theta(\infty) = 0 \tag{4}$$

3. ANALYTICAL SOLUTIONS

The first step is to consider the following class of ordinary differential equations

$$f^{(n)}(x) = C\{f(x)\}^{\beta} \tag{5}$$

When $\beta = 1$, the function $f(x) = K\exp(\alpha x)$ satisfies the equation for some α. When $\beta \neq 1$, it comes that $f(x) = K(x + x_0)^{\lambda}$, in which x_0 is a constant, satisfies Eq. (5) for any x_0 and for some λ.

Now, supposing that $\beta \neq 1$ and that $f(0) = 1$. So,

$$f(x) = K(x + x_0)^{\lambda} = (\alpha x + 1)^{\lambda} \tag{6}$$

The problem arising from the description of a porous fin involves a function (the temperature) which must be bounded for any $x \geq 0$. Therefore $\lambda < 0$ and $\alpha > 0$.
In consequence,

$$\begin{aligned} &\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (\alpha x + 1)^{\lambda} = 0 \\ &f(0) = (\alpha 0 + 1)^{\lambda} = 1 \end{aligned} \tag{7}$$

The mathematical description of the heat transfer process in a porous fin, considering convection only, is represented by the following problem,

$$\frac{d^2\theta}{dX^2} - (\mathrm{S}_H)\theta^2 = 0 \quad \theta(0) = 1, \theta(\infty) = 0 \tag{8}$$

where S_H is a positive constant. So, the solution θ is given by

$$\theta = (\alpha X + 1)^{\lambda} \tag{9}$$

In order to calculate the constants α and λ, the solution stated at Eq. Eq. (9) is inserted into the differential equation (8). In other words,

$$\frac{d^2}{dX^2}(\alpha X + 1)^{\lambda} - (\mathrm{S}_H)(\alpha X + 1)^{2\lambda} = 0 \tag{10}$$

After some calculations, it comes that

$$\lambda(\lambda - 1)\alpha^2(\alpha X + 1)^{\lambda-2} = (\mathrm{S}_H)(\alpha X + 1)^{2\lambda} \tag{11}$$

This implies the following

$$2\lambda = \lambda - 2 \quad \Rightarrow \quad \lambda = -2$$
$$-2(-3)\alpha^2 = (S_H) \quad \Rightarrow \quad \alpha = \sqrt{\frac{S_H}{6}} \tag{12}$$

Hence, the solution is given by

$$\theta = \left(\sqrt{\frac{S_H}{6}} X + 1 \right)^{-2} = \left(\frac{\sqrt{6}}{\left(\sqrt{S_H}\right) X + \sqrt{6}} \right)^2 \tag{13}$$

It is easy to see that

$$\lim_{X\to\infty} \theta = \lim_{X\to\infty} \left(\frac{\sqrt{6}}{\left(\sqrt{\mathrm{Sh}}\right) X + \sqrt{6}} \right)^2 = 0$$
$$\theta\Big|_{X=0} = \left(\frac{\sqrt{6}}{\left(\sqrt{\mathrm{Sh}}\right)(0) + \sqrt{6}} \right)^2 = 1 \tag{14}$$

ensuring that the obtained θ is a solution of our problem.

In addition, it is easy to conclude that

$$\lim_{X\to\infty} \theta' = \lim_{X\to\infty} \left(-\frac{12\left(\sqrt{S_H}\right)}{\left(\left(\sqrt{S_H}\right) X + \sqrt{6}\right)^3} \right) = 0 \tag{15}$$

At this point, a question arises: is this solution unique? In order to answer this question, it is essential to notice that we are looking for nonnegative solutions. In other words, if the temperature of the base of the fin is higher than the environment temperature, the entire fin should be at a temperature level higher than the environment temperature. In this way, looking for nonnegative solutions, the problem may be rewritten as follows

$$\frac{d^2\theta}{dX^2} - (S_H)|\theta|\theta = 0 \quad \theta(0) = 1, \theta(\infty) = 0 \tag{16}$$

The problem introduced in Eq. (16) is equivalent to the minimization of the following functional

$$I[\omega] = \int_0^\infty \left\{ \frac{1}{2}\left(\frac{d\omega}{dX}\right)^2 + \frac{1}{3}(S_H)|\omega|^3 \right\} dX \qquad \text{with } \omega = 1 \text{ for } X = 0 \tag{17}$$

It should be emphasized that since $\left(\frac{d\omega}{dX}\right)^2$ and $|\omega|^3$ are convex, the functional is convex, and its minimum is unique. Therefore, the obtained (positive) solution is unique. This problem can be generalized as follows

$$\frac{d^2\theta}{dX^2} - C\left|\theta\right|^{\gamma}\theta = 0 \quad \theta(0) = 1, \theta(\infty) = 0, \quad with \quad \gamma > 0, \quad C > 0$$
$$\theta = \left\{ \frac{\sqrt{2\gamma+4}}{\left(\gamma X\sqrt{C} + \sqrt{2\gamma+4}\right)} \right\} \tag{18}$$

The convex functional associated with Eq. (18) is presented below in Eq. (19). Eq. (18) generalizes both Eqs. (3) and (4). In other words, Eq. (18) has exact solutions for very large porous fins in two limiting cases: no radiation or no convection.

It is worth mentioning that the functional presented in Eq. (17) is a particular case of the functional presented in Eq. (19) when $\gamma = 1$ and $C = S_H$.

$$I\left[\omega\right] = \int_0^{\infty} \left\{ \frac{1}{2}\left(\frac{d\omega}{dX}\right)^2 + \frac{1}{\gamma+2}(\mathrm{C})\left|\omega\right|^{\gamma+2} \right\} dX \qquad \text{with } \omega = 1 \text{ for } X = 0 \tag{19}$$

4. RESULTS

At this point, a convex functional, proposed by Martins-Costa et al. (2020) is introduced, in order to allow the numerical computations.

$$I[\omega] = \int_0^{\infty} \left\{ \frac{1}{2}\left(\frac{d\omega}{dX}\right)^2 + \frac{S_H}{3}\left|\omega\right|^3 + \frac{G}{5}\left|\omega + C_T\right|^5 - G\,C_T^4\omega \right\} dX \qquad \text{with } \omega = 1 \text{ for } X = 0 \tag{20}$$

The minimization of $I[\omega]$ in Eq. (20) corresponds to the solution to the problem stated in Eq. (1). Figures 1 to 3 compare the analytical solution ($G = 0$), obtained by Eq. (18), with results obtained by minimizing a convex functional presented in Eq. (20) via a Conjugate Gradient method. In all computations, the temperature ratio is $C_T = 1$ $\left(T_b = 2T_{\infty}\right)$. It is important to remark that the exact solution shows very good agreement with the computations of the minimization of the functional introduced in Eq. (20) (Martins-Costa et al., 2020). These, in turn, show excellent agreement with the results obtained by Gorla and Bakier (2011).

In Fig. 1, the porous parameter is $\mathrm{S}_H = 10$. It may be observed that as the radiation parameter increases from $G = 0.01$ to $G = 0.1$ (in the numerical solutions), the deviation from the analytical solution slightly increases.

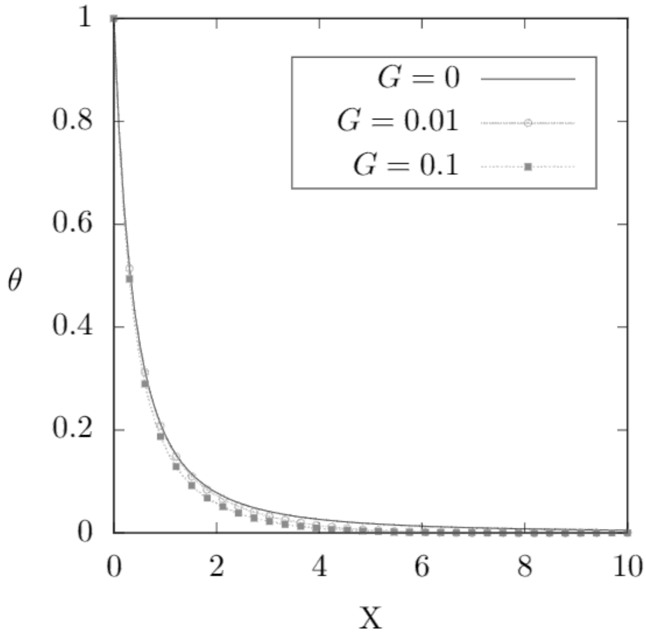

Figure 1. Comparison of results obtained with the analytical solution ($G=0$), considering $C_T=1$, $S_H=10$, and distinct values for the radiation parameter – namely $G=0.01$ and $G=0.1$.

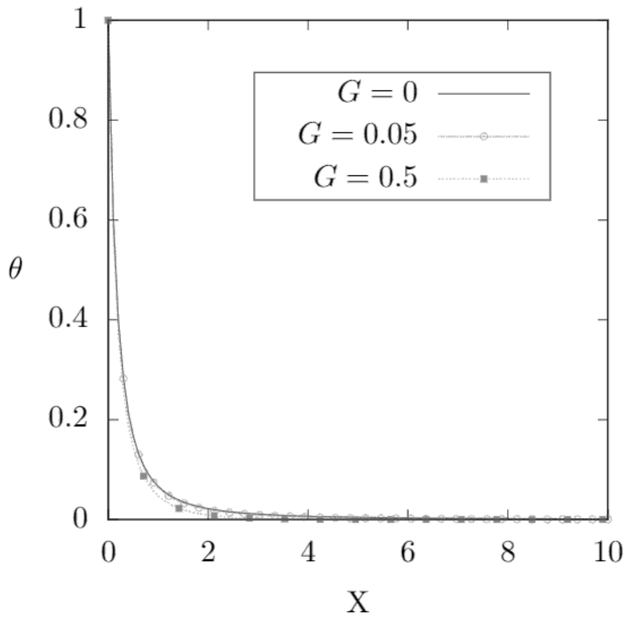

Figure 2. Comparison of results obtained with the analytical solution ($G=0$), considering $C_T=1$, $S_H=50$, and distinct values for the radiation parameter – namely $G=0.05$ and $G=0.5$.

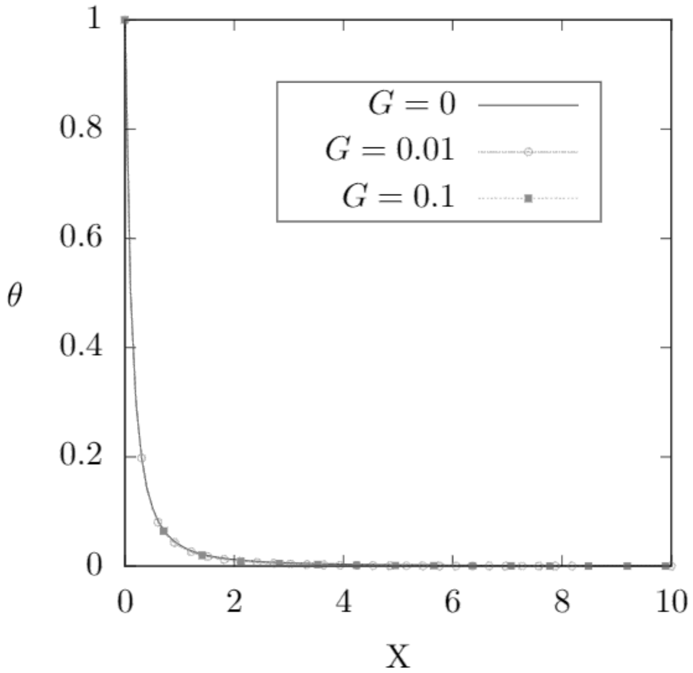

Figure 3. Comparison of results obtained with the analytical solution ($G = 0$), considering $C_T = 1$, $S_H = 100$, and distinct values for the radiation parameter – namely $G = 0.01$ and $G = 0.1$.

Figure 2 considers larger values if compared with Fig. 1. The porous parameter is $S_H = 50$ and the radiation parameter $G = 0.05$ and $G = 0.5$ (for the numerical computations). A small deviation from the analytical can be observed in the numerical computations only when $G = 0.5$.

Finally, in Fig. 3 the same values of radiation parameter used in Fig. 1 are employed ($G = 0.01$ and $G = 0.1$), but the porous parameter is much larger ($S_H = 100$). In this case, an excellent agreement is obtained when the analytical solution is compared with the minimization of the convex functional (20) employing a Conjugate Gradient method.

5. FINAL REMARKS

It is important to emphasize that the results obtained with closed analytical solutions show good agreement with the problem accounting for convection and radiation (via the minimization of a convex functional), even for very large porous parameter S_H values. Also, it is worth mentioning that analytical solutions provide a very efficient way to analyze such kind of problem, avoiding any computational effort.

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