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ANALYTICAL SOLUTION OF THE DIRECT PROBLEM IN MULTILAYER HEAT CONDUCTION

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Abstract. *This paper presents a method of obtaining the three-dimensional (3D) analytic solution of the temperature for the multilayer heat transfer problem using Green Functions (FG), due to a transient three-dimensional thermal problem, subject to the condition of convexity on all faces. The advantage in using the FG method is that 3D problems becomes a simple multiplication of one-dimensional (1D) problems. This work not only presents the analytical solution but also its computational implementation, which allows us to better understand the physical problem. Obtaining the 3D analytical solution for the two-layer heat conduction problem in one direction requires more elaborate procedures than solving single-layer problems, both to fit the solution equation in terms of Green's functions and to obtain Two eigenvalues. A multilayer 3D problem in one direction, in perfect contact, the problem is referred to as* $X33Y3C13Z33$ *in (Haji-Sheikh, 2014). The temperature profile for the double-layer medium is obtained and verified the analytical solution through comparison with exact and numerical solutions of correlated and specific thermal problems.*

Keywords: *Analytical solution, multilayer, Green's functions, heat conduction*

1. INTRODUCTION

Specifically, in the field of mechanical engineering among the existing phenomena, we study the heat transfer by conduction that occurs due to the temperature gradient in solid medium and that can be modeled mathematically by the diffusion equation.

It is proposed here to obtain the analytical solution for the heat conduction equation, which is given by a partial differential equation. Several methods can be used for this task, among them the Green Functions (FG) method is used since the boundary conditions vary over time, which immediately discards the method of separating variables.

One of the advantages in the use of integral solutions by FG is the possibility of constructing, without additional difficulties, multidimensional solutions from the one-dimensional Green functions. In this case, the versions of the 2D and 3D solution equations are absolutely equivalent to the one-dimensional equation and the GF can be obtained from products of 1D solutions in different directions (Oliveira, 2015).

The objective of the present work is the investigation and development of an analytical solution for heat conduction problems in multilayer media, or also called compounds, using the FG technique.

It is observed that the work not only presents the formulation, the development and the obtaining of the multi-layered analytical solution equation, but also its computational implementation, allowing a better physical understanding of the problem.

2. MATERIALS AND METHODS

The development of the multilayer 3D analytical solution is presented below, which is represented by $X33Y3C13Z33$ in (Haji-Sheikh, 2014).

2.1 Analytical solution of the direct problem $X33Y2C12Z33$

The 3D heat conduction problem shown in Fig. 1 is a heat conduction problem whose all faces, except where the heat flow occurs, are subjected to a heat exchange by convection. Note that the problem consists of two layers in the direction of the axis y, whose thermophysical properties are different in each layer, delimited by $y = b$.

The problem represented by Fig. 1 with homogenized equations is given by:

$$\frac{\partial^2\theta_1}{\partial x^2} + \frac{\partial^2\theta_1}{\partial y^2} + \frac{\partial^2\theta_1}{\partial z^2} + g(x, y, z, t) = \frac{1}{\alpha_1}\frac{\partial\theta_1}{\partial t} \tag{1a}$$

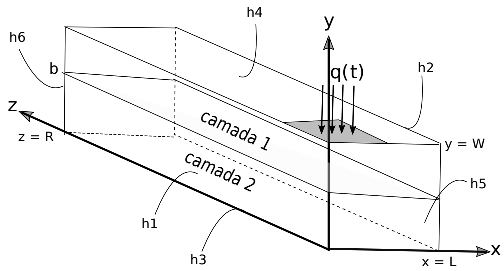

Figure 1. Problema $X33Y2C12Z33$

$$\frac{\partial^2\theta_2}{\partial x^2}+\frac{\partial^2\theta_2}{\partial y^2}+\frac{\partial^2\theta_2}{\partial z^2}=\frac{1}{\alpha_2}\frac{\partial\theta_2}{\partial t} \tag{1b}$$

Subject to the boundary conditions on the x axis

$$-k_1\frac{\partial\theta_1}{\partial x}\bigg|_{x=0}=-h_1\theta_1; \qquad -k_1\frac{\partial\theta_1}{\partial x}\bigg|_{x=L}=h_2\theta_1 \tag{1c}$$

Subject to the boundary conditions on the y axis

$$-k_1\frac{\partial\theta_1}{\partial y}\bigg|_{y=0}=-h_3\theta_1; \qquad -k_2\frac{\partial\theta_2}{\partial y}\bigg|_{y=W}-h_4\theta_2=0 \tag{1d}$$

Subject to the boundary conditions on the z axis

$$-k_1\frac{\partial\theta_1}{\partial z}\bigg|_{z=0}=-h_5\theta_1; \qquad -k_1\frac{\partial\theta_1}{\partial z}\bigg|_{z=R}=h_6\theta_1 \tag{1e}$$

and to the continuity conditions

$$\theta_1|_{y=b}=\theta_2|_{y=b}; \qquad -k_1\frac{\partial\theta_1}{\partial y}\bigg|_{y=b}=-k_2\frac{\partial\theta_2}{\partial y}\bigg|_{y=b} \tag{1f}$$

And to the initial condition

$$\theta_1(x,y,z,0)=\theta_2(x,y,z,0)=F(x,y,z)-T_\infty \tag{1g}$$

The expression for the temperature in terms of the Green function is similar to that described for the case of one layer, but the functional form of the general solution and the Green function are obtained from more elaborate procedures. The general solution of the problem given by the equations (1a) - (1g) is shown in equation (1), assuming that the inhomogeneity $g(x,y,z,t)=q(t)\delta(y-W)$. This is due to the fact that the boundary conditions (heat flux) are removed by the procedure described in the superposition. Thus, the solution applies to the term for power generation.

The solution for temperature in each region i is given by

$$\begin{aligned}\theta_i(x,t)=\sum_{j=1}^{M}\Bigg\{&\int_0^L\int_{y_j}^{y_{j+1}}\int_0^R G_{ij}(x,y,z,t|x',y',z',0)F_j(x',y',z',0)dx'dy'dz'\\&+\frac{\alpha_j}{k_j}\int_0^t\int_{L_1}^{L_2}\int_{R_1}^{R_2} g_j(x,y',z,\tau)G_{ij}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau\Bigg\}\end{aligned} \tag{2}$$

Where the first term refers to the initial temperature term $(\theta(x,y,z,0))$ and the second term is related to the condition of heat flow boundary imposed to the arbitrary area $0\le L_1\le x\le L_2\le L$ e $0\le R_1\le z\le R_2\le R$ em $y=W$ as shown by Fig.(1).

It is observed that $y_j\le y\le y_{j+1}$, for $j=1,2,...,M$, are the boundaries of each layer, and, $G_{ij}(x,y,z,t|x',y',z',\tau)$ is the Green function for multilayer problems.

If $M=1$, the solution is for the case of a single layer within the range defined $0\le y\le W$, where $y_1=0$ e $y_2=W$, therefore the solution given by Eq. (2) is algebraically equal of a single layer.

If $M=2$, defines two layers given by the following intervals $0\le y\le b$ e $b\le y\le W$ where we have respectively layer 1 and layer 2, where $y_1=0$, $y2=b$ and $y_3=W$.

$$\begin{aligned}\theta_1(x,y,z,t) &= \int_0^L \int_{y_1}^{y_2} \int_0^R G_{11}(x,y,z,t|x',y',z',0)F_1(x',y',z',0)dx'dy'dz' \\ &+ \frac{\alpha_1}{k_1}\int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} g_1(x,y',z,\tau)G_{11}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau \\ &+ \int_0^L \int_{y_2}^{y_3} \int_0^R G_{12}(x,y,z,t|x',y',z',0)F_2(x',y',z',0)dx'dy'dz' \\ &+ \frac{\alpha_2}{k_2}\int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} g_2(x,y',z,\tau)G_{12}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau\end{aligned} \tag{3a}$$

$$\begin{aligned}\theta_2(x,y,z,t) &= \int_0^L \int_{y_1}^{y_2} \int_0^R G_{21}(x,y,z,t|x',y',z',0)F_1(x',y',z',0)dx'dy'dz' \\ &+ \frac{\alpha_1}{k_1}\int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} g_1(x,y',z,\tau)G_{21}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau \\ &+ \int_0^L \int_{y_2}^{y_3} \int_0^R G_{22}(x,y,z,t|x',y',z',0)F_2(x',y',z',0)dx'dy'dz' \\ &+ \frac{\alpha_2}{k_2}\int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} g_2(x,y',z,\tau)G_{22}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau\end{aligned} \tag{3b}$$

As the heat flow is applied to the surface, this implies that the last part of the equations (3a) - (3b) are void, this is

$$\begin{aligned}\theta_1(x,y,z,t) &= \int_0^L \int_{y_1}^{y_2} \int_0^R G_{11}(x,y,z,t|x',y',z',0)F_1(x',y',z',0)dx'dy'dz' \\ &+ \frac{\alpha_1}{k_1}\int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} g_1(x,y',z,\tau)G_{11}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau \\ &+ \int_0^L \int_{y_2}^{y_3} \int_0^R G_{12}(x,y,z,t|x',y',z',0)F_2(x',y',z',0)dx'dy'dz'\end{aligned} \tag{4a}$$

$$\begin{aligned}\theta_2(x,y,z,t) &= \int_0^L \int_{y_1}^{y_2} \int_0^R G_{21}(x,y,z,t|x',y',z',0)F_1(x',y',z',0)dx'dy'dz' \\ &+ \frac{\alpha_1}{k_1}\int_0^t \int_{L_1}^{L_2} \int_{R_1}^{R_2} g_1(x,y',z,\tau)G_{21}(x,y,z,t|x',W,z',\tau)dx'dz'd\tau \\ &+ \int_0^L \int_{y_2}^{y_3} \int_0^R G_{22}(x,y,z,t|x',y',z',0)F_2(x',y',z',0)dx'dy'dz'\end{aligned} \tag{4b}$$

Obtain the Green function $G_{ij}(x,y,z,t|x',y',z'\tau)$ by observing the types of boundary conditions in the directions of x, y, being three independent one-dimensional problems. Then, we get $G_j(x,y,z,t|x',y',z'\tau)$ as a product of these Green functions, that is, $G_{ij}(x,y,z,t|x',y',z'\tau) = GX33GY2C13GZ33$.

In the x and y direction, we have the boundary conditions of type three, which means convection condition, and the one-dimensional Green function is easily found in (**?**), but in y direction we have a unidimensional problem consisting of two layers submitted the convection condition in both directions, in this case requires obtaining the Grenn function.

In the x direction, we have

$$\begin{aligned}G_{X33}(x,t|x',\tau) &= \frac{2}{L}\sum_{m=1}^{\infty} e^{-\alpha_m^2\alpha(t-\tau)/L^2}\left[\alpha_m cos\left(\frac{\alpha_m(x)}{L}\right) + B_1 sen\left(\frac{\alpha_m(x)}{L}\right)\right] \\ &\times \frac{\left[\alpha_m cos\left(\frac{\alpha_m(x')}{L}\right) + B_1 sen\left(\frac{\alpha_m(x')}{L}\right)\right]}{(\alpha_m^2 + B_1^2)\left[1 + \frac{B_2}{(\alpha_m^2+B_2^2)}\right]} + B_1\end{aligned} \tag{5}$$

where tan $\alpha_m = \frac{\alpha_m(B_1+B_2)}{\alpha_m^2 - B_1B_2}$, $B_1 = \frac{h_1L}{k_1}$ e $B_2 = \frac{h_2L}{k_1}$

In the z direction, we have

$$G_{Z33}(z,t|z',\tau) = \frac{2}{R}\sum_{p=1}^{\infty} e^{-\gamma_p^2\alpha(t-\tau)/R^2}\left[\gamma_p cos\left(\frac{\gamma_p(x)}{R}\right) + B_5 sen\left(\frac{\gamma_p(z)}{R}\right)\right] \times \frac{\left[\gamma_p cos\left(\frac{\gamma_p(z')}{R}\right) + B_5 sen\left(\frac{\gamma_p(z')}{R}\right)\right]}{(\gamma_p^2 + B_5^2)\left[1 + \frac{B_6}{(\alpha_p^2+B_6^2)}\right]} + B_5 \tag{6}$$

where tan $\gamma_p = \frac{\gamma_p(B_5+B_6)}{\gamma_p^2 - B_5 B_6}$, $B_5 = \frac{h_5 R}{k_1}$ e $B_6 = \frac{h_6 R}{k_1}$.
In the y direction we have two layers and the Green multilayer function G_{ij} is given by (**?**)

$$G_{ij}(y,t|y',\tau) = \sum_{n=1}^{\infty} e^{-\lambda_n^2(t-\tau)}\frac{1}{N_y}Y_{in}(y)Y_{jn}(y'), \tag{7}$$

3. ACKNOWLEDGEMENTS

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4. REFERENCES

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5. RESPONSIBILITY NOTICE