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PHYSICAL NON-LINEAR ANALYSIS OF THE PLANE TRUSSES BY ITERATIVE METHODS WITH CUBIC CONVERGENCE

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Abstract. *The nonlinear analysis of structural problems using numerical techniques has gained a notoriety due to the technological advance that possibility a lower computational cost with a satisfactory degree of safety when compared with experimental tests. The structural design of steel truss requires the collapse load and its response, applied load versus deformation. Currently, plane truss is used in many engineering practical application as more complex structures due to new materials with a higher resistance. Then, it is produced structures more economical due to the reducing of the materials demand and consequently the global cost, presenting a non-linear behavior in the relation stress versus deformation. Structures that have plastic behavior only after the yelding was modeled using a parameter for hardening module. The non-linear analysis can be analyzed by incremental-iterative method. The current work aim is compare the solution methods Newton-Raphson, Modified Newton-Raphson and Potra-Pták. To analyses the performance of the methods, steel trusses problems with physical non-linearity are analyzed by algorithms using Finite Element Methods developed in Fortran90. The both Newton-Raphson methods are largely used in non-linear analysis and have quadratics convergence and the Potra-Pták is a new method that has cubic convergence. According with the result, the Potra-Pták method become an advantageous comparing to others iterative methods.*

Keywords: *Nonlinear Physical Analysis, Iterative Methods, Steel Trusses, hardening module*

1. INTRODUCTION

The trusses are as a simple and economical structural solution for many engineering situations, especially for walkways, bridges and roofs of stadiums and sheds. This structure has the great advantage of being able to cover large unobstructed areas with low dead weight. The behavior of this kind of structure is very sensitive to geometric and material nonlinearities.

Many structures have a linear behavior in the initial state. However, this only can be observed when the material has a linear elastic response with small displacements. When the load imposed on the structure exceeds the value of the load of yielding the deformation becomes an elastic and a plastic part as long as the structure does not have brittle rupture. Latter the deformation become permanent in the material defining the plastic regime. The sources of non-linearity are due to the non-linear behavior of the material, to the geometric non-linearity or to a combined effect of these (Pintea, 2012). The accurate analysis of plane trusses requires accurate constitutive relationships, which account for several failure modes of member such as buckling, yielding, inelastic postbuckling, unloading, and reloading (Thai and Kim, 2011).

Santos (2002) analyzed plane trusses, which had also been analyzed in the work of Rodrigues (1997), both observed that the results obtained by linear and non-linear geometric behavior showed negligible differences. On the other hand, when physical non-linearity was considered, the displacements had a significant increase in relation to linear analysis, demonstrating the importance of considering such a source of non-linearity.

It is necessary to use iterative methods to solve the nonlinear systems that occur during the structural analysis, due to the non-linear behavior of the material. In the case of analysis of these structures, the iterative method of Newton-Raphson is widely used (Yang and Kuo, 1994). In recent years, with the development of efficient and fast computers, the investigation of non-linear problems and numerical methods for their resolution has increased dramatically. According to Souza et al. (2018), before the 1980s order of convergence higher than Newton-Raphson method the

iterative method require the computation of higher order derivatives. It is known that greater the computational cost lower its applicability. There are methods that have a cubic convergence rate, which are better than Newton-Raphson method in this aspect, such as methods belonging to the Chebyshev-Halley class (Candela and Marquina, 1990) and the Potra-Pták method (Potra and Pták, 1984).

In this paper, we present algorithm for the incremental and iterative procedures based on Newton-Raphson and Potra-Ptak methods. The analyzes were performed in trusses problems with physical non linearity. The Finite Element formulation is used. The comparison between the results obtained by the proposed program and the results of the literature is done to show the ability of the proposed program to capture the inelastic nonlinear responses of plane trusses structures under static loads.

2. THEORETICAL BASIS FOR THE PHYSICAL NON LINEAR ANALYSIS

2.1 Formulation for non-linear physical analysis

The Finite Element Method (FEM) aims to transform the continuous elements into discrete elements with a finite number of degrees of freedom, obtaining the displacements at any point of the continuous element in terms of a finite number of displacements at the nodal points, multiplied by appropriate interpolation functions. According to Bathe (1996) the FEM is an extension of the displacement method used for many years in Civil Engineering.

The Figure 1a shows a truss element in the global coordinate system and Figure 1b shows this same element in the local coordinate system.
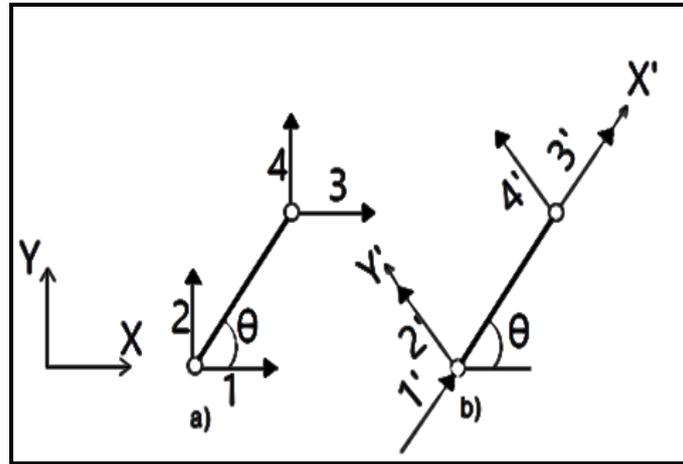

Figure 1. Truss element a) global system b) local system

The total potential energy π for a truss element, Eq. (1), is written in terms of the internal deformation energy (U) and the potential energy caused by the actions acting on the structural system (V).

$$\pi = U - V = \frac{1}{2} \int_V \varepsilon^T \sigma \, dV - \sum_i u^{iT} F^i \tag{1}$$

From this functional energy to the ideal truss the principle of stationarity applies

$$\delta\pi = \frac{1}{2} \int_V \delta\varepsilon^T \sigma \, dV - \sum_i \delta u^{iT} F^i \tag{2}$$

Considering that all displacements are along the axis x, these functions being u_i and u_j, it is possible to write $u(x)$ this way:

$$u(x) = N_1(x)u_i + N_2(x)u_j \tag{3}$$

The form functions, $N_i(x)$, considering a linear element with two nodes are given by:

$$N_1(x) = \frac{-x + L}{L} \tag{4}$$

$$N_2(x) = \frac{x}{L} \tag{5}$$

In the truss element, the axial deformation is given by:

$$\varepsilon = \frac{du}{dx} \tag{6}$$

and the strain field is defined as follows:

$$\varepsilon = Bu_i \tag{7}$$

Substituting equations 3, 4 and 5 into 6, derivation results in:

$$\varepsilon = \frac{dN_1}{dx}u_i + \frac{dN_2}{dx}u_j \tag{8}$$

and in matrix form:

$$\varepsilon = \begin{bmatrix} \dfrac{dN_1}{dx} & \dfrac{dN_2}{dx} \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{9}$$

The matrix of equation 9 is called a gradient matrix containing the derivatives of the form functions, stored as follows

$$B = \begin{bmatrix} \dfrac{dN_1}{dx} & \dfrac{dN_2}{dx} \end{bmatrix} \tag{10}$$

Considering a constitutive relation appropriate to the model of the material with its matrix that represents the matrix of elastic constants, we have:

$$\sigma = D\varepsilon \tag{11}$$

The matrix equation that defines the static equilibrium of the structure in every domain is given by:

$$Ku = F \tag{12}$$

Where K is the stiffness matrix of the element, obtained by the following expression:

$$K = \int_{V^e} B^T DBdV \tag{13}$$

being: V the volume of the element.

2.2 Constituve Modeling

Considering the ideal truss model it is possible to disregard the forces of bodies and the distributed surface forces. In this case, if the truss element is in a linear elastic regime the deformations are purely elastic, $\varepsilon = \varepsilon^e$, and if the element is in a plastic regime, the total deformation takes into account the plastic deformations $\varepsilon = \varepsilon^e + \varepsilon^p$.

The physical non-linearity analysis of steel trusses structures subject to static actions can be conceived with the numerical implementation of models with linear hardening. For this, it is necessary the development of mathematical equations that simulate the structural behavior of the steel and the creation of a suitable computational algorithm that stores all the previous history of the relation tension *versus* deformation of the structural elements. In this work we chose to use a simple model that takes into account the hardening of the material through a Isotropic Hardening Parameter. The Figure 2 show the model used bilinear elastoplastic to analyzes the members of trusses, where an elastic behavior with elastic modulus E and a linear hardening plastic region with tangent modulus E_t are distinguished.
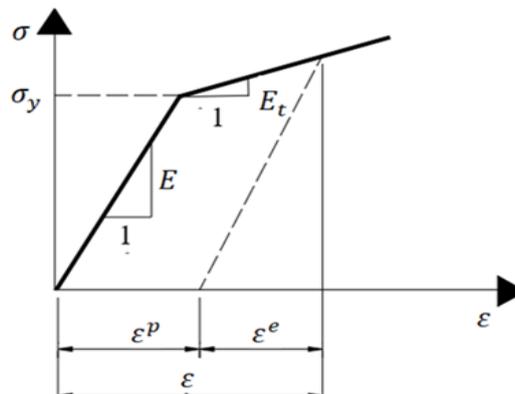

Figure 2. Bilinear elastoplastic model

2.3 Incremental Elastoplastic Analysis

Considering that the increase in deformation can be decomposed into an elastic increase and another plastic, Fig. 3, we have that:

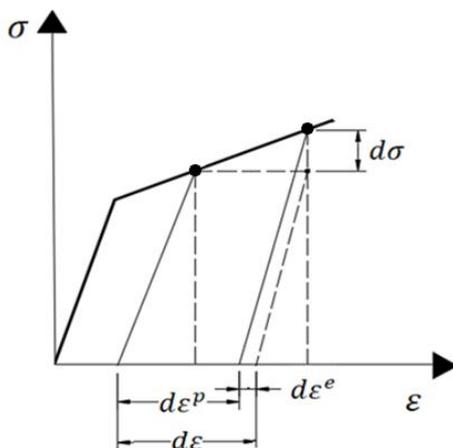$$de = de^e + de^p \tag{14}$$

Figure 3. Incremental elastoplastic model

The material initially deforms according to the modulus of elasticity E until the element reaches the yield stress, σ_y.

In this case, the elastic deformation is obtained by equation 15 and plastic deformation $d\varepsilon^p = 0$.

$$d\varepsilon^e = \frac{1}{E} d\sigma \tag{15}$$

If the system is loaded with a load value bigger than the yield load, it will deform according to the tangent modulus E_t. In this case, the increase of tension is followed by an increase of the elastic deformation and of the plastic deformation, given by the equation 16.

$$d\varepsilon^p = d\varepsilon - \frac{1}{E} d\sigma \tag{16}$$

During the implementation of an algorithm for physical non-linear it is necessary to verify if the element is in the elastic or plastic regime, for this a plastification function is defined. For the truss element, this function can be expressed in terms of the module of the tension acting on the element and the yield stress.

$$f^0(\sigma) = |\sigma| - \sigma_y \tag{17}$$

If it $f^0(\sigma)$ is less than zero the element is in elastic regime and the Hooke's Law is obeyed, and the addition of stress is given by:

$$d\sigma = Ed\varepsilon \tag{18}$$

Otherwise, the element is in plastic regime and the addition of soils is given by:

$$d\sigma = E_t d\varepsilon = \left(\frac{E}{E + E_p} E_p \right) d\varepsilon \tag{19}$$

For ideally plastic materials $E_t = 0$ and $E_p = 0$.

2.4 Incremental-Iterative Method of Potra-Pták

According with Crisfield (1997), the determination of the response in the post-critical interval is essential when it is desired to study the non-linear behavior in order to know the collapse load of the structure. The load versus full displacement curve describes the variation of the overall behavior of the structural system as control parameters such as the applied external force and the displacement are varied.

According to Soleymani et al. (2012), the iterative method of Potra-Pták requires two steps to solve a non-linear problem, keeping the stiffness matrix constant within each iteration. For non-linear structural analysis the method requires two function evaluations.

Calculate the correction vector of the displacements of the first step, Δy_k:

$$\Delta y_k = K_{k-1}^{-1} g\left(u_{k-1} + \Delta_{k-1}\right) \tag{20}$$

From this result, a new vector of unbalanced loads is calculated

$$g_k = g(u_{k-1} + \Delta y_k) \tag{21}$$

and a new displacement correction vector, Δu_k, maintaining the constant stiffness matrix

$$\Delta u_k = K_{k-1}^{-1} g\left(u_{k-1} + \Delta y_k\right) \tag{22}$$

determining the displacement vector of each iteration

$$u_k = u_{k-1} + \Delta u_k \tag{23}$$

The incremental-iterative process looks for the configuration of the displacement corresponding to the time. At the end of each iteration, the displacements must be within a certain tolerance of the actual displacement solution. Thus, a convergence criterion is based on the convergence of the displacements.

$$\left\| \Delta u_k^i \right\| = \xi \left\| {}^{t+\Delta t} u_{k-1}^i \right\| \tag{24}$$

3. EXAMPLE

3.2 Elastoplastic Analysis of Hyperstatic Plane Truss

Leite (2000) analyzed the truss shown in Figure 4 for physical nonlinear analysis with the same constitutive relation presented in this article. The truss elements have longitudinal modulus of elasticity $E = 20500\,kN/cm^2$, yield stress $\sigma_e = 34,5\,kN/cm^2$, cross-sectional area $A = 12,51 cm^2$ and length L = 200 cm. The load P = 1050 kN was applied on node 4 in the vertical direction, from top to bottom.
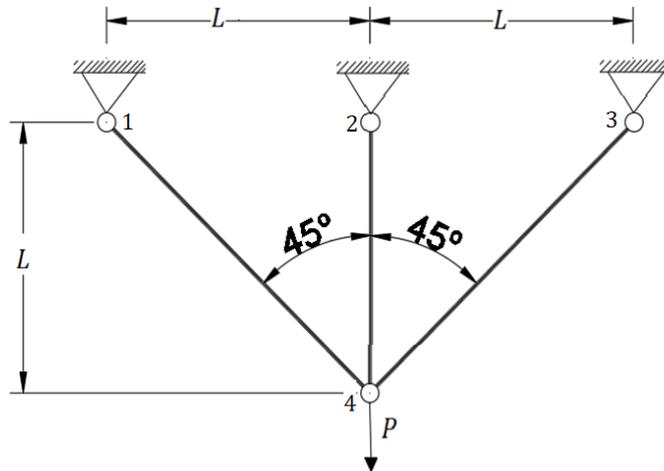

Figure 4. Plane Truss in elastoplastic regime

Using the displacement method as the analytical solution, it is possible to determine the normal stresses supported by each bar.

After the determination of the coefficients of the stiffness matrix and the establishment of the equilibrium equations, according to the respective degrees of freedom, we can determine the displacement at the force application node.

$$\delta_4 = \frac{PL}{EA\left(1 + \dfrac{\sqrt{2}}{2}\right)} \tag{25}$$

The normal force for the vertical bar to go into plastic process is given by:

$$P_e = \frac{\sigma_e A}{\left(2 - \sqrt{2}\right)} \tag{26}$$

The breaking stress is obtained when the two inclined bars reach the flow load:

$$P_r = \left(1 + \sqrt{2}\right) \sigma_e A \tag{27}$$

From equation 26 and 27, the theoretical values for and can be obtained: $P_e = 736.78kN$ and $P_r = 1041.96kN$.

The Figures 5, 6 and 7 show the curve load versus displacement of the physical non-linear analysis with the implemented program, based on the Isotropic Hardening Parameter, by Newton-Raphson Standard method, Newton Raphson modified method and Potra-Pták method, respectively. For the simulations with the incremental iterative technique, we adopted the increase of force equal $\Delta P = -10.5kN$ or $\Delta P = -105kN$ and tolerance equal $\xi = 10^{-5}$.
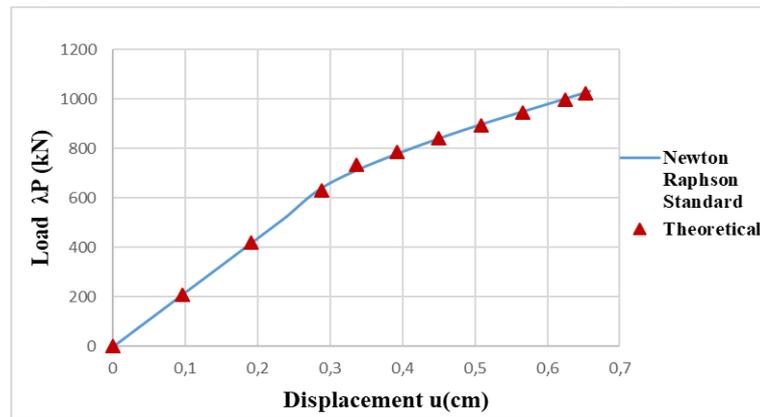

Figure 5. Physical non-linear analysis by Newton-Raphson's standard iterative method
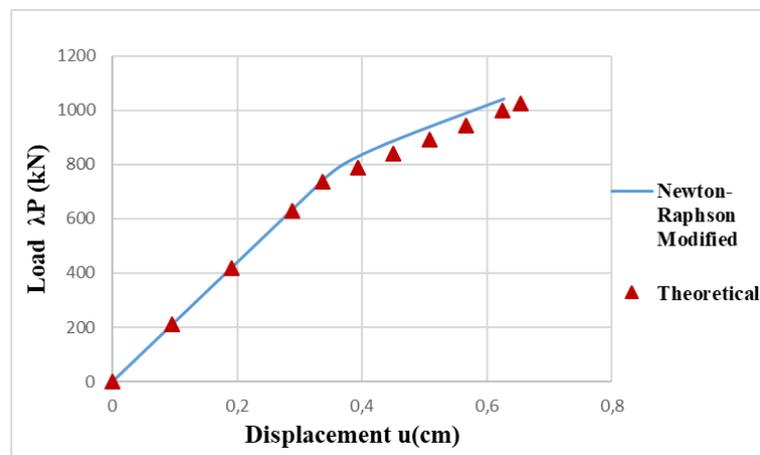

Figure 6. Physical non-linear analysis by Newton-Raphson's standard iterative method
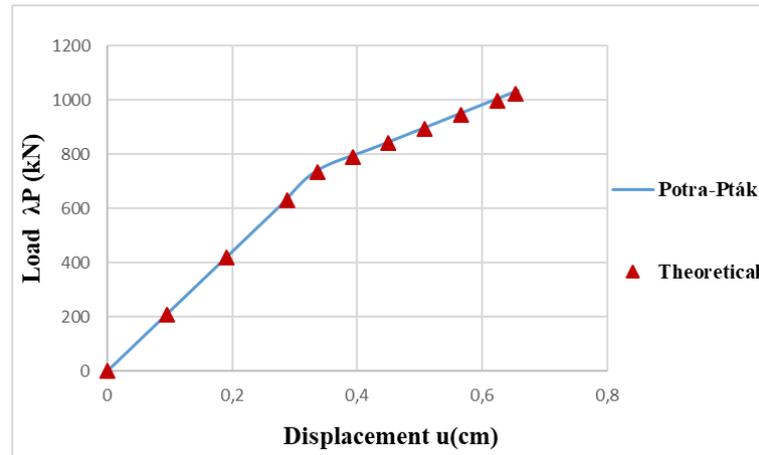
Figure 7. Physical non-linear analysis by Potra-Pták iterative method

It can be seen that the physical non-linear analysis using the three iterative methods reached the yielding load and the burst load very close to the theoretical values. Validating the implementation of the Potra-Pták method for non-linear structures analysis.

The table 1 presents the number of increment and number of iterations of each iterative method.

Table 1. Comparison between iterative methods for nonlinear analysis

Iterative Method '	$\Delta P = -10.5kN$		$\Delta P = -10,5kN$	
	Number of increments	Accumulated iterations	Number of increments	Accumulated iterations
Newton-Raphson Standart	100	198	34	66
Newton-Raphson Modified	100	226	34	74
Potra-Pták	100	101	34	35

Two parameters of increment of load, one small and one greater, were used in order to verify the effectiveness of the method of Potra-Pták before the method of Newton-Raphson. As expected, the modified method presented a number of iterations higher than the standard method. The small number of iterations required for the convergence of the Potra-Ptak method, due to its cubic convergence, is observed. For this example, it was found that the Potra-Ptak method was quite efficient.

3.3 Elastoplastic Analysis of Isostatic Plane Truss

This example, also analyzed by Rodrigues (1997), Santos (2002) and Souza (2015), is composed of a steel truss, Fig. 8, of 13 bars with cross-sectional area $A = 1cm^2$ and length $L = 200cm$. The structure is requested by a concentrated force of intensity $60kN$ in the lower central node, in the vertical direction and downward. The elements that make up the structure show bilinear elastoplastic behavior. The modulus of elasticity of all elements is $E = 21000\,kN/cm^2$ and the tangent stiffness module $E_t = 5000\,kN/cm^2$. The material yield stress is $\sigma_y = 24\,kN/cm^2$.
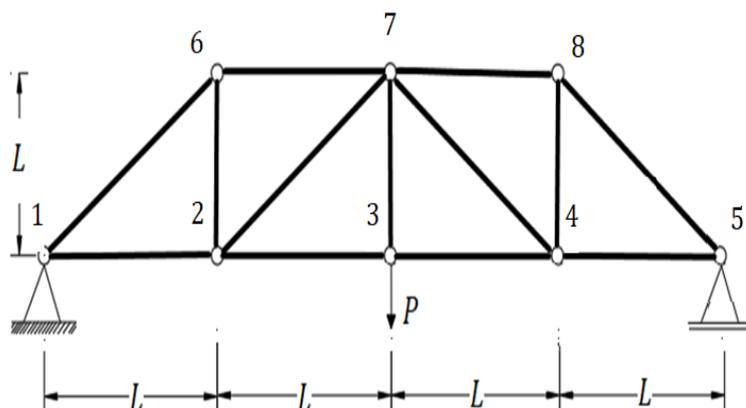
Figure 8. Truss in elastoplastic regime with hardening

For the simulations with the incremental iterative technique, we adopted the increase of force equal $\Delta P = -5kN$ and tolerance equal $\xi = 10^{-5}$.
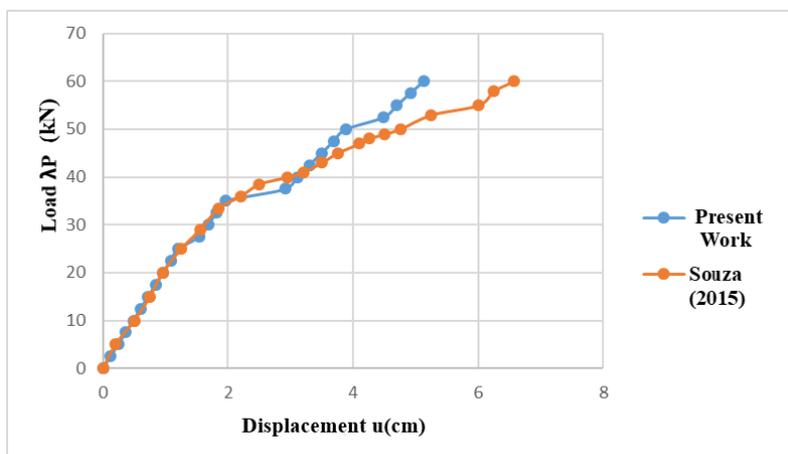

Figure 9. Load versus displacement curve with tangent stiffness modulus

From the results obtained by the implemented program it is possible to notice from Fig. 9 that the curve load versus displacement is in agreement with the results obtained from the research of Souza (2015), until the formation of the second plastic hinge, in some elements, with displacement of $u = 2.93cm$. After this displacement, the curve tends to distance itself from the required result. At the end of the incremental-iterative process the value of $60kN$ the displacement for the force of was $u = 5.48cm$.

The value of the displacement reached by the program was lower than the others, overestimating the capacity of the structure to withstand the deformations, after the 40 kN load and the plastification of the diagonals. This fact can be explained by by the use of linear deformation in the analyzes. The elastoplastic model with a hardening parameter adopted in the present work was very efficient for small deformations.

The truss was analyzed by the Newton-Raphson Standard iterative method and the Potra-Pták method. For both methods, 25 increments were required. It was observed that the number of iterations using the iterative method of Potra-Pták was 26 while that of Newton-Raphson was 48. This tendency also was observed in other examples analyzed, besides results of the literature, proving its efficiency for non-linear analysis physical

4. CONCLUSION

The numerical results showed that a smaller number of iterations required for convergence, from a given tolerance, a tendency that the of Potra-Pták iterative method, when compared with Newton-Raphson's method, is more efficient. The efficiency of the implemented method is verified by the trend of having half iterations than the standard Newton Raphson method or modified Newton Raphson method.

The Isotropic Hardening Parameter model implemented was able to obtain good results, reaching the theoretical and numerical values of yelding load and collapse load. Therefore, for simple cases it is possible to avoid models that have a high computational cost, such as the plastic label method and the plastic zone method.

5. ACKNOWLEDGEMENTS

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