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ANALYTICAL IMPULSE RESPONSE IDENTIFICATION FOR A MOVING HEAT SOURCE PROBLEM USING GREEN'S FUNCTIONS

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Abstract. Dynamical systems has three variables to be study: Perturbation, transfer function and impulse response, thus we can solve these kind of problems using knowing two of three variables. To do it, we need to know the system impulse response that can be determined knowing the analytical solution of the heat conduction problem that can be determined using Green's functions method with convolution theorem. This method is called Transfer Function Based on Green's Function (TFBGF), we propose through modifications in original TFBGF method to determine the impulse response function to a moving heat source model.

Keywords: Impulse response, inverse problem, analytical solution, moving heat source, Green's function

1. INTRODUCTION

Analytical approach shows an important tool to develop engineering solutions, and also, can be used to validate numerical solutions or make approximations. This fact makesa a little easier to alanlyze and understand physical problems **?**. The thermal model complexity, when we develop analitycal solutions are, normally, in transient multidimensional problems submitted to nonhomogeneities such as prescribed flow boundary conditions, heat generation or boundary conditions as temperature varying with the time and space.

Transient, multidimensional analitycal solutions whith heat generation or heat flux using Green's functions was showed by Beck *et al.* (2010). This work presented the theoretical development and examples of multidimensional applications, change of variables, the Green's function method with finite and semi-infinite geometries.

Moving heat source are presents in numerous pratical engineering problems. For example, machining processes as cutting, milling or welding process. Moving heat source are also present in biological heating as the metabolism or heat thermal treatment. All these cases, the heat input identification are a complex task and represents an important factor in the process optimization.

Many researchers have been studying welding heat flow problems, analytically, numerically and experimentally. In most cases, input heat flux values are assumed known, obtained from literature or determined by calorimetric techniques. Although, the moving heat flux are unknow to many processes these methods are called IHCP (inverse heat conduction problem). Thus researches use techniques to obtain heat flux or heat generation

Moving heat source estimation without use minimization least squares, or optimization techniques presents a great advantage of the technique proposed here. The moving heat source can be obtained directly from the measured temperatures, since the 3D transient analytical solution is obtained and the TFBGF method, (Fernandes, 2013), can be applied.

The temperature measurements are obtained using thermocouples at accessible regions of the workpiece surface while the theoretical temperatures are calculated from a 3D transient heat conduction thermal model with a moving heat source. The thermal model solution is obtained analytically (direct problem) in (Ribeiro, 2015). Then the inverse problem, it means, the moving heat source estimation, uses the TFBGF method (Fernandes *et al.*, 2015). This method is based on Green's function and in the equivalence between thermal and dynamic systems.

The technique is a simple approach without iterative process and extremely fast. From the knowledge temperature profile (hypothetical or experimental) and transfer function is possible estimate the moving heat source by an inverse procedure using the fast Fourier transform FFT. To do it, the TFBFG was adapted.

2. GOVERNING EQUATION

The thermal problem can be described by the $3D$ heat conduction equation in the fixed x, y, z coordinate system, assuming constant thermal properties, as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{q_g}{k}\delta(x - vt)\delta(y - P_y)\delta(z - P_z) = \frac{1}{\alpha}\frac{\partial T}{\partial t} - u\frac{\partial T}{\partial x} \tag{1}$$

subjected to the first kind boundary conditions

$$T(0, y, z, t) = T(L_1, y, z, t) = T_0$$
$$T(x, y, z, t) = T(x, L_2, z, t) = T_0 \tag{2}$$

and the initial condition

$$T(x, y, z, 0) = T_0 \tag{3}$$

There are two manner of considering the moving heat souce effects. One of them is consider the heat flux releases at surface and in this sense the heat flux need to be considered as a boundary condition. The second hypoteses, used here, is to consider the heat flux as a heat source generation. This assuptions is a better approximation for pratical engineering applications.

Machining, grinding, cutting, and sliding of surfaces, that has energy generated as a result of friction heating in a area with depth penetrarion, it means a volume heat source. In this sense, in this work, the heat source moving will be considered a point heat source of constant strength, releasing it is energy continuously over time while moving along the x axis in positive x direction with a constant velocity u, in a stationary medium that is initially at zero temperature.

After performing variables change using:

$$T(x, y, z, t) \quad = W(x, y, z, t)exp\left(\frac{ux}{2\alpha} - \frac{u^2 t}{4\alpha}\right) \tag{4}$$

To solve this problem it is convenient change x variable to ξ Özişik (1993) due coordinate system move with the source. The new variable ξ is given by:

$$\xi = x - ut \tag{5}$$

Thus, considering Eqs. (4) and (5) the governing equation is obtaned as

$$\frac{\partial^2 W}{\partial \xi^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} + \frac{q_g}{k}\delta(\xi - P_\xi)\delta(y - P_y)\delta(z - P_z) = \frac{1}{\alpha}\frac{\partial W}{\partial t} \tag{6}$$

subjected to the boundary conditions

$$W(0, y, z, t) = W(L_1, y, z, t) = 0 \tag{7}$$
$$W(\xi, 0, z, t) = W(\xi, L_2, z, t) = 0 \tag{8}$$
$$W(\xi, y, 0, t) = W(\xi, y, L_3, t) = 0 \tag{9}$$

and

$$k\frac{\partial W}{\partial z}\Big|_{z=L_3} + W(L_3, t)\left(h + \frac{ku}{2\alpha}\right) = hW_\infty e^{\frac{-uL_3}{2\alpha} - \frac{u^2 t}{4\alpha}} \tag{10}$$

where

$$h + \frac{ku}{2\alpha} = h_{eff} \tag{11}$$

The term h_{eff} in Eq.(11) is the effective heat convection coefficien Beck *et al.* (2010).
Equation (6) can then be solved by Green's function method Beck and McMasters (2004) as

$$W = \int_{\tau=0}^{t} \int_{\xi'=0}^{L_1} \int_{y'=0}^{L_2} \int_{z'=0}^{L_3} G^*(t-\tau)q_g(\tau)dz'dy'd\xi'd\tau \tag{12}$$

where

$$G^*(\xi,y,z,t|\xi',y',z',t-\tau) = G(t-\tau)\delta(\xi'-P_\xi)\delta(y'-P_y)\delta(z'-P_z)$$
$$\times e^{\frac{uP_\xi}{2\alpha}-\frac{u^2\tau}{4\alpha}} \tag{13}$$

and the Green's fucntion for this problem can be obtained in Beck *et al.* (2010)

$$G(\xi,y,z,t|\xi',y',z',t-\tau) = \frac{8}{L_1L_2L_3} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} e^{-\beta_m^2 u}e^{-\beta_n^2 u}e^{-\beta_p^2 u}$$
$$\times \, sen(\beta_m x)sen(\beta_m x')sen(\beta_n y)$$
$$\times \, sen(\beta_n y')sen(\beta_p z)sen(\beta_p z')\frac{\beta_p^2+B^2}{\beta_p^2+B^2+B} \tag{14}$$

Where $\beta_m = \frac{m\pi}{L_1}$, $\beta_n = \frac{n\pi}{L_2}$, $-B = \beta_p cot(\beta_p)$, $B = \frac{hL_3}{k}$ e $u = \alpha(t-\tau)$.
Original temperature can then be recorved by Eq.(4).

$$T(x,y,z,t) = W(x,y,z,t)exp\left(\frac{ux}{2\alpha}-\frac{u^2t}{4\alpha}\right) \tag{15}$$

Figure (2) presents the temperatures generated by $3D$ analytical solution using a gaussian heat source presented in figure (1) at the fixed point $(0.02, 0.05, L_3)$. This coordinate can simulate a temperature acquisition system.

Moreover, using $\alpha = 23.1 \times 10^{-6} m^2 s^{-1}$, $k = 80.2 W m^{-1} K^{-1}$ heat source speed $u = 0.001 m s^{-1}$ at initial position $(0.00, 0.05, L_3)$ for the parallelepiped with edges $L_1 = L_2 = L_3 = 0.1m$
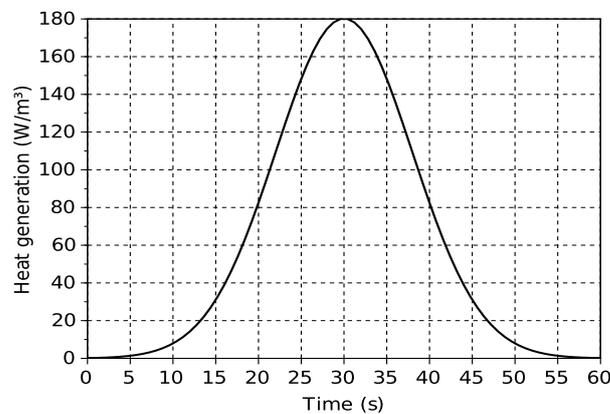

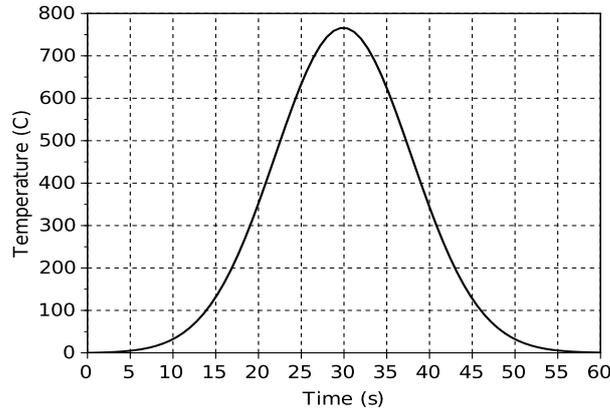Figure 1. Gaussian heat source generation

Figure 2. Temperatures for a gaussian distribution

3. THE TFBGF METHOD

For any dynamic system (Fig. 3), the relation between input $X(t)$ and output $Y(t)$ can be given by the convolution equation:
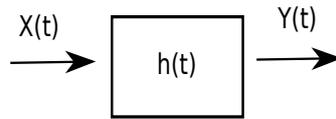

Figure 3. Diagram for dynamical systems

This dynamical system is constituted by three variables $X(t)$, $h(t)$ and $Y(t)$ where $X(t)$ refers, in thermal systems, to heat flux, heat generation or another way to excite the system, $h(t)$ refers to system behavior, is called transfer function Fernandes (2013), $Y(t)$ is the system response, in thermal cases, temperature.

Here we propose in an analytic way to determine the system behavior, $h(t)$ function. Through $h(t)$ function we can deternine the system input and solve an inverse problem about estimate heat flux, heat source or moving versions of them.

Altough, starting from convolution theorem:

$$Y(t) = h(t) * X(t) = \int_{-\infty}^{+\infty} h(t - \tau)X(\tau)d\tau \qquad (16)$$

Note that the right hand of Eq. (16), the convolution integration is given by Green's function form, where, $h(t - \tau)$ times $X(t)$ function is observed.

That in Laplace domain, the convolution theorem is expressed by the multiplication

$$Y(x, s) = H(x, s) \cdot X(s). \qquad (17)$$

In this sense, input X in domain s can be calculated by

$$X(s) = (1/H(x, s)) \cdot Y(x, s). \qquad (18)$$

or in the time domain by the deconvolution

$$X(t) = \mathcal{L}^{-1}\{1/h(x, t)\} * Y(x, t). \qquad (19)$$

Comparing Eq.(13) and (16) it can be identified an equivalent thermal system where the output of the system can be represented by the auxiliar temperature W, the transfer function by the modified Green's function G^* and heat moving source q_g can be considered the input of the system. It means,

$$G^*(\xi, y, z, t|\xi', y', z', t - \tau) = h(t - \tau)$$

and

$$q_g(t) = X(t) \tag{20}$$

An imediate application of the TFBGF is identify an thermal system which has a dynamic equivalence. In this way, considering the system output as temperature W, transfer function as G^* and the moving heat source can be considered the input of the system.

4. ANALYTICAL IMPULSE RESPONSE IDENTIFICATION

The proposed methodology for identification of the analytical impulse response is based on the theory of dynamical systems of one input and one output.

It can be observed that for any input $X(t)$ and the output $Y(t)$, the transfer fucntion, $H(t)$, remains the same.

In this case, if the input $X(\tau) = \delta(\tau)$ is applied, the transfer function, then, can be obtained by the auxiliar temperature distribution, $W(t)$. It means, $Y(t) = W(t) = h(t) = G^*(t)$.

Therefore, by comparing Eq.(13) and Eq.(16), and using transformation Eq. (4). In this case we use $q_g(\tau) = \delta(\tau)$. After the integral performing the Transfer function can then be calculated as

$$
\begin{aligned}
h(\xi, y, z, t) = {} & \frac{\alpha}{k} \frac{8}{L_1 L_2 L_3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} e^{-(\beta_m^2 + \beta_n^2 + \beta_p^2)\alpha t} \\
& \times sen(\beta_m \xi) sen(\beta_m P_\xi) sen(\beta_n y) sen(\beta_n P_y) \\
& \times sen(\beta_p z) sen(\beta_p P_z) e^{\frac{u P_\xi}{2\alpha}} \left(\frac{\beta_p^2 + B^2}{\beta_p^2 + B^2 + B} \right)
\end{aligned}
\tag{21}
$$

That is the analytical impulse response to a moving heat source.

5. CONCLUSION

To sum up, it was possible identificate in analitycal way the impulse response through transfer function for heat conduction equation with moving heat source, change variables were needed to made adaptations in classical TFBGF theory. To execute the IHCP thecnique proposed here we need know temperatures invoved in these processes, but can be consider hypothetical or experimental. If use hypothetical temperatures are necessary know the analytical or numerical temperature solution.

Therefore, knonwing both of them, temperatures and impuse response is possible to evaluate the TFBGF thecnique and estimate heat flux, heat source or moving versions of them. The TFBGF thecnique showed a strong tool for IHCP applications because not dependent of minimization or numeric thecniques, just perform a division of temperatures by impulse response using the fast Fourier transform FFT to take both of them to Laplace domain.

6. ACKNOWLEDGEMENTS

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